

Reinforcement of a thin plate by a thin layer.

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Abstract

We study the bending of a thin plate, stiffened with a thin elastic layer, of thickness δ . We describe the complete construction of an asymptotic expansion with respect to δ of the solution of the Kirchhoff-Love model and give optimal estimates for the remainder. We identify approximate boundary conditions, which take into account the effect of the stiffener at various orders. Thanks to the tools of multi-scale analysis, we give optimal estimates for the error between the approximate problems and the original one. We deal with a layer of constant stiffness, as well as with a stiffness in δ^{-1} .

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1 Introduction

The structures studied in engineering are often made of materials covered with thin layers. Their mathematical modelling is a problem of outstanding practical importance. However, from a numerical point of view, such problems require the discretization of the thin layer which needs very thin meshes and may lead to very expansive calculations.

An alternative well-known approach consists in deriving approximate boundary conditions that incorporate in an approximate way the effect of the thin layer and permit to remove the mesh constraints on the discretization. More precisely, we seek an approximate problem posed on the interior domain (i.e. not including the thin layer) but taking into account its effect via these new conditions.

The idea of introducing this type of boundary conditions which can be substituted to the thin layer has been widely used in numerous studies, mainly in electro-magnetics and mechanics, see for instance [6, 3] for the Helmholtz equation in acoustics, [2, 8] for Maxwell equations, and [12, 11, 13, 7, 14] in structure mechanics, see also [16, 1, 15].

It is also worth noting that there is a hierarchy in these boundary conditions: the greater the order, the better the approximation. Moreover, they lead to non standard boundary value problems in which the boundary conditions involve tangential derivatives of order greater or equal to that of the interior differential operator.

The purpose of this paper is to describe the application of an asymptotic method, for identifying approximate boundary conditions within the framework of linear elasticity. To begin with, we consider a two-dimensional model; referred as the Kirchhoff-Love model, for an elastic plate surrounded by a thin elastic layer. The middle surface of the plate is denoted by Ω_+ . The boundary of Ω_+ consists of two disjoint parts, Γ_0 and Γ , assumed to be smooth. For $\delta > 0$ sufficiently small, the elastic layer Ω_-^δ derives from a uniform dilation of Γ_0 in the normal direction, with thickness δ :

$$\Omega_-^\delta = \{x + r \mathbf{n}(x) ; x \in \Gamma_0 \text{ and } 0 < r < \delta\},$$

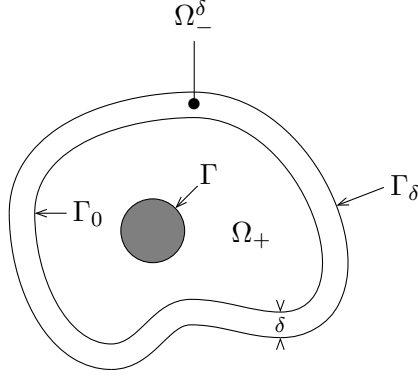


Figure 1: The plate and the thin layer.

where $\mathbf{n}(x)$ denotes the normal vector at point x on Γ_0 , outer from Ω_+ ; the external boundary of the domain Ω_- is Γ_δ and the whole domain is $\Omega^\delta = \Omega_+ \cup \Gamma_0 \cup \Omega_-$ (see figure 1).

These two elastic bodies are perfectly “bonded” along their common boundary Γ_0 , thus forming together an elastic multi-structure, viewed as an elastic plate of middle surface Ω^δ . This plate is clamped along its interior boundary Γ and is motion free on its exterior boundary Γ_δ . The equations given by the Kirchhoff-Love model for the displacement w^δ (which stands for the bending of the plate) read as follows (see [10, 9])

$$\begin{cases} D^+ \Delta^2 w_+^\delta = f & \text{in } \Omega_+, \\ D^- \Delta^2 w_-^\delta = 0 & \text{in } \Omega_-, \\ [w^\delta] = 0; [\partial_n w^\delta] = 0 & \text{on } \Gamma_0, \\ M^+(w_+^\delta) = M^-(w_-^\delta); T^+(w_+^\delta) = T^-(w_-^\delta) & \text{on } \Gamma_0, \\ M(w^\delta) = 0; T(w^\delta) = 0 & \text{on } \Gamma_\delta, \\ w^\delta = 0; \partial_n w^\delta = 0 & \text{on } \Gamma, \end{cases} \quad (1)$$

where ∂_n denotes the normal derivative along $\mathbf{n} = (n_1, n_2)$ and $[]$ the jump across Γ_0 ; the loading f belongs to $L^2(\Omega_+)$ (we will see that further regularity is required for the asymptotic analysis). The trace operators M and T denote respectively the *bending moment* and the *shear force*, and have the following expressions:

$$\begin{aligned} M &= D [\Delta + (1 - \nu) (2n_1 n_2 \partial_{12} - n_1^2 \partial_2^2 - n_2^2 \partial_1^2)], \\ T &= D [\partial_n \Delta + (1 - \nu) \partial_\tau ((n_1^2 - n_2^2) \partial_{12} + n_1 n_2 (\partial_2^2 - \partial_1^2))], \end{aligned}$$

where $D = \frac{2E}{3(1-\nu^2)}$, E being the *Young's modulus* and $\nu \in (0, \frac{1}{2})$ the *Poisson's ratio*; ∂_τ denotes the tangential derivative. We assume that the elastic coefficients E and ν are piecewise constant: $E = E_+$ in Ω_+ and E_- in Ω_- ; $\nu = \nu_+$ in Ω_+ and ν_- in Ω_- . The Poisson's ratio is independent of δ and we will successively consider the two cases for the Young's modulus:

- the coefficients E^+ and E^- are independent of δ ;
- the coefficient E^+ does not depend on δ and $E^- = \mathcal{O}(\delta^{-1})$.

The second case is more interesting because it stands for a layer which is at the same time very thin and very stiff.

Hence, D is piecewise constant, and we set

$$D_+ = \frac{2E_+}{3(1-\nu_+^2)} \quad \text{and} \quad D_- = \frac{2E_-}{3(1-\nu_-^2)}.$$

The relations along Γ_0 , which formally express the continuity of the displacement w , of M , T and ∂_n along the common portion of the two boundaries are called transmission conditions. The first condition along Γ_0

shows in particular that we are modelling a situation where the inserted portion of the layer is “perfectly bonded” to the plate : we are thus excluding situations where the inserted portion could slide along, or part away from, the plate.

As was already pointed out, our aim is to identify and justify boundary conditions on Γ_0 for problem (1), which approximate the effect of the thin layer. We use the technique of multi-scale expansions (see [17, 18, 4]) to build an asymptotic expansion (in powers of δ) of w^δ , solution of (1), as δ tends to 0; then we have to estimate the remainder after cut-off at a given order. The approximate boundary conditions are then obtained by considering the series given by its asymptotic expansion, truncated at a given order. The conditions satisfied by this approximation on Γ_0 give the desired boundary condition.

The article is organized as follows. First, we set the variational framework used to solve problem (1) and we give the expressions of the operators in local coordinates on the boundary Γ_0 . Then we build the asymptotic expansion of w^δ and give optimal estimates of the remainder, based on a priori estimates. Finally, we derive and justify approximate boundary conditions for problem (1) from the first terms of the expansion.

In the following, for $m \in \mathbb{N}$, $H^m(\omega)$ denotes the standard Sobolev space of order m in the open set ω , endowed with its natural norm $\|\cdot\|_{m,\omega}$:

$$\|w\|_{m,\omega}^2 = \sum_{|\alpha| \leq m} \|\partial^\alpha w\|_{L^2(\omega)}^2.$$

2 Outline of the results

We expose here the main results of our paper (for detailed statements, error estimates, and proofs, see sections below). Basically, we sum up the approximate boundary conditions obtained for Problem (1), i.e. such that the solution \tilde{w}_+ of

$$\begin{cases} D_+ \Delta^2 \tilde{w}_+ = f_+ & \text{in } \Omega_+, \\ \text{Approximate boundary condition (ABC)} & \text{on } \Gamma_0, \\ \tilde{w}_+ = 0; \partial_n \tilde{w}_+ = 0 & \text{on } \Gamma, \end{cases}$$

gives an approximation of w_+^δ .

2.1 Case of a constant Young’s modulus

When the coefficients E^+ and E^- do not depend on δ , problem (1) converges towards the limit case $\delta = 0$. This boundary value problem can be seen as the order 0-approximate problem for (1). The boundary condition on Γ_0 is merely the Neumann conditions inherited from the stiffener:

$$(ABC) : M_+(\tilde{w}_+) = 0 \text{ and } T_+(\tilde{w}_+) = 0 \text{ on } \Gamma_0.$$

Of course such an approximation is not accurate since it simply omits the effect of the thin layer. A better approximation of w_+^δ is given by the first order condition

$$(ABC) : M_+(\tilde{w}_+) + \delta Q_0(\tilde{w}_+) = 0 \text{ and } T_+(\tilde{w}_+) + \delta P_0(\tilde{w}_+) = 0 \text{ on } \Gamma_0,$$

where the operators P_0 and Q_0 are given in the local coordinates (s is the arclength, $c(s)$ the curvature, ∂_s and ∂_n the tangential and normal derivatives, respectively)

$$\begin{aligned} Q_0 &= -D_- \left[2(1 - \nu_-) \partial_s (\partial_s \partial_n - c(s) \partial_s) - (1 - \nu_-^2) c(s) (\partial_s^2 + c(s) \partial_n) \right], \\ P_0 &= -D_- \left[(1 - \nu_-^2) \partial_s^2 (\partial_s^2 + c(s) \partial_n) + 2(1 - \nu_-) \partial_s [c(s) (\partial_s \partial_n - c(s) \partial_s)] \right], \end{aligned}$$

Here, the approximate boundary condition depends the thickness δ of the layer and, through P_0 and Q_0 , takes nontrivially into account the effect of the stiffener. Tangential derivatives of \tilde{w}_+ and $\partial_n \tilde{w}_+$ are involved, they may be interpreted as the bending and torsion contributions of the thin layer. The wellposedness of this problem, as well as optimal error estimates in strong energy norm, are given in section 5. Higher order conditions may also be derived, but technicality increases drastically and their mechanical interpretation gets less clear.

2.2 Case of a Young's modulus in δ^{-1}

It is more natural to consider a Young's modulus blowing up in the stiffener as δ goes to 0. Actually the right scaling is $E_- \sim \delta^{-1}$, where the mechanical coefficient exactly compensates the thickness of the layer. It turns out that the limit as $\delta \rightarrow 0$ of Problem (1) involves itself the operators P_0 and Q_0 defined above and the associated zero-order approximate boundary conditions is nontrivial, it writes

$$(ABC) : M_+(\tilde{w}_+) + Q_0(\tilde{w}_+) = 0 \text{ and } T_+(\tilde{w}_+) + P_0(\tilde{w}_+) = 0 \text{ on } \Gamma_0,$$

It is worth noticing that the major effect of the stiffener contained in operators P_0 and Q_0 is now seen since order 0 whereas it only appears at order 1 for a constant Young's modulus. This corresponds to the natural physical idea that the high rigidity of the stiffener emphasizes its effect on the plate. Besides, a condition of order 1 is derived in section 6, which involves higher order tangential operators, see (24).

2.3 Behavior in the stiffener

The approximate boundary conditions presented previously give information about the global mechanical behavior of the stiffened plate. Indeed, they provide a hierarchy of approximate problems which replaces the effect of the stiffener with more or less accuracy. Nevertheless, such approximate problems only give a representation of the displacement inside the plate, and nothing in the stiffener.

Actually the asymptotic expansion built in section 4.2 (and leading to the approximate boundary conditions just under discussion) also gives a precise description inside the stiffener. Precisely, the displacement in the layer admits the following asymptotic behavior in local coordinates (we do not give any sense to the convergence of the series, the notation only means that the error is small when truncating at a fixed order)

$$w_-^\delta(x) = \sum_{\ell \geq 0} \delta^\ell P_\ell(r) \psi_\ell(s),$$

where (r, s) are the normal-tangential coordinates in Ω_-^δ , P_ℓ is a polynomial of degree less than ℓ , and ψ_ℓ are smooth functions on the boundary Γ_0 . Such an expression has the particularity to be cartesian in the Frénet coordinates, with polynomial dependence towards the normal variable. This allow to easily obtain bounds on the displacement and its derivatives and can therefore be useful for addressing local properties of the material such as crack initiation (in this case, the dependence of the tangential functions ψ_ℓ needs also to be precised).

3 Preliminaries

3.1 Existence, uniqueness, a priori estimate

In order to obtain remainder estimates for the asymptotic expansion in section 4.2, we will consider more general right hand-sides in the problem (1):

$$\left\{ \begin{array}{ll} D^+ \Delta^2 w_+^\delta = f_+ & \text{in } \Omega_+, \\ D^- \Delta^2 w_-^\delta = f_- & \text{in } \Omega_-^\delta, \\ [w^\delta] = 0 ; [\partial_n w^\delta] = 0 & \text{on } \Gamma_0, \\ M^+(w_+^\delta) = M^-(w_-^\delta) + g_1 ; T^+(w_+^\delta) = T^-(w_-^\delta) + g_2 & \text{on } \Gamma_0, \\ M(w^\delta) = h_1 ; T(w^\delta) = h_2 & \text{on } \Gamma_\delta, \\ w^\delta = 0 ; \partial_n w^\delta = 0 & \text{on } \Gamma. \end{array} \right. \quad (2)$$

We define the following functional space

$$W = \{ \psi \in H^2(\Omega^\delta) ; \psi = \partial_n \psi = 0 \text{ on } \Gamma \}.$$

Let $w \in W$ be a solution of the problem (2). Integrating by parts, we get for $\psi \in W$,

$$\int_{\Omega^\delta} D(\Delta^2 w) \psi \, dx = a(w, \psi) + \int_{\partial\Omega^\delta} [T(w)\psi - M(w)\partial_n \psi] \, d\sigma,$$

where the bilinear form a is given by

$$a(w, \psi) = \int_{\Omega^\delta} D \left[(\partial_1^2 w + \nu \partial_2^2 w) \partial_1^2 \psi + 2(1 - \nu) \partial_{12} w \partial_{12} \psi + (\partial_2^2 w + \nu \partial_1^2 w) \partial_2^2 \psi \right] dx. \quad (3)$$

The variational formulation of problem (2) reads

$$\forall \psi \in V, \quad a(w, \psi) = \langle F, \psi \rangle, \quad (4)$$

with the linear form F :

$$\langle F, \psi \rangle = \int_{\Omega_+} f_+ \psi dx + \int_{\Omega_-} f_- \psi dx + \int_{\Gamma_0} (g_2 \psi - g_1 \partial_n \psi) d\sigma + \int_{\Gamma_\delta} (h_2 \psi - h_1 \partial_n \psi) d\sigma.$$

The following theorem gives a coarse estimate – but sufficient for our purpose – for problem (4)

Theorem 1 *Let $f_+ \in L^2(\Omega_+)$, $f_- \in L^2(\Omega_-)$, $g_1, g_2 \in L^2(\Gamma_0)$ and $h_1, h_2 \in L^2(\Gamma_\delta)$. There exists a unique solution $w^\delta \in W$ for Problem (4). Moreover, we have the following a priori estimate, with a constant C , independent of $\delta \in (0, 1)$:*

$$\|w^\delta\|_{2, \Omega^\delta} \leq C \left(\|f_+\|_{0, \Omega_+} + \|f_-\|_{0, \Omega_-} + \|g_1\|_{0, \Gamma_0} + \|g_2\|_{0, \Gamma_0} + \|h_1\|_{0, \Gamma_\delta} + \|h_2\|_{0, \Gamma_\delta} \right). \quad (5)$$

PROOF. This is a straightforward application of the Lax-Milgram lemma: the form F is obviously continuous on W , its norm in W' being bounded by the right hand-side of inequality (5). Furthermore the bilinear form a is continuous and coercive on W : thanks to Dirichlet conditions on Γ , a Poincaré inequality holds in Ω^δ , independent of δ (since the measure of the domain Ω^δ is uniformly bounded for $0 < \delta < 1$). ■

We have seen that equations (1) define a well-posed problem in $H^2(\Omega^\delta)$, for fixed $\delta \in (0, 1)$. In the next section, we focus on the asymptotic analysis of the solution w^δ when the thickness δ of the layer goes to 0.

3.2 Expressions of the operators in local coordinates

Depending on the thickness δ , the functional setting of our problem is not suited for giving a precise meaning to an asymptotic expansion of the solution. Hence, the first step of the analysis is a scaling inside the thin layer in order to remove the dependence of the space domain on the small parameter δ . So, we will perform a dilation in the normal direction of the layer Ω_-^δ (of ratio δ^{-1}) to get a fixed geometry. To achieve this goal, the use of Frénet coordinates is needed. The operators involved in problem (1) can be expanded into powers of δ , which is the first step towards the construction of an asymptotic expansion for w^δ .

3.2.1 Frénet coordinates

We denote by \mathbf{t} and \mathbf{n} the vectors respectively tangent and normal on Γ . They are directly orthogonal:

$$\mathbf{t} = \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} \quad \text{and} \quad \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}.$$

We recall the Frénet formulæ defining the curvature $c(s)$ at the point on Γ_0 with arclength s .

$$\frac{d\mathbf{t}}{ds} = -c(s) \mathbf{n} \quad \text{and} \quad \frac{d\mathbf{n}}{ds} = c(s) \mathbf{t}.$$

We denote by $c(s, r)$ the curvature on $\Gamma_r = \{x + r\mathbf{n}(x) ; x \in \Gamma_0\}$ at point (s, r) ; we have the identity

$$c(s, r) = \frac{c(s)}{1 + rc(s)}.$$

As a mere consequence of the Frénet formulæ, we get the expressions of the cartesian derivatives $\partial_1 = \frac{\partial}{\partial x_1}$ and $\partial_2 = \frac{\partial}{\partial x_2}$ in the local coordinates:

$$\begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix} = \begin{pmatrix} \frac{n_2}{(1+rc(s))} & n_1 \\ \frac{-n_1}{(1+rc(s))} & n_2 \end{pmatrix} \begin{pmatrix} \partial_s \\ \partial_r \end{pmatrix}$$

We deduce the expression of the bilaplacian

$$\begin{aligned}\Delta^2 &= \frac{1}{(1-rc(s))} \partial_s \left[\frac{1}{(1+rc(s))} \partial_s \left(\frac{1}{(1+rc(s))^2} \partial_s^2 - \frac{rc'(s)}{(1+rc(s))^3} \partial_s \right) + c(s,r) \partial_r + \partial_r^2 \right] \\ &\quad + c(s,r) \partial_r \left[\frac{1}{(1+rc(s))^2} \partial_s^2 - \frac{rc'(s)}{(1+rc(s))^3} \partial_s + c(s,r) \partial_r + \partial_r^2 \right] \\ &\quad + \partial_r^2 \left[\frac{1}{(1+rc(s))^2} \partial_s^2 - \frac{rc'(s)}{(1+rc(s))^3} \partial_s + c(s,r) \partial_r + \partial_r^2 \right]\end{aligned}$$

as well as the expressions of the trace operators M and T :

$$\begin{aligned}M &= D [\nu \Delta + (1-\nu) \partial_r^2], \\ T &= D \left[\partial_n \Delta + (1-\nu) \frac{1}{1+rc(s)} \partial_s \left(-\frac{c(s)}{(1+rc(s))^2} \partial_s + \frac{1}{(1+rc(s))} \partial_{rs} \right) \right].\end{aligned}$$

3.2.2 Expansion of the operators into powers of δ

Thanks to Frénet coordinates, the thin layer reads in a tensorial way:

$$\Omega_-^\delta \simeq \Gamma_0 \times (0, \delta).$$

Introducing the scaled variable – or fast variable – $y = \frac{r}{\delta}$, we obtain a fixed domain $\Gamma_0 \times (0, 1)$. We notice that the normal dilation is performed in the Frénet variables, not in the physical domain Ω_-^δ itself. Indeed, since we did not assume any convexity, the domain Ω_-^1 might not be well-defined (nevertheless, we could work in $\Omega_-^{\delta_0}$ for δ_0 sufficiently small).

Let W^δ denote the function defined in $\Gamma_0 \times (0, 1)$ by

$$W^\delta(s, y) = w^\delta(s, r).$$

The dilation $r \mapsto y$ maps the exterior layer Ω_-^δ into a fixed domain; the small parameter δ is now involved in the equations. Hence, the biharmonic operator expands into powers of δ :

$$\Delta^2 = \frac{1}{\delta^4} A^{-4} + \frac{1}{\delta^3} A^{-3} + \frac{1}{\delta^2} A^{-2} + \frac{1}{\delta} A^{-1} + A^0 + \delta A^1 + \dots$$

The first terms are given by – we write c for $c(s)$

$$\begin{aligned}A^{-4} &= \partial_y^4; \\ A^{-3} &= 2c \partial_y^3; \\ A^{-2} &= 2\partial_s^2 (\partial_y^2) + c^2 \partial_y^2 - yc \partial_y^3 - c^2 \partial_y^2 (y \partial_y); \\ A^{-1} &= \partial_s^2 (c \partial_y) - 2yc \partial_s^2 (\partial_y^2) - yc' \partial_s (\partial_y^2) + c \partial_y (\partial_s^2) - y c^3 \partial_y^2 \\ &\quad - c^3 \partial_y (y \partial_y) + y^2 c^3 \partial_y^3 - 2c \partial_y^2 (y \partial_s^2) - c' \partial_y^2 (y \partial_s) + c^3 \partial_y^2 (y^2 \partial_y); \end{aligned}$$

$$\begin{aligned}
A^0 &= \partial_s^4 - 2yc \partial_s^2 (c \partial_y) - y \partial_s^2 (c^2 \partial_y) + 3y^2 c^2 \partial_s^2 (\partial_y^2) - yc' \partial_s (c \partial_y) \\
&\quad + 3y^2 cc' \partial_s (\partial_y^2) - 2c^2 \partial_y (y \partial_s^2) - yc^2 \partial_y (\partial_s^2) - cc' \partial_y (y \partial_s) + c^4 \partial_y (y^2 \partial_y) \\
&\quad + yc^4 \partial_y (y \partial_y) + y^2 c^4 \partial_y^2 - y^3 c^4 \partial_y^3 + 3c^2 \partial_y^2 (y^2 \partial_s^2) + 3cc' \partial_y^2 (y^2 \partial_s) \\
&\quad - c^4 \partial_y^2 (y^3 \partial_y) ; \\
A^1 &= -2y \partial_s^2 (c \partial_s^2) + 2yc \partial_s^4 - y \partial_s^2 (c' \partial_s) + y^2 \partial_s^2 (c^3 \partial_y) + 2y^2 c \partial_s^2 (c^2 \partial_y) \\
&\quad + 3y^2 c^2 \partial_s^2 (c \partial_y) - 4y^3 c^3 \partial_s^2 (\partial^2) - yc' \partial_s^3 + y^2 c' \partial_s (c^2 \partial_y) \\
&\quad + 3y^2 cc' \partial_s (c \partial_y) - 6y^3 c^2 c' \partial_s (\partial_y^2) + 3c^3 \partial_y (y^2 \partial_s^2) + 2yc^3 \partial_y (y \partial_s^2) \\
&\quad - y^2 c^3 \partial_y (\partial_s^2) + 3c^2 c' \partial_y (y^2 \partial_s) + yc^2 c' \partial_y (y \partial_s) - c^5 \partial_y (y^3 \partial_y) \\
&\quad - yc^5 \partial_y (y^2 \partial_y) - y^2 c^5 \partial_y (y \partial_y) - y^3 c^5 \partial_y^2 + y^4 c^5 \partial_y^3 \\
&\quad - 4c^3 \partial_y^2 (y^3 \partial_s^2) - 6c^2 c' \partial_y^2 (y^3 \partial_s) + c^5 \partial_y^2 (y^4 \partial_y) .
\end{aligned}$$

The trace operators also expand into powers of δ :

$$\begin{aligned}
M &= \frac{1}{\delta^2} M^{-2} + \frac{1}{\delta} M^{-1} + M^0 + \delta M^1 + \delta^2 M^2 + \dots \\
T &= \frac{1}{\delta^3} T^{-3} + \frac{1}{\delta^2} T^{-2} + \frac{1}{\delta} T^{-1} + T^0 + \delta T^1 + \delta^2 T^2 + \dots
\end{aligned}$$

with

$$\begin{aligned}
M^{-2} &= D_- \partial_y^2 ; \\
M^{-1} &= D_- \nu_- c \partial_y ; \\
M^0 &= D_- \nu_- (\partial_s^2 - yc^2 \partial_y) ; \\
M^1 &= D_- \nu_- (-2yc \partial_s^2 - yc' \partial_s + \nu_- y^2 c^3 \partial_y) ; \\
M^2 &= D_- \nu_- (3y^2 c^2 \partial_s^2 + 3y^2 cc' \partial_s - y^3 c^4 \partial_y) ; \\
T^{-3} &= D_- \partial_y^3 ; \\
T^{-2} &= D_- c \partial_y^2 ; \\
T^{-1} &= D_- [\partial_y (\partial_s^2) - \partial_y (yc^2 \partial_y) + (1 - \nu_-) \partial_y (\partial_s^2)] ; \\
T^0 &= D_- [\partial_y (-2yc \partial_s^2 - yc' \partial_s + y^2 c^3 \partial_y) + (1 - \nu_-) \partial_s (c \partial_s) \\
&\quad + (1 - \nu_-) (2yc \partial_y \partial_s^2 + yc' \partial_y \partial_s)] ; \\
T^1 &= D_- [\partial_y (3y^2 c^2 \partial_s^2 + 3y^2 cc' \partial_s - y^3 c^4 \partial_y) + (1 - \nu_-) \partial_s (-2yc^2 \partial_s + y^2 c^2 \partial_{ys}) \\
&\quad + (1 - \nu_-) yc \partial_s (c \partial_s) + (1 - \nu_-) y^2 c^2 \partial_s (\partial_{ys}) + (1 - \nu_-) y^2 c \partial_s (c \partial_{ys})] ; \\
T^2 &= D_- [\partial_y (-4y^3 c^3 \partial_s^2 + 6y^2 cc' \partial_s + y^4 c^5 \partial_y) + (1 - \nu_-) \partial_s (-3y^2 c^3 \partial_s - y^3 c^3 \partial_{ys}) \\
&\quad - (1 - \nu_-) yc \partial_s (2yc^2 \partial_s) - (1 - \nu_-) y^2 c^2 \partial_s (c \partial_s) - (1 - \nu_-) yc \partial_s (y^2 c^2 \partial_{ys}) \\
&\quad - (1 - \nu_-) y^2 c^2 \partial_s (yc \partial_{ys}) - (1 - \nu_-) y^3 c^3 \partial_s (\partial_{ys})] .
\end{aligned}$$

The expansions into powers of δ given in this section are formal (we do not give a sense to the convergence of the series). This point of view is generally adopted in multi-scale analysis; it allows to build (and prove the convergence of) the asymptotic expansion, see next sections.

4 Construction of the asymptotic expansion

First, we consider the case where the Young's modulus is independent of the thickness δ . In the previous section, we have introduced the scaled variable $y = \frac{r}{\delta}$, which maps the geometry onto a fixed domain; the

small parameter δ is now involved only in the equations, see the expansions of the operators in section 3.2.2. We aim here at describing an algorithmic procedure to build the terms of the asymptotic expansion of the solution of problem (1).

Since the expansions of the operators Δ^2 , M and T only involve integer powers of δ (see §3.2.2), we seek an expansion of the form

$$w_+^\delta = \sum_{n \geq 0} \delta^n w_+^n \quad \text{and} \quad W_-^\delta = \sum_{n \geq 0} \delta^n W_-^n, \quad (6)$$

we recall that the function W_-^δ denotes the transform of w_-^n in semi-scaled variables (s and r stand for the tangential and normal variables, respectively):

$$w_-^\delta(s, r) = W_-^\delta(s, \delta^{-1}r).$$

Inserting the ansatz (6) into the equations (1) and identifying the terms with same power of δ , we obtain

$$\left\{ \begin{array}{l} D_+ \Delta^2 w_+^n = f_+ \text{ if } n = 0 \text{ and } 0 \text{ else} \quad \text{in } \Omega_+, \\ A^{-4} W_-^n = - \sum_{\substack{k+\ell=n-4 \\ k \geq -3}} A^k W_-^\ell \quad \text{for } 0 \leq y \leq 1, \\ W_-^n = w_+^n \quad ; \quad \partial_n W_-^n = \partial_n w_+^{n-1} \quad \text{on } \Gamma_0, \\ M_+(w_+^n) = \sum_{k+\ell=n} M^k W_-^\ell \quad \text{on } \Gamma_0, \\ T_+(w_+^n) = \sum_{k+\ell=n} T^k W_-^\ell \quad \text{on } \Gamma_0, \\ M^{-2}(W_-^n) = - \sum_{k+\ell=n-2} M^k W_-^\ell \quad \text{on } \Gamma_-, \\ T^{-3}(W_-^n) = - \sum_{k+\ell=n-3} T^k W_-^\ell \quad \text{on } \Gamma_-, \\ w_+^n = 0 \quad ; \quad \partial_n w_+^n = 0 \quad \text{on } \Gamma. \end{array} \right. \quad (7)$$

4.1 The first terms of the expansion

We detail here the construction of the first terms of the expansion. In the equations (7), the transmission conditions for the normal derivatives, as well as the operators M and T , involve different order terms, thanks to the shift in the powers of δ . It allows an alternative resolution in each subdomain Ω_-^δ and Ω_+ . The operator T being of order 3, we need to know W_-^ℓ for $\ell = 0, 1, 2, 3$ to define the term $T_+(w_+^0)$. For this reason, we start by looking for the first four exterior terms.

For $n = 0$, equations (7) give the following problem for the exterior part W_-^0

$$\left\{ \begin{array}{l} D_- \partial_y^4 W_-^0(s, y) = 0 \quad \text{for } 0 \leq y \leq 1, \\ D_- \partial_y^3 W_-^0(s, 1) = 0, \\ D_- \partial_y^2 W_-^0(s, 1) = 0, \end{array} \right. \quad (8)$$

which leads to $W_-^0(s, y) = \alpha^0(s)y + \beta^0(s)$, where α^0 and β^0 are functions of the arclength s . Taking into account the transmission condition for the normal derivatives on Γ_0 , we deduce $\partial_n W_-^0(s, 0) = 0$, whence

$$W_-^0(s, y) = \beta^0(s).$$

At order 1, the equations satisfied by W_-^1 are the same as before, see (8). Thus

$$W_-^1(s, y) = \alpha^1(s)y + \beta^1(s).$$

At order 2, a non-zero right hand-side appears, due to the terms $M^0W_-^0$ and $M^{-1}W_-^1$: W_-^2 solves the problem

$$\begin{cases} D_- \partial_y^4 W_-^2(s, y) &= 0 \text{ for } 0 \leq y \leq 1, \\ D_- \partial_y^3 W_-^2(s, 1) &= 0, \\ D_- \partial_y^2 W_-^2(s, 1) &= -D_- \theta_-^0(s, 1), \end{cases}$$

with $D_- \theta_-^0(s, y) = M^0W^0 + M^{-1}W^1 = D_- [\nu_- \partial_s^2 \beta^0(s) + \nu_- c(s) \alpha^1(s)]$ (which, here, does not depend on the transverse variable y). Thus, we can find two functions of the arclength α^2 and β^2 such that

$$W_-^2(s, y) = -\frac{1}{2} \theta_-^0(s, 1) y^2 + \alpha^2(s) y + \beta^2(s).$$

We now look at order 3; we set

$$\begin{aligned} \zeta_-^1(s, y) &= T^0W_-^0 + T^{-1}W_-^1 + T^{-2}W_-^2, \\ \theta_-^1(s, y) &= M^1W_-^0 + M_0W_-^1 + M^{-1}W_-^2. \end{aligned}$$

Then W_-^3 solves

$$\begin{cases} D_- \partial_y^4 W_-^3(s, y) &= 0 \text{ for } 0 \leq y \leq 1, \\ D_- \partial_y^3 W_-^3(s, 1) &= -\zeta_-^1(s, 1), \\ D_- \partial_y^2 W_-^3(s, 1) &= -\theta_-^1(s, 1), \end{cases}$$

so that it reads

$$W_-^3(s, y) = -\zeta_-^1(s, 1) \left(\frac{1}{6}y^3 - \frac{1}{2}y^2\right) - \frac{1}{2} \theta_-^1(s, 1) y^2 + \alpha^3(s) y + \beta^3(s),$$

where the functions α^3 and β^3 have to be determined.

We now write the problem solved by the first interior term:

$$\begin{cases} D_+ \Delta^2 w_+^0 = f_+ & \text{in } \Omega_+, \\ M_+(w_+^0) = \theta_+^0(s); T_+(w_+^0) = \zeta_+^0(s) & \text{on } \Gamma_0, \\ w_+^0 = 0; \partial_n w_+^0 = 0 & \text{on } \Gamma, \end{cases} \quad (9)$$

where the data (θ_+^0, ζ_+^0) is defined by

$$\begin{aligned} \theta_+^0(s) &= (M^{-2}W_-^2 + M^{-1}W_-^1 + M^0W^0)(s, 0) \\ &= D_- [-\theta_-^0(s, 1) + \theta_-^0(s, 0)] \\ &= 0, \\ \zeta_+^0(s) &= (T^{-3}W_-^3 + T^{-2}W_-^2 + T^{-1}W_-^1 + T^0W^0)(s, 0) \\ &= D_- [-\zeta_-^1(s, 1) + \zeta_-^1(s, 0)] \\ &= 0, \end{aligned}$$

by definition of θ^1 and ζ^1 , since these functions do not depend on y . As a consequence, problem (9) is nothing but the bi-harmonic problem in Ω_+ with homogeneous Dirichlet conditions on Γ and homogeneous Neumann conditions on Γ_0 ; it uniquely defines w_+^0 . The transmission conditions on Γ_0 allow to determine the functions β^0 and α^1 :

$$\beta^0(s) = w_+^0|_{\Gamma_0} \quad \text{et} \quad \alpha^1(s) = \partial_n w_+^0|_{\Gamma_0}.$$

Coming back to the exterior part, we set

$$\begin{aligned} \phi_-^2(s, y) &= (A^{-3}W_-^3 + A^{-2}W_-^2 + A^{-1}W_-^1 + A^0W_-^0)(s, y), \\ \zeta_-^2(s, y) &= (T^1W_-^0 + T^0W_-^1 + T^{-1}W_-^2 + T^{-2}W_-^3)(s, y), \\ \theta_-^2(s, y) &= (M^2W_-^0 + M^1W_-^1 + M^0W_-^2 + M^{-1}W_-^3)(s, y), \end{aligned}$$

so that W_-^4 solves the problem

$$\begin{cases} D_- \partial_y^4 W_-^4(s, y) &= -\phi_-^2(s, y) \text{ for } 0 \leq y \leq 1, \\ D_- \partial_y^3 W_-^4(s, 1) &= -\zeta_-^2(s, 1), \\ D_- \partial_y^2 W_-^4(s, 1) &= -\theta_-^2(s, 1). \end{cases}$$

Thus W_-^4 admits the following expression

$$W_-^4 = -\frac{1}{2} \left(\frac{1}{12} y^4 - \frac{1}{3} y^3 + \frac{1}{2} y^2 \right) \phi_-^2(s, y) - \zeta_-^2(s, 1) \left(\frac{1}{6} y^3 - \frac{1}{2} y^2 \right) - \frac{1}{2} \theta_-^2(s, 1) y^2 + \alpha^4(s) y + \beta^4(s).$$

The term W_-^4 being known, we can define w_+^1 as the solution of the interior problem

$$\begin{cases} D_+ \Delta^2 w_+^1 = 0 & \text{in } \Omega_+, \\ M_+(w_+^1) = \zeta_-^1(s, 1) - \theta_-^1(s, 1) + \theta_-^1(s, 0) & \text{on } \Gamma_0, \\ T_+(w_+^1) = \phi_-^1(s, 1) - \zeta_-^2(s, 1) + \zeta_-^2(s, 0) & \text{on } \Gamma_0, \\ w_+^1 = 0; \partial_n w_+^1 = 0 & \text{on } \Gamma. \end{cases}$$

To determine completely W_-^1 , we need to precise the function β^1 . This can be done thanks to the transmission condition of order 0 across Γ_0 : for $y = 0$, $W_-^1 = w_+^1$. Finally

$$\beta^1(s) = w_+^1|_{\Gamma_0} \quad \text{and} \quad \alpha^1(s) = \partial_n w_+^1|_{\Gamma_0}.$$

We have seen in this section the way we can define a few terms of the asymptotic expansion: we first compute the first four exterior terms $(W_-^\ell)_{0 \leq \ell \leq 3}$ (up to an affine function in y), which are needed to write the problem solved by the first interior term w_+^0 . the knowledge of the latter allows to fix completely the first exterior term.

4.2 The complete expansion: remainder estimates

The procedure described in the previous section can be generalized at any order; it leads to the identification of all the terms in the asymptotic expansion (6). Using a priori estimates, we can prove the following result.

Theorem 2 *We assume the curve Γ_0 – defining the boundary of domain Ω_+ – and the right hand-side f infinitely smooth. Then the solution w^δ of problem (1) admits the asymptotic expansion*

$$w^\delta = \sum_{n \geq 0} \delta^n w^n, \quad (10)$$

where $w^n|_{\Omega_+}(x) = w_+^n(x)$ and $w^n|_{\Omega_-}(s, r) = W_-^n(s, \delta^{-1}r)$. The terms w_+^n and W_-^n do not depend on the parameter δ and are defined by problems (12) and (13) below.

The identity (10) is valid in the sense of asymptotic expansions, i.e. for any integer N , there exists a constant C_N such that the remainder of order N

$$r^N(\delta) = w^\delta - \sum_{n=0}^N w^n$$

satisfies the (optimal) estimate

$$\|r_+^N(\delta)\|_{2, \Omega_+} + \delta^{\frac{3}{2}} \|r_-^N(\delta)\|_{2, \Omega_-} \leq C_N \delta^{N+1}. \quad (11)$$

PROOF. We assume the terms of the expansion built up to order $N - 1$. We can write the problem solved by the exterior term W_-^N :

$$\begin{cases} D_- \partial_y^4 W_-^N(s, y) &= -\phi_-^{N-2}(s, y) \text{ for } 0 \leq y \leq 1, \\ D_- \partial_y^3 W_-^N(s, 1) &= -\zeta_-^{N-2}(s, 1), \\ D_- \partial_y^2 W_-^N(s, 1) &= -\theta_-^{N-2}(s, 1), \end{cases} \quad (12)$$

where the right hand-sides ϕ_-^{N-2} , ζ_-^{N-2} , and θ_-^{N-2} are given by

$$\begin{aligned}\phi_-^{N-2}(s, y) &= D_- \sum_{k+\ell=N-4} A^k W_-^\ell, \\ \zeta_-^{N-2}(s, y) &= \sum_{k+\ell=N-3} T^k W_-^\ell, \\ \theta_-^{N-2}(s, y) &= \sum_{k+\ell=N-2} M^k W_-^\ell.\end{aligned}$$

(these quantities only involve W_-^ℓ for $\ell \leq N-1$). Problem (12) then determines the term W_-^N , up to an affine function in y , denoted by $\alpha^N(s)y + \beta^N(s)$.

In the same way, we can define – up to an affine function in y – the terms W_-^ℓ for $\ell = N+1, N+2$ and $N+3$. It allows to write the problem solved by w_+^N :

$$\begin{cases} D_+ \Delta^2 w_+^N = 0 & \text{in } \Omega_+, \\ M_+(w_+^N) = \sum_{k+\ell=N} M^k W_-^\ell & \text{on } \Gamma_0, \\ T_+(w_+^N) = \sum_{k+\ell=N} T^k W_-^\ell & \text{on } \Gamma_0, \\ w_+^N = 0; \partial_n w_+^N = 0 & \text{on } \Gamma. \end{cases} \quad (13)$$

It is a bi-harmonic problem with homogeneous Dirichlet conditions on Γ and non-homogeneous Neumann conditions on Γ_0 , which uniquely defines the term w_+^N .

The transmission conditions across Γ_0 : $W_-^N = w_+^N$ and $\partial_y W_-^N = \partial_n w_+^{N-1}$ fix the functions α^N and β^N :

$$\alpha^N(s) = \partial_n w_+^{N-1}|_{\Gamma_0} \quad \text{and} \quad \beta^N(s) = w_+^N|_{\Gamma_0}.$$

Thus, starting from the knowledge of (w_+^n, W_-^n) for $n \leq N-1$, we have built the terms w_+^N and W_-^N .

We now prove the remainder estimate (11): by construction, the remainder of order N satisfies

$$\begin{cases} D^+ \Delta^2 r_+^N(\delta) = f & \text{in } \Omega_+, \\ D^- \Delta^2 r_-^N(\delta) = \mathcal{O}(\delta^{N-3}) & \text{in } \Omega_-^\delta, \\ [r^N(\delta)] = 0; [\partial_n r^N(\delta)] = \mathcal{O}(\delta^N) & \text{on } \Gamma_0, \\ M^+(r_+^N(\delta)) = M^-(r_-^N(\delta)) + \mathcal{O}(\delta^{N-1}) & \text{on } \Gamma_0, \\ T^+(r_+^N(\delta)) = T^-(r_-^N(\delta)) + \mathcal{O}(\delta^{N-2}) & \text{on } \Gamma_0, \\ M^-(r_-^N(\delta)) = \mathcal{O}(\delta^{N-1}); T^-(r_-^N(\delta)) = \mathcal{O}(\delta^{N-2}) & \text{in } \Gamma_\delta, \\ r_+^N(\delta) = 0; \partial_n r_+^N(\delta) = 0 & \text{in } \Gamma, \end{cases} \quad (14)$$

In order to use the a priori estimate (5) given in theorem 1, we first need to lift the jump of the normal derivative across the interface Γ_0 . If we denote by τ_N this jump, we have $\tau_N = \mathcal{O}(\delta^N)$.

Let then z_+^N be the solution in $H^2(\Omega_+)$ of the bi-harmonic problem in Ω_+ with Dirichlet conditions on Γ

$$\begin{cases} D_+ \Delta^2 z_+^N = 0 & \text{in } \Omega_+, \\ z_+^N = 0; \partial_n z_+^N = \tau_N & \text{on } \Gamma_0, \\ z_+^N = 0; \partial_n z_+^N = 0 & \text{in } \Gamma. \end{cases}$$

The function z_+^N satisfies the following estimate

$$\|z_+^N\|_{2, \Omega_+} = \mathcal{O}(\delta^N).$$

In the same way, let z_-^N be the solution of the exterior problem

$$\begin{cases} D_+ \Delta^2 z_-^N = 0 & \text{in } \Omega_-^\delta, \\ z_-^N = z_+^N; \partial_n z_-^N = 0 & \text{on } \Gamma_0, \\ M^-(z_-^N) = 0; T^-(z_-^N) = 0 & \text{on } \Gamma_\delta. \end{cases}$$

We can estimate the norm of z_-^N too:

$$\|z_-^N\|_{2,\Omega_\delta^-} = \mathcal{O}(\delta^{N-\frac{3}{2}}),$$

the factor $\delta^{-\frac{3}{2}}$ is due to the fact that the domain Ω_δ^- depends on δ .

Coming back to problem (14), we can apply a priori estimate (5) to the function

$$\tilde{r}^N(\delta) = r^N(\delta) - z^N,$$

which belongs to $H^2(\Omega^\delta)$: we get

$$\|\tilde{r}^N(\delta)\|_{2,\Omega^\delta} \leq C \delta^{N-\frac{5}{2}},$$

because the L^2 -norm of the exterior right hand-side is the limiting term; it is of order δ^{N-2} (we gain a factor $\delta^{1/2}$ thanks to the measure of the domain Ω_δ^-). We deduce the estimate for the remainder of order N :

$$\|r_+^N(\delta)\|_{2,\Omega_+} + \|r_-^N(\delta)\|_{2,\Omega_\delta^-} \leq C \delta^{N-\frac{5}{2}}. \quad (15)$$

We can easily improve is, writing

$$r^N(\delta) = r^{N+4}(\delta) + \sum_{n=N+1}^{N+4} \delta^n w^n.$$

Estimate (15) applied to the remainder of order $N+4$ gives

$$\|r_+^{N+4}(\delta)\|_{2,\Omega_+} + \|r_-^{N+4}(\delta)\|_{2,\Omega_\delta^-} \leq C \delta^{N+\frac{3}{2}},$$

and bounding the norms of w^n in Ω_+ (it is of order δ^n) and in Ω_δ^- (in $\delta^{n-\frac{3}{2}}$), we finally get for $r^N(\delta)$

$$\|r_+^N(\delta)\|_{2,\Omega_+} + \delta^{\frac{3}{2}} \|r_-^N(\delta)\|_{2,\Omega_\delta^-} \leq C \delta^{N+1},$$

which is the stated estimate. ■

Remark 1. The expansion obtained in theorem 2 does not belong globally to $H^2(\Omega^\delta)$, but only piecewise in Ω_+ and Ω_δ^- . It is possible to “repair” this drawback (see previous proof), but we loose a power of δ in the estimate.

Remark 2. The use of multi-scale analysis leads to optimal estimates, as we have seen in theorem 2. The error is due to the first omitted term but truncating the series, it is generically non-zero and estimate (11) is optimal.

Remark 3. Expansion (10) is a two-scale expansion: the interior terms w_+^n are naturally in the original cartesian variables, whereas the terms in the layer involve the semi-scaled variables $(s, y = \delta^{-1}r)$. There is no corner-layer term in the interior domain Ω_+ , it only appears in Ω_δ^- . The obtained expansion allows to describe more precisely the corner-layer in the stiffener: each term W_-^n has a tensorial structure:

$$W_-^n(s, y) = \sum_{\ell=0}^n y^\ell \phi_\ell(s),$$

where the function ϕ_ℓ only depends on the arclength s . The dependence with respect to the transverse variable is polynomial. This remark can be used at a numerical level: the use of high-degree finite elements is particularly adapted to the approximation in the stiffener, even if we only use one row of elements in the thin layer (the elements hence become anisotropic).

5 Approximate boundary conditions

Even if one might approximate numerically w^δ , solution of (1), the computations become awkward when δ is very small. One would rather replace the effect of the stiffener by a boundary condition on Γ_0 , called *approximate boundary condition*.

In the present section, we will see how to identify such a condition using the asymptotic expansion obtained earlier. This method also leads to a validation of the approximate boundary condition.

The idea is to approximate w^δ by the series given by its asymptotic expansion (truncated at a given order). The condition satisfied by this approximation on Γ_0 gives the desired boundary condition.

5.1 Condition of order 0

Here, we only keep one term of the asymptotic expansion of w^δ . We recall the problem solved by the first interior term w_+^0 (see 4.1)

$$\begin{cases} D_+ \Delta^2 w_+^0 = f_+ & \text{in } \Omega_+, \\ M_+(w_+^0) = 0 ; T_+(w_+^0) = 0 & \text{on } \Gamma_0, \\ w_+^0 = 0 ; \partial_n w_+^0 = 0 & \text{on } \Gamma, \end{cases}$$

The first approximate boundary condition is obvious: it is nothing but the homogeneous Neumann conditions on Γ_0 . This is not surprising, since it corresponds to the limit case without stiffener ($\delta = 0$). The exterior boundary condition is simply imposed on $\Gamma_\delta = \Gamma_0$. Thus, we obtain a model where the effect of the thin layer is completely neglected.

The remainder estimate proved in theorem 2 allows to evaluate the difference between the solution w_+^δ of the initial problem, and w_+^0 , solution of the 0-order approximate problem:

$$\|w_+^\delta - w_+^0\|_{2,\Omega_+} = \mathcal{O}(\delta).$$

5.2 Condition of order 1

As was already pointed out, the approximate problem of order 0 does not take into account the effect of the thin layer. This model is of no interest since our aim is to obtain an approximate problem that incorporates this effect. For this reason, we must go further in the asymptotic expansion and derive the condition of order 1. To this end, we keep the first two terms of the expansion: we define $w^{[1]}$ as

$$w^{[1]} = w^0 + \delta w^1.$$

Using again the results of section 4.1, we set

$$\begin{aligned} Q_0 &= -D_- [2(1 - \nu_-) \partial_s (\partial_s \partial_n - c(s) \partial_s) - (1 - \nu_-^2) c(s) (\partial_s^2 + c(s) \partial_n)], \\ P_0 &= -D_- [(1 - \nu_-^2) \partial_s^2 (\partial_s^2 + c(s) \partial_n) + 2(1 - \nu_-) \partial_s [c(s) (\partial_s \partial_n - c(s) \partial_s)]], \end{aligned}$$

so that w_+^1 solves

$$\begin{cases} D_+ \Delta^2 w_+^1 = 0 & \text{in } \Omega_+, \\ M_+(w_+^1) + Q_0(w_+^0) = 0 & \text{on } \Gamma_0, \\ T_+(w_+^1) + P_0(w_+^0) = 0 & \text{on } \Gamma_0, \\ w_+^1 = 0 ; \partial_n w_+^1 = 0 & \text{on } \Gamma. \end{cases}$$

We deduce the following relations for $w_+^{[1]}$ on Γ_0 :

$$\begin{aligned} M_+(w_+^{[1]}) + \delta Q_0(w_+^{[1]}) &= \mathcal{O}(\delta^2) \quad \text{on } \Gamma_0, \\ T_+(w_+^{[1]}) + \delta P_0(w_+^{[1]}) &= \mathcal{O}(\delta^2) \quad \text{on } \Gamma_0. \end{aligned}$$

To obtain the approximate boundary condition of order 1, we choose to omit the $\mathcal{O}(\delta^2)$ term in the expressions given above. Doing so, we obtain the new approximate problem

$$\begin{cases} D_+ \Delta^2 w = f & \text{in } \Omega_+, \\ M_+(w) = -\delta Q_0(w) & \text{on } \Gamma_0, \\ T_+(w) = -\delta P_0(w) & \text{on } \Gamma_0, \\ w = 0; \partial_n w = 0 & \text{on } \Gamma. \end{cases} \quad (16)$$

As can be seen, the approximate problem of order 1 differs from that of order 0 by the appearance of the operators P_0 and Q_0 in the formulation of the boundary conditions posed on Γ_0 , which is the common portion of the boundaries of the plate and the stiffener. These operators express the effect of the thin layer : they depend on the physical characteristics of the material constituting this later.

Theorem 3 *Let X be the space*

$$X = \{w \in H^2(\Omega_+); w \in H^2(\Gamma_0) \text{ and } w = \partial_n w = 0 \text{ in } \Gamma\}.$$

Problem (16) is well posed in X , it is associated with the variational form

$$\forall \psi \in X, \quad a(w, \psi) + \delta b(w, \psi) = \int_{\Omega_+} f \psi \, dx, \quad (17)$$

where the bilinear and linear forms a and b are respectively given by (3) et

$$b(u, v) = 2D_-(1 - \nu_-) \int_{\Gamma_0} \gamma_\tau(w) \gamma_\tau(\psi) \, d\sigma + D_-(1 - \nu_-^2) \int_{\Gamma_0} \gamma_\mu(w) \gamma_\mu(\psi) \, d\sigma, \quad (18)$$

with

$$\gamma_\tau = \partial_s^2 - c(s) \partial_n \quad \text{and} \quad \gamma_\mu = \partial_s \partial_n - c(s) \partial_s. \quad (19)$$

It is remarkable that the expression $a(w, w) + \delta b(w, w)$ described above, which is the total energy of the plate Ω_+ , is nothing but the sum of the strain energy of the plate in bending in the Kirchhoff theory $a(w, w)$, and of the strain energy "inherited" from the stiffener. Indeed, $\delta b(w, w)$ involves tangential derivatives of the traces of w and $\partial_n w$, which stand, respectively, for the energy in bending and in torsion of the thin layer. Let then w_1^δ be the variational solution of (16).

In order to estimate the difference between w_+^δ , interior part of the solution of the original problem (1), and w_1^δ , we could use an a priori estimate on problem (16), and apply it for the function $w_+^\delta - w_+^{[1]}$. This method leads to (combining this estimate with that of the remainder $w_+^\delta - w_+^{[1]}$)

$$\|w_+^\delta - w_1^\delta\|_{2, \Omega_+} \leq C \delta^{\frac{3}{2}}.$$

The loss of the factor $\delta^{1/2}$ is due to the a priori estimate, which does depend on δ . A better way consists in determining the asymptotic expansion of the function w_1^δ and compare it with the expansion obtained in theorem 2 for w_+^δ . Since problem (16) does not involve any layer, it is easy to build the asymptotic expansion of its solution.

Theorem 4 *We assume that the curve Γ_0 – defining the boundary of the domain Ω_+ – and the right hand-side infinitely smooth. The following asymptotic expansion holds for the solution w_1^δ of problem (16):*

$$w_1^\delta = \sum_{n \geq 0} \delta^n w_1^n, \quad (20)$$

The terms $(w_1^n)_n$ do not depend on the parameter δ and are built according to equations (22) below. The equality (20) is valid in the sense of asymptotic expansions, i.e. for every integer N , there exists a constant C_N such that the remainder

$$r_+^N(\delta) = w_1^\delta - \sum_{n=0}^N w_1^n$$

satisfies the (optimal) estimate

$$\|r_+^N(\delta)\|_{2, \Omega_+} \leq C_N \delta^{N+1}. \quad (21)$$

PROOF. Inserting the polynomial ansatz (20) in the equations (16), we get

$$\begin{cases} D_+ \Delta^2 w_1^N = f \text{ ou } 0 & \text{in } \Omega_+, \\ M_+(w_1^N) = -Q_0(w_1^{N-1}) & \text{on } \Gamma_0, \\ T_+(w_1^N) = -P_0(w_1^{N-1}) & \text{on } \Gamma_0, \\ w_1^N = 0 ; \partial_n w_1^N = 0 & \text{on } \Gamma. \end{cases} \quad (22)$$

This problem defines w_1^N from w_1^{N-1} (under the convention $w^{-1} = 0$). The remainder estimate is proven by the same way as in theorem 2. ■

It is straightforward to check that the first terms of the expansions (10) and (20) are the same:

$$w_+^0 = w_1^0 \quad \text{and} \quad w_+^1 = w_1^1.$$

The following terms w_+^2 and w_1^2 do not generically equal, so that we get an optimal estimate for the difference $w_+^\delta - w_1^\delta$.

Theorem 5 *The difference between the interior part of the solution of the transmission problem (1), and the solution of the 1-order approximate problem, cf. (16) – satisfies*

$$\|w_+^\delta - w_1^\delta\|_{2,\Omega_+} \leq C \delta^2.$$

This estimate is generically optimal.

PROOF. One only needs to use the estimates (11) and (21) at order 2. ■

The latter result illustrates the efficiency of the multi-scale analysis for the study of problems depending on a small parameter. It shows that problem (16) is an approximation of the original problem (in the interior domain Ω_+) up to $\mathcal{O}(\delta^2)$. The major interest of replacing the original problem by the approximate one is, as we already mentioned, the complexity of discretizing the original problem. Since approximate problem (16) does not involve the thin layer, we can use a coarser mesh, independent of the parameter δ .

6 Case where the Young's modulus depends on the thickness

Until now, we have made the assumption that the Young's modulus E is piecewise constant in Ω^δ , independent of the thickness δ . In this section, we suppose that it behaves as δ^{-1} in the thin layer (E_+ still independent of δ). This expresses that the elastic material constituting the layer Ω_-^δ must be more rigid than that constituting the plate Ω_+ .

More precisely, we suppose that the Young's modulus E equals $\delta^{-1}E_-$ in the layer, and E_+ in Ω_+ , with E_- and E_+ independent of δ .

With much less details, we carry out the construction and the analysis of the approximate boundary conditions. The rule brought out above gives a hierarchy of approximate boundary conditions in the present case too. Indeed, the techniques developed in the previous sections still apply in this situation. The only difference is the shift of one power of δ for the operators in the stiffener. This shift only really affects the transmission conditions for M and T across the interface Γ_0 – see problem (7). In order to define the first interior term w_+^0 , we need here to determine the first five terms in the stiffener, instead of the first four terms, previously. No extra difficulty appears.

Applying the basic Ansatz used in section 4, one can check that the first interior term w_+^0 solves the following problem:

$$\begin{cases} D_+ \Delta^2 w_+^0 = f & \text{in } \Omega_+, \\ M_+(w_+^0) + Q_0(w_+^0) = 0 & \text{on } \Gamma_0, \\ T_+(w_+^0) + P_0(w_+^0) = 0 & \text{on } \Gamma_0, \\ w_+^0 = 0 ; \partial_n w_+^0 = 0 & \text{on } \Gamma \end{cases}$$

This allows to write the approximate problem of order 0, that is:

$$\begin{cases} D_+ \Delta^2 w_0^\delta = f, & \text{in } \Omega^+ \\ M_+(w_0^\delta) = -Q_0(w_0^\delta) & \text{in } \Gamma_0, \\ T_+(w_0^\delta) = -P_0(w_0^\delta) & \text{in } \Gamma_0, \\ w_0^\delta = 0 ; \partial_n w_0^\delta = 0 & \text{in } \Gamma. \end{cases} \quad (23)$$

Hence, a simple inspection of the above equations reveals that, unlike in the previous case, the operators P_0 and Q_0 appear in the right-hand sides of their formulations. Indeed, this is in sharp contrast with the analysis made before, where it was found that these operators were not involved until order 1.

We thus realize that the effect of the thin layer, in the present case, is seen at order 0 via the new boundary conditions posed on Γ_0 and is completely embodied by the operators P_0 and Q_0 . Indeed, one observes that these operators depend solely on the elastic material constituting the layer, through its characteristics E and ν .

In the previous analysis, where the thin layer characteristics were assumed to be constant, the limit problem, as $\delta \rightarrow 0$ (which corresponds to the approximate problem of order 0) was simply obtained by omitting the thin layer. As it can be seen, a completely different limit behavior occurs if the Young's modulus of the thin layer approaches $+\infty$ sufficiently rapidly as $\delta \rightarrow 0$ (i.e. if it behaves like δ^{-1}). This difference comes from the fact that, in this situation, the material constituting the thin layer is more rigid than that constituting the plate: the high rigidity of the layer emphasizes its effect on the displacement.

In order to get a "limit" problem that takes into account the effect of the thin layer as its thickness goes to zero, it turns out to be sufficient to compensate this thickness by a specific increase as $\delta \rightarrow 0$. The appropriate scaling is just $E = \delta^{-1} E_-$.

As has been already done in the previous section, we can go further in the construction of the approximate boundary conditions and obtain the approximate problem of order 1. Indeed, a condition of higher order leads to a better approximation of the exact solution of the initial problem. Following the same procedure, we obtain that the second term w_+^1 of the asymptotic expansion satisfies

$$\begin{cases} D_+ \Delta^2 w_+^1 = f & \text{in } \Omega_+, \\ M_+(w_+^1) + Q_0(w_+^1) + Q_1(w_+^0) = 0 & \text{on } \Gamma_0, \\ T_+(w_+^1) + P_0(w_+^1) + P_1(w_+^0) = 0 & \text{on } \Gamma_0, \\ w_+^0 = 1 ; \partial_n w_+^1 = 0 & \text{on } \Gamma, \end{cases}$$

where the differential operators P_1 and Q_1 are defined by

$$\begin{aligned} Q_1 = -D_- \left[\frac{1}{2} (3\nu_- + 1) (\nu_- - 1) \partial_s^2 [\partial_s^2 w_+^0 + c(s) \partial_n] \right. \\ \left. + 3(1 - \nu_-) \partial_s [c(s) (\partial_s \partial_n w_+^0 - c(s) \partial_s)] \right. \\ \left. + \frac{1}{2} (3\nu_- + 1) (\nu_- - 1) c(s) \partial_s [\partial_s \partial_n - c(s) \partial_s] \right. \\ \left. - \frac{1}{2} (2\nu_- + 1) (\nu_-^2 - 1) c^2(s) (\partial_s^2 + c(s) \partial_n) \right], \end{aligned}$$

$$\begin{aligned} P_1 = -D_- \left[\frac{1}{2} (3\nu_- + 1) (\nu_- - 1) \partial_s [c(s) \partial_s (\partial_s^2 + c(s) \partial_n)] \right. \\ \left. - 3(1 - \nu_-) \partial_s [c^2(s) (\partial_s \partial_n - c(s) \partial_s)] \right. \\ \left. - \frac{1}{2} (3\nu_- + 1) (\nu_- - 1) \partial_s^3 [\partial_s \partial_n - c(s) \partial_s] \right. \\ \left. + \frac{1}{2} (2\nu_- + 1) (\nu_-^2 - 1) \partial_s^2 [c(s) (\partial_s^2 + c(s) \partial_n)] \right]. \end{aligned}$$

We immediately deduce the approximate problem of order 1 :

$$\begin{cases} D_+ \Delta^2 w_1^\delta = f & \text{in } \Omega^+, \\ M_+(w_1^\delta) = -Q_0(w_1^\delta) - \delta Q_1(w_1^\delta) & \text{on } \Gamma_0 \\ T_+(w_1^\delta) = -P_0(w_1^\delta) - \delta P_1(w_1^\delta) & \text{on } \Gamma_0 \\ w_1^\delta = 0 ; \partial_n w_1^\delta = 0 & \text{on } \Gamma \end{cases} \quad (24)$$

As we can see, the approximate problem of order 1 differs from that of order 0 by the additive terms $\delta Q_1(w_1^\delta)$ and $\delta P_1(w_1^\delta)$ that lead to a better approximation. These two problems are well-posed in *ad hoc* variational spaces (at least for small values of δ for the second one) and define two solutions w_0^δ (actually not depending on δ), and w_1^δ . With the same tools as above, we can show that they lead to approximations of the solution w_+^δ of 1, with the optimal estimates:

$$\begin{aligned} \|w_+^\delta - w_0^\delta\|_{2,\Omega_+} &\leq C \delta, \\ \|w_+^\delta - w_1^\delta\|_{2,\Omega_+} &\leq C \delta^2. \end{aligned}$$

Physical interpretation. Within the framework of the linearized elasticity, the mechanical interpretation of the solutions of the above approximate problems is natural: they solve the two dimensional Kirchhoff-love model of a plate Ω_+ clamped along its part of boundary Γ and subjected to an applied body force f acting on its interior. The remaining portion of the boundary Γ_0 is no longer free but subjected to external actions that represent the effect of the thin layer: the new additive terms ($Q_0(w_1^\delta)$ and $P_0(w_1^\delta)$ for the first problem and $Q_0(w_1^\delta) + \delta Q_1(w_1^\delta)$, $P_0(w_1^\delta) + \delta P_1(w_1^\delta)$ for the second one) have the physical interpretation of inserted moments and forces, all of which acting on the part of the boundary Γ_0 .

Finally, it is worth noting that the derivation of higher order approximate conditions can be achieved by following the same procedure. This way, we obtain a hierarchy of boundary value problems giving in each case a model that incorporates the effect of the thin layer at different order of accuracy. However, the explicit calculations become more and more intricate.

7 Conclusion

We have derived and validated approximate boundary conditions for the problem of reinforcement of a thin plate. The use of multi-scale asymptotic analysis led us to optimal estimates of the errors between the solution of the original problem, and the solutions of the problems with approximate boundary conditions. The technique we developed here can apply to various situations of the same kind, for example the case of Dirichlet conditions (which is actually simpler), or Neumann conditions without any embedding (compatibility conditions then appear, making the analysis harder).

An interesting perspective is to investigate the situation where the plate has corners, which is generally the case in numerical applications. Our analysis can not directly apply because of the loss of regularity at each step of the construction of the expansion. We need to take into account the singularities arising from the corners, and treat them in a different way, using ideas from [5, 17, 18, 4].

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