Multi-scale Asymptotic expansion for a singular problem of a free plate with thin stiffener

Leila Rahmani* and Grégory Vial†

March 3, 2014

Abstract

In this paper, we consider a partially clamped plate which is stiffened on a portion of its free boundary. Our aim is to build an asymptotic expansion of the displacement, solution of the Kirchhoff-Love model, with respect to the thickness of the stiffener. Due to the mixed boundary conditions, singularities appear, obstructing the construction of the terms of the asymptotic expansion in the same way as if the plate was surrounded by the stiffener on its whole boundary. Using a splitting into regular and singular parts, we are able to formulate an asymptotic expansion involving profiles which allow to take into account the singularities.

1 Introduction

1.1 Formulation of the problem

This paper deals with the study of a linear model of a bending plate which is reinforced by a thin stiffener on a portion of its boundary. Such structures are widely used in many engineering applications and their mathematical analysis has received a lot of attention in recent years. Besides, problems involving thin layers have been extensively investigated by several authors and a large amount of research has been carried out in this area (see [EN93, BL96, ALG98, AHH00, HJ01, AKS06, Poi08, BL08, Poi09] for acoustic or electromagnetic problems, [AB86, AMM06] in the thermic framework, [SP74, LT92, GKL99, MS03, KMO04, Rah04, Rah09] for mechanical applications). These studies are in particular devoted to the question of the derivation of approximate boundary conditions that replace "in an approximate way" the effect of the thin layer. The use of these boundary conditions, which can be substituted to the thin layer, allows to overcome the numerical difficulties arising from the discretization of this later. Indeed, dealing now with a domain without thin layer, it is no more necessary to use very thin meshes and allows cheaper computations.

In a previous work (see [RV08]), we have addressed the problem of a plate surrounded by a thin stiffener and derived approximate boundary conditions. The technique we have used is based on a construction of an asymptotic expansion of the solution of the problem in powers of the thickness δ of the layer. This expansion is built via an alternative resolution between the plate and the stiffener, and can be achieved only under strong elliptic regularity. This smoothness assumption is not fulfilled anymore if the stiffener only lies on a portion of the plate boundary. The goal of the present work is to investigate the later case, showing how to deal with singularities appearing at the tips of the stiffener. More precisely, our aim is to provide

*Université Mouloud Mammeri de Tizi-Ouzou, (Algérie) – rahmani_lei@yahoo.fr

†Université de Lyon, CNRS UMR 5208, École Centrale de Lyon, 36 avenue Guy de Collongue, 69134 Ecully Cedex, France.

a $\delta-$expansion of the solution of the partial stiffened plate model that takes into account the singularities arising at each stage of its construction.

We now describe the geometrical setting. Let Ω_+ be a smooth bounded domain of $\mathbb{R}^2$, describing the geometry of the mean surface of the plate, with boundary of $\Omega_+ = \Gamma^0 \cup \Gamma$. We suppose that the intersection of $\bar{\Gamma}^0$ and $\bar{\Gamma}$ is formed by two points $\mathbf{O}_1$ and $\mathbf{O}_2$. For $\delta > 0$ sufficiently small, the elastic stiffener Ω_-^δ derives from a uniform dilation of Γ^0 in the normal direction, with thickness δ:

$$\Omega_-^\delta = \{\mathbf{x} + z\,\mathbf{n}(\mathbf{x})\ ;\ \mathbf{x} \in \Gamma^0 \text{ and } 0 < z < \delta\},$$

where $\mathbf{n}(\mathbf{x})$ denotes the normal vector at point $\mathbf{x} \in \Gamma^0$, outer from Ω_+; the boundary of the domain Ω_-^δ is $\Omega_-^\delta = \Gamma_1^\delta \cup \Gamma_2^\delta \cup \Gamma^0 \cup \Gamma^\delta$, where $\Gamma^\delta = \left\{\mathbf{x} + \delta \mathbf{n}(\mathbf{x})\ ;\ \mathbf{x} \in \Gamma^0\right\}$. Finally, we set $\Omega^\delta = \Omega_+ \cup \Gamma^0 \cup \Omega_-^\delta$, see Fig. 1. For technical reasons, we assume that the boundary of Ω_+ coincides with a *straight line* in a neighborhood of Γ^0.

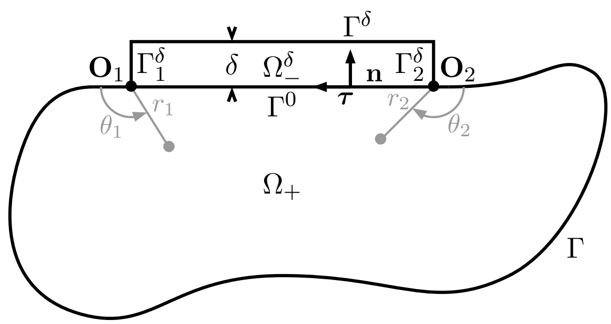

Figure 1: Geometric setting: the plate Ω_+ and the stiffener Ω_-^δ.

The whole plate (corresponding to the domain Ω^δ) is clamped along the boundary $\Gamma \cup \Gamma_1^\delta \cup \Gamma_2^\delta$ and is motion free on Γ^δ. The Kirchhoff-Love model – see [LL88] – leads to the following fourth order elliptic problem for the displacement w^δ

$$\begin{cases} D_+ \Delta^2 w_+^\delta = f_+ & \text{in } \Omega_+ \\ D_- \Delta^2 w_-^\delta = f_- & \text{in } \Omega_-^\delta, \\ \left[w^\delta\right] = 0\ ;\ [\partial_n w^\delta] = 0 & \text{on } \Gamma^0, \\ M_+(w_+^\delta) = M_-(w_-^\delta) + g_1\ ;\ T_+(w_+^\delta) = T_-(w_-^\delta) + g_2 & \text{on } \Gamma^0, \\ M_-(w^\delta) = h_1\ ;\ T_-(w^\delta) = h_2 & \text{on } \Gamma^\delta, \\ w_-^\delta = 0\ ;\ \partial_n w_+^\delta = 0 & \text{on } \Gamma, \\ w_-^\delta = 0\ ;\ \partial_n w_-^\delta = 0 & \text{on } \Gamma_1^\delta \cup \Gamma_2^\delta, \end{cases} \tag{P$^\delta$}$$

where ∂_n denotes the normal derivative along $\mathbf{n} = (n_x, n_y)$ and [ ] the jump across Γ^0, the components of $\mathbf{x}$ being denoted by (x, y). The trace operators M and T denote respectively the *bending moment* and the *shear force*, and have the following expressions:

$$\begin{aligned} M &= D\left[\Delta + (1-\nu)\left(2n_x n_y \partial_{xy} - n_x^2 \partial_y^2 - n_y^2 \partial_x^2\right)\right] &&= D\left[\nu \partial_x^2 + \partial_y^2\right], \\ T &= D\left[\partial_n \Delta + (1-\nu)\partial_\tau\left((n_x^2 - n_y^2)\partial_{xy} + n_x n_y(\partial_y^2 - \partial_x^2)\right)\right] &&= D\left[\partial_y^3 + (2-\nu)\partial_{xxy}\right], \end{aligned}$$

where $D = \frac{2E}{3(1-\nu^2)}$, E being the *Young's modulus* and $\nu \in (0, \frac{1}{2})$ the *Poisson's ratio*; ∂_τ denotes the tangential derivative. We assume that the elastic coefficients E and ν are piecewise constant and that are independent of the thickness δ: $E = E_+$ in Ω_+ and E_- in Ω_-^δ; $\nu = \nu_+$ in Ω_+ and ν_- in Ω_-^δ.It would be interesting though to consider a large stiffness, with a Young modulus in δ^{-1} for instance. This would lead to extra-difficulties we do not want to address here.

Problem (P^δ) admits a unique solution in $\mathrm{H}^2(\Omega^\delta)$, associated with the variational formulation $a(w, \psi) = F(\psi)$, where the bilinear form is given by

$$a(w, \psi) = \int_{\Omega^\delta} D\Big[\left(\partial_1^2 w + \nu \partial_2^2 w\right) \partial_1^2 \psi + 2(1-\nu)\partial_{12} w\, \partial_{12}\psi + \left(\partial_2^2 w + \nu \partial_1^2 w\right) \partial_2^2 \psi \Big] \mathrm{d}x$$

and the linear form by

$$\langle F, \psi \rangle = \int_{\Omega_+} f_+\, \psi \mathrm{d}x + \int_{\Omega_-^\delta} f_-\, \psi \mathrm{d}x + \int_{\Gamma^0} (-g_2\, \psi + g_1\, \partial_n \psi)\mathrm{d}\sigma + \int_{\Gamma^\delta} (-h_2\, \psi + h_1\, \partial_n \psi)\mathrm{d}\sigma.$$

Generally, the construction of an asymptotic expansion for a singularly perturbed problem requires regularity on the limit problem (without layer, i.e. $\delta = 0$). Roughly, using an operator formulation, we are led to solve an equation

$$L_\delta w_\delta = f_\delta,$$

where the operator $L_\delta : E \to F$ (E and F are two Banach spaces) and the right-hand side $f_\delta \in F$ both expand (at least formally) into powers of δ, namely

$$L_\delta = \sum_n \delta^n L^n \quad \text{and} \quad f_\delta = \sum_n \delta^n f^n,$$

Inserting the *ansatz* $w_\delta = \sum_n \delta^n w^n$, we find the sequence of equations

$$L^0 w^n = - \sum_{\ell + p = n, \ell > 0} L^\ell w^p + f^n.$$

Since $L^0 = L_0$ and $f^0 = f_0$, the first term w^0 is nothing else but the limit solution w_0. For singularly perturbed problems, either the operator L^0 is not invertible from E onto F, or the operators L^ℓ ($\ell > 0$) are not continuous from E into F. In the later case, the right-hand side $L^\ell w^0$ does not necessarily belong to F for $\ell > 0$, obstructing the construction of the next terms. Nevertheless, the asymptotic expansion may be still defined up to a certain order, provided the solutions of the limit problem $L^0 w = f$ are sufficiently smooth for smooth right-hand sides f.

In our situation, the limit problem reads

$$\left\{ \begin{array}{lcl} D_+ \Delta^2 w_+^0 & = & f_+ \text{ in } \Omega_+, \\ M_+(w_+^0) & = & \varphi_1 \text{ on } \Gamma^0, \\ T_+(w_+^0) & = & \varphi_2 \text{ on } \Gamma^0, \\ w_+^0 = \partial_n w_+^0 & = & 0 \text{ on } \Gamma, \end{array} \right. \tag{P^0}$$

with $\varphi_i = g_i + h_i$. If $\Gamma^0 = \emptyset$, since the domain Ω_+ is smooth, the solution w_+^0 of the biharmonic Dirichlet problem belongs to $\mathrm{H}^{N+4}(\Omega_+)$ as soon as the right hand-side f_+ is in $\mathrm{H}^N(\Omega_+)$. Because of the change of boundary conditions between Γ and Γ^0, singularities appear at the "corner" points $\mathbf{O}_1$, $\mathbf{O}_2$ – see section 2.1 – and the solution w_+^0 is generically not smooth even if f_+ is infinitely differentiable up to the boundary.

Such singularly perturbed problems have been considered from a mathematical point of view for model problems for example in [CD96, BMNP01, Via03, CCDV06]. We show how the ideas of these works may be adapted to the biharmonic problem (P^δ). Extra difficulties arise from the free boundary conditions and the high technicality related to fourth order operators. The asymptotic expansion involves three scales: the *physical* variable $\mathbf{x} = (x, y)$ in the plate Ω_+, the *stretched* variable in the layer $(x, y/\delta)$, and the *scaled* variable $(x/\delta, y/\delta)$ around the points $\mathbf{O}_1$, $\mathbf{O}_2$. For such multi-scale problems, two strategies are possible: the *matching of expansions*, where the solution is described through two expansions which are matched in a transition region, or the *multi-scale technique* based on superposition via cut-off functions. We refer to [Il'92] and [MNP00] for the presentation of the two methods, respectively – see also [DTV09] for a comparison on a model problem.

1.2 Structure of the paper

The paper is organized as follows: in section 2, we recall some classical results concerning the regularity of the solutions to Problem (P^0) (see §2.1), and state the main results of the paper (definition of the *profiles* in §2.2 and construction of the asymptotic expansion for the solution to (P^δ) in §2.3). More details on profiles and expansions can be found in sections 3 and 4, where we have postponed the detailed proofs of our results.

2 Preliminaries and statement of the results

2.1 Singularities of the limit problem

As a preliminary step, we start with the description of the singularities of Problem (P^0). The singular functions at $\mathbf{O}_i$ ($i = 1, 2$) are obtained as non-zero solutions of the mixed biharmonic problem with zero right-hand sides in the half plane which coincides with Ω_+ at this point (we recall that the boundary of Ω_+ is supposed to be a straight line near Γ^0).

Definition 2.1 *The set of singular exponents for the problem* (P^0) *is*

$$\mathfrak{S} = \left\{ \frac{1}{2} + k \pm \mathbf{i}\eta_0 \ ; \ k \in \mathbb{Z} \right\},$$

where $\mathbf{i}$ *is the complex number with* $\mathbf{i}^2 = -1$*, and* η_0 *is defined by*

$$\eta_0 = \frac{1}{\pi} \log \left[a + \sqrt{a^2 - 1} \, \right],$$

with $a = \sqrt{\frac{4}{(1-\nu)(3+\nu)}}$.

The singular function associated with the problem corresponding to $\lambda \in \mathfrak{S}$ *is*

$$\mathfrak{s}^\lambda(r, \theta) = r^\lambda \varphi_\lambda(\theta), \tag{1}$$

in polar coordinates (r, θ)*. The function* φ_λ *is a linear combination of* $\sin\left((\lambda - 2)\,\theta\right)$*,* $\cos\left((\lambda - 2)\,\theta\right)$*,* $\sin\left(\lambda\theta\right)$*, and* $\cos\left(\lambda\theta\right)$*.*

From now on, for the sake of simplicity, we moreover assume the right-hand sides f_+, f_-, φ_1, and φ_2 are $\mathscr{C}^\infty$ and vanish at the points $\mathbf{O}_1$ and $\mathbf{O}_2$, as well as their derivatives at any order. For regular right hand sides, any solution w_+^0 of the mixed Dirichlet-Neumann problem (P^0) splits into a regular part and a linear combination of singular functions. The regular part has the maximum regularity allowed by the right hand sides and the ellipticity of the problem, the singular part "filling the gap" between H^2 and the regularity of the smooth part. This is expressed in the following Theorem:

Theorem 2.2 *Let $K > 1$ be an integer. We denote by $\mathfrak{S}(K)$ the set of exponents $\lambda \in \mathfrak{S}$ such that $1 < \mathrm{Re}(\lambda) < K$. The solution $w_+^0 \in \mathrm{H}^2(\Omega_+)$ of the mixed Dirichlet-Neumann problem (P^0) admits the following decomposition :*

$$w_+^0(\mathbf{x}) = w_+^{0,K}(\mathbf{x}) + \sum_{i=1}^{2} \chi(r_i) \sum_{\lambda \in \mathfrak{S}(K)} c_\lambda^i \mathfrak{s}^\lambda(r_i, \theta_i), \qquad (c_\lambda^i \in \mathbb{C}), \tag{2}$$

where the regular part $w_+^{0,K} \in \mathrm{H}^{K+1}(\Omega_+)$ satisfies $w_+^{0,K}(\mathbf{x}) = \mathcal{O}(\|\mathbf{x} - \mathbf{O}_i\|^K)$ as $\mathbf{x} \to 0$, and (r_i, θ_i) are the polar coordinates centered in $\mathbf{O}_i$, see Fig. 1. The function χ is a cut-off equal to 1 in a neighborhood of 0 and vanishing away from 0.

If the solution w_+^δ is real, then we have $c_{\bar\lambda}^i = \overline{c_\lambda^i}$, since $\mathfrak{s}^{\bar\lambda} = \overline{\mathfrak{s}^\lambda}$.

This is a classical result, for more details on singularities of the bihamonic operator (and, more generally, of elliptic operators) we refer to [Kon67, BR80, Gri85, Dau88, Nic93].

2.2 Profiles of the transmission problem

As already stated, the singular functions $\mathfrak{s}^\lambda$ solve the totally homogeneous biharmonic problem in a half-plane, which is the local model for Ω_+ near the point $\mathbf{O}_i$. Since we consider here a transmission problem in Ω^δ, we need counterparts of the singular functions for the local model of Ω^δ near $\mathbf{O}_i$ (see next section to understand the precise role of such *profiles*). This model domain turns out to be a half-plane with a semi-strip, denoted by Q in Fig. 2. A first idea consists in defining an extension $\mathfrak{s}_*^\lambda$ of $\mathfrak{s}^\lambda$ so that the jump $[\mathfrak{s}_*^\lambda]$

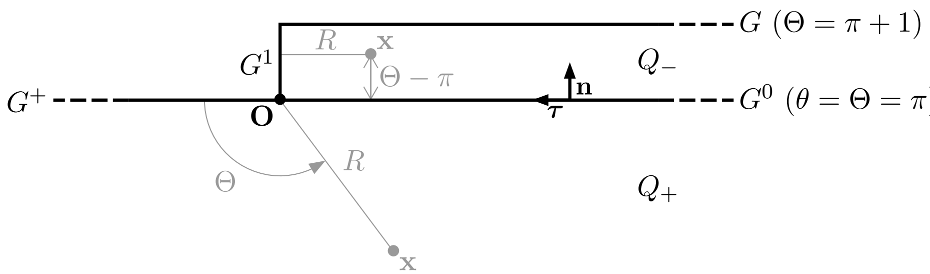

Figure 2: The infinite domain Q.

across G^0 is zero, i.e.

$$\mathfrak{s}_*^\lambda(R, \Theta) = \begin{cases} \mathfrak{s}^\lambda(R, \Theta) & \text{in } Q_+, \\ \mathfrak{s}^\lambda(R, \pi) & \text{in } Q_-, \end{cases} \tag{3}$$

where (R, Θ) are the polar coordinates in Q_+, and the cartesian coordinates in Q_- ($\Theta \in (\pi, \pi + 1)$ in this domain), see Fig. 2. But the function $\mathfrak{s}_*^\lambda$ does not satisfy neither the transmission conditions on G^0, nor the boundary conditions on G, nor is biharmonic in Q^-! Therefore another function $\mathfrak{R}^\lambda$ – called *profile* – has to be introduced, which satisfies all the conditions. Precisely, $\mathfrak{R}^\lambda$ solves the following problem in Q.

$$\left\{\begin{array}{ll} D_+\Delta^2\mathfrak{R}^\lambda_+ = 0 & \text{in } Q_+ \\ D_-\Delta^2\mathfrak{R}^\lambda_- = 0 & \text{in } Q_-, \\ [\mathfrak{R}^\lambda] = 0\ ;\ [\partial_n\mathfrak{R}^\lambda] = 0 & \text{on } G^0, \\ M_+(\mathfrak{R}^\lambda_+) = M_-(\mathfrak{R}^\lambda_-)\ ;\ T_+(\mathfrak{R}^\lambda_+) = T_-(\mathfrak{R}^\lambda_-) & \text{on } G^0, \\ M(\mathfrak{R}^\lambda_-) = 0\ ;\ T(\mathfrak{R}^\lambda_-) = 0 & \text{on } G, \\ \mathfrak{R}^\lambda_+ = 0\ ;\ \partial_n\mathfrak{R}^\lambda_+ = 0 & \text{on } G^+, \\ \mathfrak{R}^\lambda_- = 0\ ;\ \partial_n\mathfrak{R}^\lambda_- = 0 & \text{on } G^1, \\ \mathfrak{R}^\lambda(R,\Theta) \sim \mathfrak{s}^\lambda_*(R,\Theta) & \text{as } R \to \infty. \end{array}\right. \tag{P$^\infty$}$$

We have no explicit formula for $\mathfrak{R}^\lambda$. However, we can state an existence result, giving in addition its behavior at infinity. The following theorem will be proved in Section 4.

Theorem 2.3 *Let $\lambda \in \mathfrak{S}$ be a singular exponent of Problem* (P^0). *For any integer P, we define the set of complex number $\mathfrak{S}^\lambda_P$ by*

$$\mathfrak{S}^\lambda_P = \big\{\mu = \lambda - \ell\ ;\ \ell \in \mathbb{N}^*,\ \ell < \mathrm{Re}(\lambda) + P\big\} \cup \big\{\mu \in \mathfrak{S}\ ;\ -P < \mathrm{Re}(\mu) < \tfrac{3}{2}\big\}. \tag{4}$$

There exists a solution $\mathfrak{R}^\lambda$ of Problem (P$^\infty$) *with the expansion for any P:*

$$\mathfrak{R}^\lambda = \mathfrak{s}^\lambda_* + \sum_{\mu\in\mathfrak{S}^\lambda_P} \mathfrak{R}^{\lambda,\mu} + \mathcal{O}(R^{-P}), \quad \textit{as } R \to \infty, \tag{5}$$

where $\mathfrak{R}^{\lambda,\mu}(R,\Theta) = R^\mu \sum_{\ell,\,\mathit{finite}} \phi_\ell(\Theta)\log^\ell R$ with smooth functions ϕ_ℓ. Moreover, equality (5) *can be differentiated to obtain an expression for $\partial^\alpha\mathfrak{R}^\lambda$, the remainder being $\mathcal{O}(R^{-P-|\alpha|})$.*

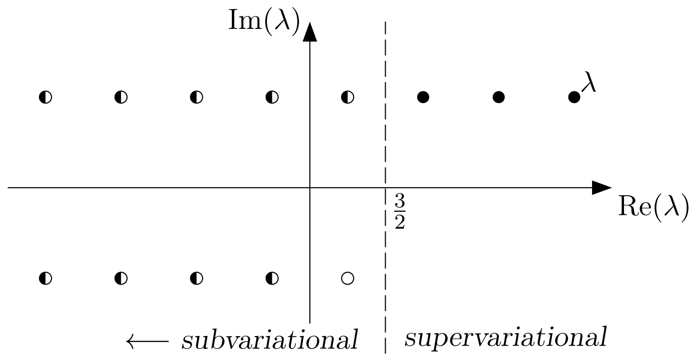

Figure 3: Complex numbers μ appearing in expansion (5).

2.3 Outline of the results

The rest of the paper is devoted to the derivation (i.e. construction) and the validation (i.e. rigorous error estimates) of the asymptotic expansion of the solution w^δ of Problem (P^δ). In this expansion smooth terms are superposed with a singular part involving the *profiles* $\mathfrak{R}^\lambda$ in the scaled variable $\mathbf{x}/\delta$. Precisely, we will show the following result in Section 3.

Theorem 2.4 *Let us fix the target precision*[1] N*, and denote by* K *another integer such that* $K > N + 7/2$*. The variational solution* w^δ *of* (P^δ) *expands into the asymptotic expansion*

$$\begin{aligned} w^\delta_+(\mathbf{x}) &= \sum_{n=0}^{N} \delta^n w_+^{n,K-n}(\mathbf{x}) + \sum_{i=1}^{2} \chi(r_i) \sum_{n=0}^{N} \delta^n \left(\sum_{\lambda\in\mathfrak{S}(K-n)} c^i_{n,\lambda}[\log\delta]\, \delta^\lambda \mathfrak{R}^\lambda_+(\tfrac{r_i}{\delta},\theta_i) \right) + r^N_{\delta,+}(\mathbf{x}), \\ w^\delta_-(\mathbf{x}) &= \sum_{n=0}^{N} \delta^n W_-^{n,K-n}(x,\tfrac{y}{\delta}) + \sum_{i=1}^{2} \chi(r_i) \sum_{n=0}^{N} \delta^n \left(\sum_{\lambda\in\mathfrak{S}(K-n)} c^i_{n,\lambda}[\log\delta]\, \delta^\lambda \mathfrak{R}^\lambda_-(\tfrac{r_i}{\delta},\theta_i) \right) + r^N_{\delta,-}(\mathbf{x}), \end{aligned}$$

where

- *the remainder* r^N_δ *satisfies* $\|r^N_\delta\|_{\mathrm{H}^2(\Omega^\delta)} = o(\delta^N)$,
- *the terms* $w_+^{n,K-n}$ *and* $W_-^{n,K-n}$ *are smooth near the points* $\mathbf{O}_i$*, and come from regular-singular splitting of solutions of alternate problems in* Ω_+ *and* Ω^δ_-*, they are flat near* $\mathbf{O}_i$ *as* r_i^{K-n}*,*
- *the coefficients* $c^i_{n,\lambda}[\log\delta]$ *are complex numbers, the brackets denoting a* trigonometrico-polynomial dependence *in* $\log\delta$*,*
- *the function* χ *is a cut-off equal to* 1 *near* 0*.*

3 Multiscale asymptotic expansion

The aim of this section is to build an asymptotic expansion in δ of the solution w^δ of the problem (P^δ). The first step of the analysis is a scaling inside the stiffener in order to remove the dependance of the space domain on the small parameter δ. So, we perform a dilatation in the normal direction of Ω^δ_- (of ratio δ^{-1}) to get a fixed geometry. The operators involved in the problem (P^δ) can be expanded into powers of δ. Introducing the scaled variable – or fast variable – $Y = \frac{y}{\delta}$, we obtain a fixed domain $\Gamma^0 \times (0,1)$.

3.1 The first terms of the expansion

Hereafter, we make the assumption that f_+ is very smooth, infinitely flat near the points O_i and, for the sake of simplicity, we suppose that $g_1 \equiv g_2 = h_1 \equiv h_2 \equiv 0$.

Let W^δ denote the function defined in $\Gamma^0 \times (0,1)$ by $W^\delta(x,Y) = w^\delta(x,y)$. The dilatation $y \mapsto Y$ maps the exterior layer Ω^δ_- into a fixed domain ; the small parameter δ is now involved in the equations. The bilaplacian reads :

$$\Delta^2 = \frac{1}{\delta^4}\left(\partial_Y^4 - \left(\delta^2 A_2 + \delta^4 A_4\right)\right),$$

[1] meaning that we aim at building an expansion with remainder of size $o(\delta^N)$.

where $A_2 = -2\partial_x^2\partial_Y^2$ and $A_4 = -\partial_x^4$. The trace operators are expanded into powers of δ as well :

$$M_- = \frac{D_-}{\delta^2}\left(\partial_Y^2 - \delta^2 M_2\right)\ ;\ T_- = \frac{D_-}{\delta^3}\left(\partial_Y^3 - \delta^2 T_2\right),$$

where $M_2 = -\nu_-\partial_x^2$ and $T_2 = -(2-\nu_-)\partial_x^2\partial_Y$. Using the above expansions, Problem (P^δ) becomes

$$\left\{\begin{array}{ll} D_+\Delta^2 w_+^\delta = f_+ & \text{in } \Omega_+ \\ M_+ w_+^\delta = \frac{D_-}{\delta^2}\left(\partial_Y^2 W_-^\delta - \delta^2 M_2 W_-^\delta\right) & \text{on } \Gamma^0\times\{0\} \\ T_+ w_+^\delta = \frac{D_-}{\delta^3}\left(\partial_Y^3 W_-^\delta - \delta^2 T_2 W_-^\delta\right) & \text{on } \Gamma^0\times\{0\} \\ \frac{1}{\delta}\partial_Y W_-^\delta = \partial_n w_+^\delta,\ W_-^\delta = w_+^\delta & \text{on } \Gamma^0\times\{0\} \\ \partial_n w_+^\delta\ = w_+^\delta = 0 & \text{on } \Gamma \\ \frac{1}{\delta^4}\left(\partial_Y^4 W_-^\delta - \left(\delta^2 A_2 W_-^\delta + \delta^4 A_4 W_-^\delta\right)\right) = 0 & \text{in } \Gamma^0\times(0,1) \\ \frac{1}{\delta^2}\left(\partial_Y^2 W_-^\delta - \delta^2 M_2 W_-^\delta\right) = 0 & \text{on } \Gamma^0\times\{1\} \\ \frac{1}{\delta^3}\left(\partial_Y^3 W_-^\delta - \delta^2 T_2 W_-^\delta\right) = 0 & \text{on } \Gamma^0\times\{1\} \\ \partial_n W_-^\delta\ = W_-^\delta = 0 & \text{on } \{0\}\times(0,1) \end{array}\right. \tag{6}$$

We first try to start with the algorithm we have already used for the case of a plate surrounded by a thin layer (see [RV08]) and seek an expansion of the form $w_+^\delta = \sum_{n\geq 0}\delta^n w_+^n$ and $W_-^\delta = \sum_{n\geq 0}\delta^n W_-^n$.

The algorithm of construction of the functions w_+^n and W_-^n consists in inserting the above ansatz into the boundary value problem (6) and identifying the terms of same power of δ. In doing so, we realize that the four exterior terms solve the following equations:

$$\left\{\begin{array}{ll} \partial_Y^4 W_-^n = 0 & \text{in } \Gamma^0\times(0,1), \\ \partial_Y^3 W_-^n = -(2-\nu_-)\partial_x^2\,\partial_Y W_-^{n-2} & \text{on } \Gamma^0\times\{1\}, \\ \partial_Y^2 W_-^n = -\nu_-\partial_x^2\, W_-^{n-2} & \text{on } \Gamma^0\times\{1\}, \end{array}\right.$$

with the convention $W_-^k \equiv 0$ for $k<0$. The resolution of this sequence of equations leads to the computation of the terms W_-^n for $n = 0,1,2,3$ up to an affine function of Y – note that, for $n=0$, we can also use the equation $\partial_Y W^0 = 0$ for $Y=0$ thanks to the fourth line of Problem (6) –

$$\begin{aligned} W_-^0(x,Y) &= \beta^0(x), \\ W_-^1(x,Y) &= \alpha^1(x)Y + \beta^1(x), \\ W_-^2(x,Y) &= -\nu_-\partial_x^2 W_-^0\tfrac{Y^2}{2} + \alpha^2(x)Y + \beta^2(x), \\ W_-^3(x,Y) &= -(2-\nu_-)\partial_x^2\,\partial_Y W_-^1\left(\tfrac{Y^3}{6} - \tfrac{Y^2}{2}\right) - \nu_-\partial_x^2 W_-^1\tfrac{Y^2}{2} + \alpha^3(x)Y + \beta^3(x). \end{aligned}$$

Let us now consider the first term in Ω_+: it solves the following problem

$$\left\{\begin{array}{ll} D_+\Delta^2 w_+^0 = f_+ & \text{in } \Omega_+, \\ M_+ w_+^\delta = D_-\left(\partial_Y^2 W_-^2 - M_2 W_-^0\right) = 0 & \text{on } \Gamma^0\times\{0\}, \\ T_+ w_+^\delta = D_-\left(\partial_Y^3 W_-^3 - T_2 W_-^1\right) = 0 & \text{on } \Gamma^0\times\{0\}, \\ \partial_n w_+^\delta\ = w_+^\delta = 0 & \text{on } \Gamma. \end{array}\right.$$

This is nothing but the limit problem. Taking now advantage of the transmission conditions on Γ^0, we get $W_-^0 = \beta^0(x) = w_+^0|_{\Gamma^0}$, and complete the determination of W_-^0.

Because of the change in the boundary conditions, we can not expect the solution w_+^0 to be regular : singularities appear at the points of intersection of $\bar{\Gamma}_0$ and $\bar{\Gamma}$. This lack of regularity leads to the obstruction of the construction of the followings terms of the asymptotic expansion.

Our technique consists then in splitting w_+^0 according to Theorem 2.2 into a regular and a singular part which are handled separately. The main idea in our construction is, instead of considering (w_+^0, w_-^0) as a first term, to modify it by substituting the $\mathfrak{s}^\lambda$ occurring in its singular part with the profiles $\delta^\lambda \mathfrak{R}^\lambda(\frac{r_i}{\delta}, \theta_i)$. More precisely, for an integer $K > 0$, we define (we recall that (r_i, θ_i) denote the polar coordinates centered at point $\mathbf{O}_i$, see Fig. 1) :

$$\begin{aligned}
\tilde{w}_+^0(\mathbf{x}) &= w_+^{0,K}(\mathbf{x}) + \sum_{i=1}^{2} \chi(r_i) \sum_{\lambda \in \mathfrak{S}(K)} c_{0,\lambda}^i \, \delta^\lambda \mathfrak{R}_+^\lambda(\tfrac{r_i}{\delta}, \theta_i) \\
&= w_+^0(\mathbf{x}) + \sum_{i=1}^{2} \chi(r_i) \sum_{\lambda \in \mathfrak{S}(K)} c_{0,\lambda}^i \, \delta^\lambda \left(\mathfrak{R}_+^\lambda(\tfrac{r_i}{\delta}, \theta_i) - \mathfrak{s}^\lambda(\tfrac{r_i}{\delta}, \theta_i) \right) = w_+^0(\mathbf{x}) + k_{\delta+}^0(\mathbf{x}),
\end{aligned}$$

and we define the exterior term $\tilde{w}_-^0$ by

$$\tilde{w}_-^0 = W_-^{0,K}(\mathbf{x}, \frac{y}{\delta}) + \sum_{i=1}^{2} \chi(r_i) \sum_{\lambda \in \mathfrak{S}(K)} c_{0,\lambda}^i \, \delta^\lambda \mathfrak{R}_-^\lambda(\tfrac{r_i}{\delta}, \theta_i) = w_{+|\Gamma^0}^{0,K} + k_{\delta-}^0,$$

where we have used the transmission condition to get $W_-^{0,K} = w_{+|\Gamma^0}^{0,K}$. Thus, we take $\tilde{w}^0 = \left(\tilde{w}_+^0, \tilde{w}_-^0\right)$ as our starting point for the expansion of w^δ and the first remainder $r_\delta^0 = w^\delta - \tilde{w}^0$ satisfies the boundary value problem (P^δ) with data which can be expanded into positive powers of δ, that is:

$$\left\{ \begin{array}{rcll}
D_+ \Delta^2 r_{\delta,+}^0 &=& -D_+ \Delta^2 k_{\delta+}^0 & \text{in } \Omega_+, \\
D_- \Delta^2 r_{\delta,-}^0 &=& -D_- \Delta^2 k_{\delta-}^0 \; - D_- \partial_x^4 w_{+|\Gamma^0}^{0,K} & \text{in } \Omega_-^\delta, \\
M_+ r_{\delta,+}^0 &=& M_- r_{\delta,-}^0 \; + \; G_{\delta 1}^0 + D_- \nu_- \partial_x^2 w_{+|\Gamma^0}^{0,K} & \text{on } \Gamma^0, \\
T_+ r_{\delta,+}^0 &=& T_- r_{\delta,-}^0 \; + \; G_{\delta 2}^0 & \text{on } \Gamma^0 \\
r_{\delta,+}^0 &=& r_{\delta,-}^0 & \text{on } \Gamma^0, \\
\partial_n r_{\delta,+}^0 &=& \partial_n r_{\delta,-}^0 \; - \; \partial_n \, w_{+|\Gamma^0}^{0,K} & \text{on } \Gamma^0, \\
M_- r_{\delta,-}^0 &=& H_{\delta 1}^0 \; - D_- \nu_- \partial_x^2 w_{+|\Gamma^0}^{0,K} & \text{on } \Gamma^\delta \\
T_- r_{\delta,-}^0 &=& H_{\delta 2}^0 & \text{on } \Gamma^\delta, \\
r_{\delta,+}^0 = \partial_n r_{\delta,+}^0 &=& 0 & \text{on } \Gamma,
\end{array} \right. \tag{7}$$

where

$$G_{\delta 1}^0 = -M_+ k_{\delta+}^0 + M_- k_{\delta-}^0, \; G_{\delta 2}^0 = -T_+ k_{\delta+}^0 + T_- k_{\delta-}^0, \; H_{\delta 1}^0 = -M_- k_{\delta-}^0, \text{ and } \; H_{\delta 2}^0 \; = -T_- k_{\delta-}^0.$$

Theorem 3.1 *In Ω_+ and Ω_-^δ, for all number $N > 0$, the residual terms can be written as :*

$$\begin{cases} \Delta^2 k^0_{\delta+} = \sum\limits_{\ell=1}^{N} \delta^\ell F_+^{0,\ell}\left[\log\delta\right] + f_+^{0(N)};\ \Delta^2 k^0_{\delta-} = \sum\limits_{\ell=0}^{N} \delta^\ell F_-^{0,\ell}\left[\log\delta\right] + f_-^{0(N)} \\ G^0_{\delta 1} = \sum\limits_{\ell=0}^{N} \delta^\ell G_1^{0,\ell}\left[\log\delta\right] + G_1^{0(N)};\ G^0_{\delta 2} = \sum\limits_{\ell=0}^{N} \delta^\ell G_2^{0,\ell}\left[\log\delta\right] + G_2^{0(N)} \\ H^0_{\delta 1} = \sum\limits_{\ell=0}^{N} \delta^\ell H_1^{0,\ell}\left[\log\delta\right] + H_1^{0(N)};\ H^0_{\delta 2} = \sum\limits_{\ell=0}^{N} \delta^\ell H_2^{0,\ell}\left[\log\delta\right] + H_2^{0(N)} \end{cases}$$

where the remainders $f_+^{0(N)}$, $f_-^{0(N)}$, $G_1^{0(N)}$, $G_2^{0(N)}$, $H_1^{0(N)}$ and $H_2^{0(N)}$ have L^2-norms $\mathcal{O}(\delta^N)$, as $\delta \to 0$.

The terms $F_+^{0,\ell}\left[\log\delta\right]$ have the form

$$F_+^{0,\ell}\left[\log\delta\right] = F_+^{0,0,\ell}\left[\log\delta\right] + \cos\left(2\eta_0\log\delta\right) F_+^{0,1,\ell}\left[\log\delta\right] + \sin\left(2\eta_0\log\delta\right) F_+^{0,2,\ell}\left[\log\delta\right],$$

where η_0 is the number defined in definition 2.1 and $F_+^{0,k,\ell}\left[\log\delta\right], k = 0, 1, 2$ are polynomials of degree ℓ in $\log\delta$:

$$F_+^{0,k,\ell}\left[\log\delta\right] = \sum_{q=0}^{\ell} F_q^{0,k,\ell}\log^q\delta,$$

with coefficients $F_q^{0,k,\ell}$ that are $\mathscr{C}^\infty$ for $r_i \neq 0$. Similarly, $F_-^{0,\ell}, G_1^{0,\ell}$, $G_2^{0,\ell}, H_1^{0,\ell}$ and $H_2^{0,\ell}$ have the same structure as $F_+^{0,\ell}$.

Proof: From the definition of $k^0_{\delta+}$, and using the fact that $\Delta^2\mathfrak{R}_+^\lambda = \Delta^2\mathfrak{s}^\lambda = 0$ inside Q_+, we find :

$$\begin{aligned} \Delta^2 k^0_{\delta+} \;=\; & \sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0,\lambda}\,\delta^\lambda \Big[\Delta^2\chi(r_i)\left(\left[\mathfrak{R}_+^\lambda - \mathfrak{s}^\lambda\right](\frac{r_i}{\delta},\theta_i)\right) + 2\nabla(\Delta\chi(r_i))\nabla\left(\left[\mathfrak{R}_+^\lambda - \mathfrak{s}^\lambda\right](\frac{r_i}{\delta},\theta_i)\right) \\ & +2\nabla\left(\Delta\left(\left[\left(\mathfrak{R}_+^\lambda - \mathfrak{s}^\lambda\right)\right](\frac{r_i}{\delta},\theta_i)\right)\right)\nabla\chi + 2\Delta\left(\nabla\chi(r_i).\nabla\left(\left[\mathfrak{R}_+^\lambda - \mathfrak{s}^\lambda\right](\frac{r_i}{\delta},\theta_i)\right)\right)\Big] \end{aligned} \tag{8}$$

Now, we use the decomposition of $\mathfrak{R}_+^\lambda$ given in Theorem (1.3), for $P = N - [\mathrm{Re}(\lambda)]$:

$$\begin{aligned} \Delta^2 k^0_{\delta+} \;=\; & \sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0,\lambda}\,\delta^\lambda \sum_{\mu\in\mathfrak{S}_P^\lambda}\Big[\Delta^2\chi(r_i)\left(\mathfrak{R}_+^{\lambda,\mu}(\frac{r_i}{\delta},\theta_i)\right) + 2\nabla\left(\Delta\chi(r_i)\right)\nabla\left(\mathfrak{R}_+^{\lambda,\mu}(\frac{r_i}{\delta},\theta_i)\right) \\ & +2\nabla\left(\Delta\left(\mathfrak{R}_+^{\lambda,\mu}(\frac{r_i}{\delta},\theta_i)\right)\right)\nabla\chi(r_i) + 2\Delta\left(\nabla\chi(r_i).\nabla\left(\mathfrak{R}_+^{\lambda,\mu}(\frac{r_i}{\delta},\theta_i)\right)\right)\Big] + f_+^{0(\delta)}, \end{aligned} \tag{9}$$

where $f_+^{0(\delta)}$ is a remainder. This decomposition can obviously be differentiated.

Let us consider a term $\mathfrak{R}_+^{\lambda,\mu}(R,\Theta) = R^\mu \sum_{\ell,\,\text{finite}} \phi_\ell(\Theta)\log^\ell R$. Recalling the fact that $R = \frac{r}{\delta}$, $\mathfrak{R}_+^{\lambda,\mu}$ may be written as:

$$\mathfrak{R}_+^{\lambda,\mu}(\frac{r_i}{\delta},\theta_i) = \delta^{-\mu} f^{\lambda,\mu}\left[\log\delta\right](r_i,\theta_i)$$

where the coefficients of the polynomial $f^{\lambda,\mu}\left[\log\delta\right]$ are $\mathscr{C}^\infty$ for $r_i \neq 0$.

The same form and the same power of δ hold for the gradient $\nabla\left(\mathfrak{R}_+^{\lambda,\mu}(\frac{\mathbf{x}}{\delta})\right)$ and for the Laplacian $\Delta\left(\mathfrak{R}_+^{\lambda,\mu}(\frac{\mathbf{x}}{\delta})\right)$. Therefore, the sum (9) becomes

$$\begin{aligned}\Delta^2 k^0_{\delta+} &= \sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0,\lambda}\,\delta^{\lambda}\sum_{\mu\in\mathfrak{S}^\lambda_P}\delta^{-\mu}\mathfrak{F}^{\lambda,\mu}\left[\log\delta\right](r_i,\theta_i)+f_+^{0(\delta)} = \\ &\sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0,\lambda}\sum_{\mu\in\mathfrak{S}^\lambda_P}\delta^{\lambda-\mu}\mathfrak{F}^{\lambda,\mu}\left[\log\delta\right](r_i,\theta_i)+f_+^{0(\delta)},\end{aligned}$$

where $\mathfrak{F}^{\lambda,\mu}\left[\log\delta\right](\mathbf{x})$ has a polynomial dependance in $\log\delta$ and $f_+^{0(\delta)}$ is a remainder. Exploiting the definition of $\mathfrak{S}^\lambda_P(K)$, the sum above can be split into two parts: a part where we consider the $\mu\in B_1=\{\lambda-\ell,\ell\in N^*,\ell<\mathrm{Re}(\lambda)+P\}$ (that are the translated of λ), and a second part into which the μ belong to $\mathfrak{S}^\lambda_P\setminus B_1$. Thus, we obtain:

$$\begin{aligned}\Delta^2 k^0_{\delta+} &= \sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0,\lambda}\left(\sum_{\mu\in B_1}\delta^{\lambda-\mu}\mathfrak{F}^{\lambda,\mu}\left[\log\delta\right](r_i,\theta_i)+\sum_{\mu\in\mathfrak{S}^\lambda_P\setminus B_1}\delta^{\lambda-\mu}\mathfrak{F}^{\lambda,\mu}\left[\log\delta\right](r_i,\theta_i)\right)+f_+^{0(\delta)} \\ &= \sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0,\lambda}\left(\sum_{\ell=1}^{N}\delta^{\ell}\mathfrak{F}^{\lambda,\lambda-\ell}\left[\log\delta\right](r_i,\theta_i)+\sum_{\mu\in\mathfrak{S}^\lambda_P\setminus B_1}\delta^{\lambda-\mu}\mathfrak{F}^{\lambda,\mu}\left[\log\delta\right](r_i,\theta_i)\right)+f_+^{0(\delta)} \\ &= \sum_{\ell=1}^{N}\delta^{\ell}\left(\sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0\lambda}\mathfrak{F}^{\lambda,\lambda-\ell}\left[\log\delta\right](r_i,\theta_i)\right)+ \\ &\quad \sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0,\lambda}\sum_{\mu\in\mathfrak{S}^\lambda_P\setminus B_1}\delta^{\lambda-\mu}\mathfrak{F}^{\lambda,\mu}\left[\log\delta\right](r_i,\theta_i)+f_+^{0(\delta)}.\end{aligned}$$

The terms involved in the first part of the above sum have the form $\delta^{\ell}F^{0,0,\ell}\left[\log\delta\right]$ with $\ell\geq 1$ (ℓ integer), as stated in the Theorem. Besides, we can easily show that the remainder $f_+^{0(\delta)}$ is of order $\mathcal{O}(\delta^N)$. As far as concern the second part, we can see that, for each $\lambda\in\mathfrak{S}(K)$, the elements of the set $\mathfrak{S}^\lambda_P\setminus B_1$ are nothing but the translated of $\bar{\lambda}$, whose the real part is comprised between $-N+\mathrm{Re}(\lambda)$ and $\frac{3}{2}$. Thus, $\mu=\bar{\lambda}-\ell$, with $\mathrm{Re}(\lambda)-\ell<\frac{3}{2}$ and obtain:

$$\begin{aligned}&\sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c^i_{0,\lambda}\left(\sum_{\mu\in\mathfrak{S}^\lambda_P\setminus B_1}\delta^{\lambda-\mu}\mathfrak{F}^{\lambda,\mu}\left[\log\delta\right](r_i,\theta_i)\right) \\ &\qquad = \sum_{\ell=1}^{N}\delta^{\ell}\left(\sum_{i=1}^{2}\sum_{\substack{\lambda\in\mathfrak{S}(K),\\ \mathrm{Re}(\lambda)-\ell<\frac{3}{2}}}\delta^{\mathbf{i}2\mathrm{Im}(\lambda)}c^i_{0,\lambda}\mathfrak{F}^{\lambda,\bar{\lambda}-\ell}\left[\log\delta\right](r_i,\theta_i)\right)\end{aligned}$$

Reordering this second part by putting together the terms corresponding to conjugate numbers, we obtain an expression into which terms of the form $\cos\left(2\eta_0\log\delta\right)$ and $\sin\left(2\eta_0\log\delta\right)$ appear (recall that $\mathrm{Im}(\lambda)=\pm\eta_0$). Indeed, we can write

$$c^i_{0,\lambda}\mathfrak{F}^{\lambda,\bar\lambda-p}\left[\log\delta\right]\delta^{\mathbf{i}2\mathrm{Im}(\lambda)}+c^i_{0,\bar\lambda}\mathfrak{F}^{\bar\lambda,\lambda-p}\left[\log\delta\right]\delta^{-\mathbf{i}2\mathrm{Im}(\lambda)}$$
$$=\left(c^i_{0,\lambda}\mathfrak{F}^{\lambda,\bar\lambda-p}\left[\log\delta\right]+c^i_{0,\bar\lambda}\mathfrak{F}^{\bar\lambda,\lambda-p}\left[\log\delta\right]\right)\cos(2\eta_0\log\delta)$$
$$+\,\mathbf{i}\left(c^i_{0,\lambda}\mathfrak{F}^{\lambda,\bar\lambda-p}\left[\log\delta\right]-c^i_{0,\bar\lambda}\mathfrak{F}^{\bar\lambda,\lambda-p}\left[\log\delta\right]\right)\sin(2\eta_0\log\delta).$$

Recalling the structure of $\mathfrak{F}^{\lambda,\bar\lambda-p}$, one can easily show that $\mathfrak{F}^{\lambda,\bar\lambda-p}=\overline{\mathfrak{F}^{\bar\lambda,\lambda-p}}$ and also $c^i_{0,\bar\lambda}=\overline{c^i_{0,\lambda}}$. This means that $\left(c^i_{0,\lambda}\mathfrak{F}^{\lambda,\bar\lambda-p}\left[\log\delta\right]+c^i_{0,\bar\lambda}\mathfrak{F}^{\bar\lambda,\lambda-p}\left[\log\delta\right]\right)$ and $\mathbf{i}\left(c^i_{0,\lambda}\mathfrak{F}^{\lambda,\bar\lambda-p}\left[\log\delta\right]-c^i_{0,\bar\lambda}\mathfrak{F}^{\bar\lambda,\lambda-p}\left[\log\delta\right]\right)$ are reals. Recalling the above results, we obtain the expansion stated in the theorem for $k^0_{\delta+}$.

In Ω^δ_-, each term $\mathfrak{R}^{\lambda,\mu}_-$ satisfies

$$\mathfrak{R}^{\lambda,\mu}_-\left(\frac{\mathbf{x}}{\delta}\right)=\delta^{-\mu}\left(\sum_{\ell=0}^{\mathrm{Re}(\lambda)-\mathrm{Re}(\mu)}\delta^{-\ell}\xi^{\lambda,\mu;l}\left[\log\delta\right](x)y^\ell\right),$$

and so, in the same way as for $k^0_{\delta+}$, we show the other results. ∎

Remark 3.2 *In the expressions of $G^0_{\delta1}$, $G^0_{\delta2}$, $H^0_{\delta1}$ and $H^0_{\delta2}$, even if terms of order zero in δ appear, they have, in fact, no effect at order 0. Indeed, terms appearing in $G^0_{\delta1}$ and $G^0_{\delta2}$ are systematically eliminated during the processus of identification. Whereas, because of the expansions of M_- and T_-, the terms of order zero in $H^0_{\delta1}$ and $H^0_{\delta2}$ have respectively an effect of order 2 and 3.*

It is also worth noticing, that only derivatives of the cut-off function χ appear in the expressions of $\Delta^2k^0_{\delta+}$, $\Delta^2k^0_{\delta-}$, $G^0_{\delta1}$, $G^0_{\delta2}$, $H^0_{\delta1}$ and $H^0_{\delta2}$, which vanish in a neighborhood of the singularities at $r_i=0$.

We now use these expansions in order to define the term of order 1 in our asymptotic expansion of the solution of the problem (P^δ). Making use of the expansions of Δ^2_-, M_- and T_-, recalling Problem (7), we define "partially" the exterior terms W^1_-, W^2_-, W^3_- and W^4_- (this is necessary to define the interior term w^1_+) as solutions of the followings problems

$$\left\{\begin{array}{ll}\partial^4_YW^1_-=0 & \text{in }\Gamma^0\times(0,1)\\ \partial^3_YW^1_-=0 & \text{on }\Gamma^0\times\{1\}\\ \partial^2_YW^1_-=0 & \text{on }\Gamma^0\times\{1\}\end{array}\right. ;\quad \left\{\begin{array}{ll}\partial^4_YW^2_-=0 & \text{in }\Gamma^0\times(0,1)\\ \partial^3_YW^2_-=0 & \text{on }\Gamma^0\times\{1\}\\ \partial^2_YW^2_-=\frac{1}{D_-}\mathcal{T}_0(H^{0,k}_1\;[\log\delta])-\nu_-\;\partial^2_xw^{0,K}_+ & \text{on }\Gamma^0\times\{1\}\end{array}\right.$$

$$\left\{\begin{array}{ll}\partial^4_YW^3_-=0 & \text{in }\Gamma^0\times(0,1)\\ \partial^3_YW^3_-=-(2-\nu_-)\partial^2_x\partial_YW^1_-\;+\frac{1}{D_-}\mathcal{T}_0\left(H^{0,k}_2\;[\log\delta]\right) & \text{on }\Gamma^0\times\{1\}\\ \partial^2_YW^3_-=-\,\nu_-\partial^2_xW^1_-\;+\frac{1}{D_-}\mathcal{T}_1\left(H^{0,k}_1\;[\log\delta]\right) & \text{on }\Gamma^0\times\{1\}\end{array}\right.$$

$$\left\{\begin{array}{ll}\partial^4_YW^4_-=-2\partial^2_x\partial^2_YW^2_-\;+\frac{1}{D_-}\mathcal{T}_0\left(F^{0,k}_-\;[\log\delta]\right)-\partial^4_xw^{0,K}_+ & \text{in }\Gamma^0\times(0,1)\\ \partial^3_YW^4_-=-(2-\nu_-)\partial^2_x\partial_YW^2_-\;+\frac{1}{D_-}\mathcal{T}_1(H^{0,k}_2\;[\log\delta]) & \text{on }\Gamma^0\times\{1\}\\ \partial^2_YW^4_-=-\,\nu_-\partial^2_xW^2_-\;+\frac{1}{D_-}\mathcal{T}_2(H^{0,k}_1\;[\log\delta]) & \text{on }\Gamma^0\times\{1\}\end{array}\right.$$

where $\mathcal{T}_q(\phi^{0,k})$ is the sum of the contributions of the Taylor expansions of the functions $\phi^{0,k}$ on $Y=0$, corresponding to δ^q. The problem solved by the interior term w_+^1 is thus the following :

$$\left\{\begin{array}{rcll} \Delta^2 w_+^1 &=& -F_+^{0,1}\,[\log\delta] & \text{in } \Omega_+,\\ M_+(w_+^1) &=& D_-\,\left(\partial_y^2 W_-^3+\nu_-\partial_x^2 W_-^1\right)+\mathcal{T}_1(G_1^{0,k}) & \text{on } \Gamma^0,\\ T_+(w_+^1) &=& D_-\,\left(\partial_y^3 W_-^4+(2-\nu_-)\partial_x^2\partial_y W_-^2\right)+\mathcal{T}_1(G_2^{0,k}) & \text{on } \Gamma^0,\\ w_+^1 &=& 0\;=\;\partial_n w_+^1 & \text{on } \Gamma, \end{array}\right.$$

which, once exploiting the partial resolution of W_-^1, W_-^2, W_-^3 and W_-^4 becomes

$$\left\{\begin{array}{rcll} \Delta^2 w_+^1 &=& -F_+^{0,1}\,[\log\delta] & \text{in } \Omega_+,\\ M_+(w_+^1) &=& 2D_-\,(1-\nu_-)\,\partial_x^2\partial_n\, w_{+|\Gamma^0}^{0,K}+\Psi_1 & \text{on } \Gamma^0,\\ T_+(w_+^1) &=& D_-\,\left(1-\nu_-^2\right)\partial_x^4 w_{+|\Gamma^0}^{0,K}+\Psi_2 & \text{on } \Gamma^0.\\ \partial_n w_+^1=w_+^1 &=& 0 & \text{on } \Gamma, \end{array}\right.$$

where Ψ_1 and Ψ_2 are functions whose supports are far from the points O_i.
Recall that $w_+^1 = w_+^{10}\,[\log\delta] + \cos(2\eta_0\log\delta)\,w_+^{11}\,[\log\delta] + \sin(2\eta_0\log\delta)\,w_+^{12}\,[\log\delta]$.

Once again, because of the change of the boundary conditions on Γ^0, and since the right hand sides of the problem above are "flat" near the points O_i, we split the interior term w_+^1 into regular and singular parts and replace the singular part by the profile $\mathfrak{R}_+^\lambda$. As far as concern the exterior part, we set (using the transmission conditions):

$$W_-^{1,K-1} = w_{+|\Gamma^0}^{1,K-1} + Y\partial_n w_{+|\Gamma_0}^{0,K}.$$

Proceeding as before, we can write the asymptotic expansion of w^δ at order 1:

$$w_+^\delta(\mathbf{x}) = w_+^{0,K}(\mathbf{x}) + \sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c_{0,\lambda}^i\chi(r_i)\delta^\lambda\mathfrak{R}_+^\lambda(\tfrac{r_i}{\delta},\theta_i)\;+$$
$$\delta\left(w_+^{1,K-1}+\sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K-1)} c_{1,\lambda}^i\chi(r_i)\delta^\lambda\mathfrak{R}_+^\lambda(\tfrac{r_i}{\delta},\theta_i)\right)\;+r_{\delta,+}^1(\mathbf{x}),$$

that we can rewrite in the following way:

$$w_+^\delta(\mathbf{x}) = w_+^{0,K}(\mathbf{x}) + \delta w_+^{1,K-1}(\mathbf{x}) +$$
$$\sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c_{0,\lambda}^i\chi(r_i)\delta^\lambda\mathfrak{R}_+^\lambda(\tfrac{r_i}{\delta},\theta_i)\;+\delta\sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K-1)} c_{1,\lambda}^i\chi(r_i)\delta^\lambda\mathfrak{R}_+^\lambda(\tfrac{r_i}{\delta},\theta_i)+r_{\delta,+}^1(\mathbf{x}).$$

Likewise, we write:

$$w_-^\delta(\mathbf{x}) = W_-^{0,K}(x,\frac{y}{\delta}) + \delta W_-^{1,K-1}(x,\frac{y}{\delta}) +$$
$$\sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K)} c_{0,\lambda}^i\chi(r_i)\delta^\lambda\mathfrak{R}_-^\lambda\,(\tfrac{r_i}{\delta},\theta_i)+\delta\sum_{i=1}^{2}\sum_{\lambda\in\mathfrak{S}(K-1)} c_{1,\lambda}^i\chi(r_i)\delta^\lambda\mathfrak{R}_-^\lambda(\tfrac{r_i}{\delta},\theta_i)+r_{\delta,-}^1(\mathbf{x}).$$

3.2 The general expansion, error estimates

We can extend the above construction to build an asymptotic expansion of the solution w^δ of the problem (P^δ) to any order. The aim of this section is to prove Theorem 2.4 given in section 2.3, where a full asymptotic expansion of the solution w^δ and an error estimate of the remainder are addressed. Recall that in this expansion, two kinds of terms appear:

- The flat terms $w^{k,K-k}$ whose exterior parts are functions of the semi-scaled variables $(x, \delta^{-1}y)$ and interior parts are functions in the slow variable $\mathbf{x}$. They vanish at the points O_i
- The profiles $\mathfrak{R}_i^\lambda$ which take into account the singular behavior of w^δ near the points O_i.

Proof: It is made by induction on N. Suppose that the terms of the expansion are built up order $N-1$. The exterior term $W_-^{N-1,K-(N-1)}$ is determined up to an affine function on Y, denoted by $\alpha^{N-1}(x)Y + \beta^{N-1}(x)$. The interior term w_+^{N-1} solves a mixed Dirichlet-Neumann problem for the Biharmonic operator, with all data smooth and flat near the points O_i. This allows a decomposition of the later into regular and singular parts, that is:

$$w_+^{N-1}(\mathbf{x}) = w_+^{N-1,K-(N-1)}(\mathbf{x}) + \sum_{i=1}^{2} \chi(r_i) \sum_{\lambda \in \mathfrak{S}(K-(N-1))} c^i_{N-1,\lambda} \mathfrak{s}^\lambda(r_i, \theta_i).$$

In this way, and making use of the transmission conditions, we obtain $\alpha^{N-1}(x) = \partial_n w_{+|\Gamma^0}^{N-2,K-(N-2)}$ and $\beta^{N-1}(x) = w_{+|\Gamma^0}^{N-1,K-(N-1)}$, which fix completely the term $W_-^{N-1,K-(N-1)}$.

As said before, we replace the singular parts $\mathfrak{s}^\lambda$ by their counterparts $\mathfrak{R}_+^\lambda$ and define

$$\begin{aligned}
\tilde{w}_+^{N-1} &= w_+^{N-1,K-(N-1)} + \sum_{i=1}^{2} \chi(r_i) \left(\sum_{\lambda \in \mathfrak{S}(K-(N-1))} c^i_{N-1,\lambda}[\log \delta]\, \delta^\lambda \mathfrak{R}_+^\lambda(\tfrac{r_i}{\delta}, \theta_i) \right), \\
\tilde{w}_-^{N-1} &= W_-^{N-1,K-(N-1)}(x, \frac{y}{\delta}) + \sum_{i=1}^{2} \chi(r_i) \left(\sum_{\lambda \in \mathfrak{S}(K-(N-1))} c^i_{N-1,\lambda}[\log \delta]\, \delta^\lambda \mathfrak{R}_-^\lambda(\tfrac{r_i}{\delta}, \theta_i) \right).
\end{aligned}$$

Thus, we set

$$\begin{aligned}
& k_{\delta+}^{N-1} = \sum_{i=1}^{2} \chi(r_i) \left(\sum_{\lambda \in \mathfrak{S}(K-(N-1))} c^i_{N-1,\lambda}\, \delta^\lambda \left[\mathfrak{R}_+^\lambda - \mathfrak{s}^\lambda \right] \right), \\
& k_{\delta-}^{N-1} = \sum_{i=1}^{2} \chi(r_i) \left(\textstyle\sum_{\lambda \in \mathfrak{S}(K-(N-1))} c^i_{N-1,\lambda}\, \delta^\lambda \mathfrak{R}_-^\lambda \right), \\
& G_{\delta 1}^{N-1} = -\left(M_+ k_{\delta+}^{N-1} - M_- k_{\delta-}^{N-1} \right),\ G_{\delta 2}^{N-1} = -\left(T_+ k_{\delta+}^{N-1} - T_- k_{\delta-}^{N-1} \right), \\
& H_{\delta 1}^{N-1} = -M_- k_{\delta-}^{N-1},\ H_{\delta 2}^{l} = -T_- k_{\delta-}^{N-1}.
\end{aligned}$$

Like above, we can assert that

$$\begin{aligned}
\Delta^2 k_{\delta+}^{N-1} &= \sum_{p=1}^{N} \delta^p F_+^{N-1,p}[\log\delta] + f_+^{N-1(N)},\\
\Delta^2 k_{\delta-}^{N-1} &= \sum_{p=0}^{N} \delta^p F_-^{N-1,p}[\log\delta] + f_-^{N-1(N)},\\
G_{\delta 1}^{N-1} &= \sum_{p=0}^{N} \delta^p G_1^{N-1,p}[\log\delta] + G_1^{N-1(N)},\\
G_{\delta 2}^{N-1} &= \sum_{p=0}^{N} \delta^p G_2^{N-1,p}[\log\delta] + G_2^{N-1(N)},\\
H_{\delta 1}^{N-1} &= \sum_{p=0}^{N} \delta^p H_1^{N-1,p}[\log\delta] + H_1^{N-1(N)},\\
H_{\delta 2}^{N-1} &= \sum_{p=0}^{N} \delta^p H_2^{N-1,p}[\log\delta] + H_2^{N-1(N)},
\end{aligned}$$

where the remainders $f_+^{N-1(N)}$, $f_-^{N-1(N)}$, $G_1^{N-1(N)}$, $G_2^{N-1(N)}$, $H_1^{N-1(N)}$ and $H_2^{N-1(N)}$ have L^2-norms of order $\mathcal{O}(\delta^N)$,as $\delta \to 0$. Exploiting the remainder r_δ^{N-1}, we can identify the problem solved by $W_-^{N,K-N}$, that is

$$\left\{\begin{aligned}
\partial_y^4 W_-^{N,K-N} &= A_2 W_-^{N-2,K-(N-2)} + A_4 W_-^{N-4,K-(N-4)} + \mathcal{T}_{N-4}(F_-^{\ell,k}),\\
\partial_y^3 W_-^{N,K-N} &= T_2 W_-^{N-2,K-(N-2)} + \tfrac{1}{D_-}\mathcal{T}_{N-3}(H_2^{\ell,k}),\\
\partial_y^2 W_-^{N,K-N} &= M_2 W_-^{N-2,K-(N-2)} + \tfrac{1}{D_-}\mathcal{T}_{N-2}(H_1^{\ell,k}),
\end{aligned}\right.$$

where $\mathcal{T}_q(\psi^{\ell,k})$ denotes the contribution of the Taylor developments of the functions $\psi^{\ell,k}$ $[\log\delta]$ on $Y=0$, for $\ell \leq N-1$, at order q, in the expansions of $\Delta^2 k_{\delta-}^{\ell}$, $H_{\delta 1}^{\ell}$ and $H_{\delta 2}^{\ell}$ on powers of δ. This allows to determine $W_-^{N,K-N}$, up to an affine function in Y denoted by $\alpha^N(x)Y + \beta^N(x)$ (this last one can not be fixed until the interior term w_+^N is determined).

In the same way, we can compute $W_-^{N+1,K-(N+1)}$, $W_-^{N+2,K-(N+2)}$ and $W_-^{N+3,K-(N+3)}$ up to an affine function on Y. This allows to write the problem solved by the interior term w_+^N :

$$\left\{\begin{aligned}
D_+\Delta^2 w_+^N &= -\sum_{\ell+k=N} F_+^{\ell,k}[\log\delta] && \text{in } \Omega_+,\\
M_+(w_+^N) &= D_-\left[\partial_y^2 W_-^{N+2,K-(N+2)} - M_2 W_-^{N,K-N}\right] + \mathcal{T}_N(G_1^{\ell,k}) && \text{on } \Gamma^0,\\
T_+(w_+^N) &= D_-\left[\partial_y^3 W_-^{N+3,K-(N+3)} - T_2 W_-^{N+1,K-(N+1)}\right] + \mathcal{T}_N(G_2^{\ell,k}) && \text{on } \Gamma^0,\\
w_+^N &= 0 = \partial_n w_+^N && \text{on } \Gamma.
\end{aligned}\right. \tag{10}$$

Similarly, recalling the above problem, we split again w_+^N into regular and singular parts :

$$w_+^N(\mathbf{x}) = w_+^{N,K-N}(\mathbf{x}) + \sum_{i=1}^{2}\chi(r_i)\sum_{\lambda\in\mathfrak{S}(K-N)} c_{N,\lambda}^{i}\mathfrak{s}^\lambda(r_i,\theta_i),$$

and identify $\alpha^N(x) = \partial_n w_{+|\Gamma_0}^{N-1,K-(N-1)}$ and $\beta^N(x) = w_{+|\Gamma^0}^{N,K-N}$.

We set $r_\delta^N = w^\delta - \sum\limits_{\ell=0}^{N} \delta^\ell \tilde{w}^\ell$, we obtain for any integer $P > N$

$$\left\{\begin{array}{rcll} D_+\Delta^2 r_{\delta,+}^N & = & -\sum\limits_{p=N+1}^{P} \delta^p \left(\sum\limits_{\substack{\ell+k=p\\ \ell\le N}} F_+^{\ell,k}[\log\delta] \right) + \mathcal{O}(\delta^P) & \text{in } \Omega_+ \\ D_-\Delta^2 r_{\delta,-}^N & = & \mathcal{O}(\delta^{N-3}) & \text{in } \Omega_-^\delta \\ \left[r_\delta^N\right] & = & 0 & \text{on } \Gamma^0, \\ \left[\partial_n r_\delta^N\right] & = & \mathcal{O}(\delta^N) & \text{on } \Gamma^0, \\ M_+\left(r_{\delta,+}^N\right) & = & M_-\left(r_{\delta,-}^N\right) + \mathcal{O}(\delta^{N-1}) & \text{on } \Gamma^0, \\ T_+\left(r_{\delta,+}^N\right) & = & T_-\left(r_{\delta,-}^N\right) + \mathcal{O}(\delta^{N-2}) & \text{on } \Gamma^0, \\ M_-\left(r_{\delta,-}^N\right) & = & \mathcal{O}(\delta^{N-1}) & \text{on } \Gamma^\delta, \\ T_-\left(r_{\delta,-}^N\right) & = & \mathcal{O}(\delta^{N-2}) & \text{on } \Gamma^\delta, \\ r_{\delta,+}^N & = & \partial_n r_{\delta,+}^N = 0 & \text{on } \Gamma. \end{array}\right. \tag{11}$$

As for the smooth case – see [RV08] – we obtain

$$\left\|r_{\delta,+}^N\right\|_{\mathrm{H}^2(\Omega_+)} + \left\|r_{\delta,-}^N\right\|_{\mathrm{H}^2(\Omega_-^\delta)} \le C\,[\log\delta]\,\delta^{N-\frac{5}{2}}.$$

We can improve this estimate by setting

$$r_\delta^N = r_\delta^{N+3} + \sum_{\ell=N+1}^{N+3} \delta^\ell w^\ell = r_\delta^{N+3} + \sum_{\ell=N+1}^{N+3} \delta^\ell w^{\ell,K-\ell}[\log\delta] + \sum_{i=1}^{2} \chi(r_i) \sum_{\ell=N+1}^{N+3} c_\lambda^i \delta^{\ell+\lambda} \mathfrak{R}^\lambda(\frac{r_i}{\delta}, \theta_i),$$

the terms $w_+^{\ell,K-\ell}$, $W_-^{\ell,K-\ell}$ have a polynomial dependance in $\log\delta$. We obtain, in the initial coordinates in Ω_-^δ

$$\left\|w_+^{\ell,K-\ell}\right\|_{\mathrm{H}^2(\Omega_+)} = \mathcal{O}([\log\delta]), \quad \left\|w_-^{\ell,K-\ell}\right\|_{\mathrm{H}^2(\Omega_-^\delta)} = \mathcal{O}(\delta^{-\frac{3}{2}}[\log\delta]).$$

Moreover, the behavior at infinity of the profiles gives

$$\left\|\delta^\lambda \mathfrak{R}^\lambda(\frac{r_i}{\delta}, \theta_i)\right\|_{\mathrm{H}^2(\Omega_+)} = \mathcal{O}\left([\log\delta]\right) \quad \text{and} \quad \left\|\delta^\lambda \mathfrak{R}^\lambda(\frac{r_i}{\delta}, \theta_i)\right\|_{\mathrm{H}^2(\Omega_-^\delta)} = \mathcal{O}\left([\log\delta]\right).$$

Thus, we obtain

$$\left\|r_{\delta,+}^N\right\|_{\mathrm{H}^2(\Omega_+)} + \delta^{\frac{3}{2}} \left\|r_{\delta,-}^N\right\|_{\mathrm{H}^2(\Omega_-^\delta)} = o(\delta^N).$$

■

4 Construction of the profiles

This section is devoted to the construction and analysis of the so called profiles, which enter the asymptotic expansion as counterparts of the interior singularities. As already mentioned, they solve the totally homogeneous problem (P^∞) in the infinite domain Q, cf. Fig. 2.

We first establish existence of profiles in a variational framework (sections 4.1-4.2), and then expand them at infinity using the Mellin transform (section 4.3). Altogether, we shall prove Theorem 2.3.

4.1 Existence of profiles: variational framework

We will make use of the non-homogeneous problem

$$\left\{\begin{array}{ll} D_+\Delta^2\mathfrak{z}_+ = \mathfrak{f}_+ & \text{in } Q_+ \\ D_-\Delta^2\mathfrak{z}_- = \mathfrak{f}_- & \text{in } Q_-, \\ [\mathfrak{z}] = 0 \ ; \ [\partial_n\mathfrak{z}] = 0 & \text{on } G^0, \\ M_+(\mathfrak{z}_+) = M_-(\mathfrak{z}_-) + \mathfrak{g}_1 & \text{on } G^0, \\ T_+(\mathfrak{z}_+) = T_-(\mathfrak{z}_-) + \mathfrak{g}_2 & \text{on } G^0, \\ M_-(\mathfrak{z}_-) = \mathfrak{h}_1 \ ; \ T_-(\mathfrak{z}_-) = \mathfrak{h}_2 & \text{on } G, \\ \mathfrak{z}_+ = 0 \ ; \ \partial_n\mathfrak{z}_+ = 0 & \text{on } G^+, \\ \mathfrak{z}_- = 0 \ ; \ \partial_n\mathfrak{z}_- = 0 & \text{on } G^1, \end{array}\right. \tag{12}$$

where $\mathfrak{f}_+$, $\mathfrak{f}_-$, $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{h}_1$, $\mathfrak{h}_2$ are given functions. The problem being posed in an infinite Problem, we will make use of weighted Sobolev spaces to enter the variational framework of the Lax-Milgram Lemma. The variational problem associated with (12) reads

$$a(\mathfrak{z}, \phi) = F(\phi),$$

with the bilinear form

$$a(\mathfrak{z}, \phi) = \sum_{\pm} \int_{Q_\pm} \Big[D_\pm \left(\partial_1^2\mathfrak{z}_\pm + \mu_\pm\partial_2^2\mathfrak{z}_\pm\right) \partial_1^2\phi_\pm + 2(1-\mu_\pm)\partial_{12}\mathfrak{z}_\pm\partial_{12}v_\pm \\ + \left(\partial_2^2\mathfrak{z}_\pm + \mu_\pm\partial_1^2\mathfrak{z}_\pm\right) \partial_2^2\phi_\pm \Big] \mathrm{d}\mathbf{x},$$

and the linear form

$$F(\phi) = \sum_{\pm} \int_{Q_\pm} \mathfrak{f}_\pm\phi_\pm \mathrm{d}\mathbf{x} + \int_{G^0} \left(\mathfrak{g}_1\partial_n\phi_+ - \mathfrak{g}_2\phi_+\right) \mathrm{d}\sigma + \int_G \left(\mathfrak{h}_1\partial_n\phi_- - \mathfrak{h}_2\phi_-\right) \mathrm{d}\sigma.$$

Existence and uniqueness is ensured in the weighted variational space (the weight $\langle\mathbf{x}\rangle$ only affects the behavior at infinity: $\langle\mathbf{x}\rangle^2 = 1 + |\mathbf{x}|^2$)

$$\mathfrak{B} = \left\{ \mathfrak{z} \ ; \ \underset{|\alpha|=2}{\partial^\alpha\mathfrak{z}} \in \mathrm{L}^2(Q), \ \frac{\mathfrak{z}}{\langle\mathbf{x}\rangle^2} \in \mathrm{L}^2(Q), \ \frac{\nabla\mathfrak{z}}{\langle\mathbf{x}\rangle} \in \mathrm{L}^2(Q), \ \mathfrak{z} = 0 \ ; \ \partial_n\mathfrak{z} = 0 \ on \ G^+ \cup G^1 \right\} \subset \mathrm{H}^2_{\mathrm{loc}}(Q).$$

Using the Lax-Milgram lemma, we get the following result.

Lemma 4.1 *If the right-hand sides satisfy*

- $\langle\mathbf{x}\rangle^2\mathfrak{f}_+ \in \mathrm{L}^2(Q_+)$, $\langle\mathbf{x}\rangle^2\mathfrak{f}_- \in \mathrm{L}^2(Q_-)$,
- $\langle\mathbf{x}\rangle^{\frac{1}{2}}\mathfrak{g}_1 \in \mathrm{L}^2(G^0)$, $\langle\mathbf{x}\rangle^{\frac{1}{2}}\mathfrak{h}_1 \in \mathrm{L}^2(G)$,
- $\langle\mathbf{x}\rangle^{\frac{3}{2}}\mathfrak{g}_2 \in \mathrm{L}^2(G^0)$, $\langle\mathbf{x}\rangle^{\frac{3}{2}}\mathfrak{h}_2 \in \mathrm{L}^2(G)$,

then Problem (12) *admits a unique solution $\mathfrak{Z}$ belonging to the space $\mathfrak{B}$.*

Proof: The continuity of the bilinear form a is also clear. For the coercivity, we can take advantage of the homogeneous Dirichlet conditions on G^+, we may write down an angular Poincaré inequality, which gives the coercivity in the weighted space after radial integration. By similar arguments, we show that the condition on the right-hand sides ensures the continuity of the linear form F over $\mathfrak{B}$. For more details, see [Via03, Chap.2].

∎

Our aim is to build a solution $\mathfrak{R}^\lambda$ to Problem (P^∞), which is nothing but (12) with zero right-hand side, and the additional condition at infinity $\mathfrak{R}^\lambda \sim \mathfrak{s}_*^\lambda$ – the extension $\mathfrak{s}_*^\lambda$ of the interior singularity is defined by (3). Of course, the only solution in $\mathfrak{B}$ of Problem (12) with zero right-hand side is $\mathfrak{Z} = 0$, and it does not fulfill the asymptotic condition $\mathfrak{R} \sim \mathfrak{s}_*^\lambda$ in (P^∞). Furthermore, since $\mathfrak{s}_*^\lambda$ does not belong to the space $\mathfrak{B}$ for λ large, we will need a preliminary step before we enter the variational framework. This is done in the next paragraph.

4.2 Algorithmic construction of profiles: supervariational structure

The procedure developed here allows the use of the variational framework previously described. The terms that will be constructed naturally belong to (semi-)homogeneous spaces. Recalling that (R, Θ) denotes the polar coordinates in Q_+, and the cartesian ones in Q_- (see Figure 2), we define for $\mu \in \mathbb{C}$

$$\begin{aligned}
S^\mu(Q_+) &= \Big\{ \sum_{\ell \geq 0,\ \text{finite}} R^\mu \log^\ell R\, \phi_\ell(\Theta)\ ;\ \phi_\ell \in \mathscr{C}^\infty[0,\pi] \Big\}, \\
S^\mu(G^0) &= \Big\{ \sum_{\ell \geq 0,\ \text{finite}} c_\ell\, R^\mu \log^\ell R\ ;\ c_\ell \in \mathbb{C} \Big\}, \\
S^\mu(Q_-) &= \Big\{ \sum_{\ell \geq 0,\ \text{finite}} \Theta^\ell v_\ell(R)\ ;\ v_\ell \in S^\mu(G^0) \Big\}
\end{aligned}$$

Finally, $S^\mu(Q)$ will denote the space of functions whose restrictions to Q_+ and Q_- respectively belong to $S^\mu(Q_+)$ and $S^\mu(Q-)$, which are continuous across G^0. Since no condition is imposed on the normal derivatives on G^0, the space $S^\mu(Q)$ is not a subset of $\mathrm{H}^2(Q)$. However, this space is natural for the forthcoming algorithmic construction. Technically we will need to lift such defects in the jumps of normal derivatives. This is done thanks to the following lemma.

Definition 4.2 *Given a function φ in $\mathrm{H}^2(Q_+ \cup Q_-)$, we denote by $[\partial_n\varphi]$ the jump of its normal derivative across G^0, and we define the extension $\mathscr{E}(\varphi)$ by*

$$\mathscr{E}(\varphi) = \begin{cases} \varphi & \text{in } Q_+, \\ \varphi - [\partial_n\varphi](\Theta - \pi) & \text{in } Q_-. \end{cases}$$

The function $\mathscr{E}(\varphi)$ belongs to $\mathrm{H}^2(Q)$.

Coming back to the problem of existence for (P^∞), we can easily give a positive answer in the case where $\mathrm{Re}(\lambda) < 1$. Indeed, the function $-\mathscr{E}(\mathfrak{s}_*^\lambda)$ belongs to $S^\lambda(Q) + S^{\lambda-1}(Q)$ and satisfies Problem (12) with the right-hand sides

$$\begin{aligned}
&\mathfrak{f}_+ = 0,\ \mathfrak{f}_- = D_-\Delta^2\mathscr{E}(\mathfrak{s}_*^\lambda) \in S^{\lambda-4}(Q_-) + S^{\lambda-5}(Q_-),\\
&\mathfrak{g}_1 = -M_-(\mathscr{E}(\mathfrak{s}_*^\lambda)) \in S^{\lambda-2}(G^0) + S^{\lambda-3}(G^0),\ \mathfrak{g}_2 = -T_-(\mathscr{E}(\mathfrak{s}_*^\lambda)) \in S^{\lambda-3}(G^0) + S^{\lambda-4}(G^0),\\
&\mathfrak{h}_1 = M_-(\mathscr{E}(\mathfrak{s}_*^\lambda)) \in S^{\lambda-2}(G) + S^{\lambda-3}(G),\ \mathfrak{h}_2 = T_-(\mathscr{E}(\mathfrak{s}_*^\lambda)) \in S^{\lambda-3}(G) + S^{\lambda-4}(G).
\end{aligned}$$

For $\mathrm{Re}(\lambda) < 1$, the functions $\mathfrak{f}_-, \mathfrak{f}_+, \mathfrak{g}_1, \mathfrak{h}_1, \mathfrak{g}_2, \mathfrak{h}_2$ are decreasing enough at infinity to satisfy the conditions of Lemma 4.1 at infinity. However, to remove the singular behavior near $R = 0$, we use a radial cut-off function ψ, vanishing near 0, and equal to 1 for R large, and consider $-\psi\mathscr{E}(\mathfrak{s}_*^\lambda)$ instead. As a result, the previous right-hand sides are replaced with functions smooth functions decreasing at infinity, and thus fulfilling requirements of Lemma 4.1. This ensures existence (and uniqueness) of $\mathfrak{Z} \in \mathfrak{B}$ solution of the same problem, and $\mathfrak{R}^\lambda := \mathfrak{Z} + \psi\mathscr{E}(\mathfrak{s}_*^\lambda)$ satisfies Problem (P^∞).

Of course, this does not work for λ with larger real part. The idea is to substract "supervariational" terms to get back to this situation. Precisely, we will build show the result

Proposition 4.3 *Let $\lambda \in \mathfrak{S}$. There exist an integer p, terms $\mathfrak{R}^{\lambda,\mu} \in S^\mu(Q)$ (for $\mu \in \lambda - \mathbb{N}$) and $\mathfrak{Z}_p^\lambda \in \mathfrak{B}$ such that*

$$\psi\left(\mathfrak{R}^{\lambda,\lambda} + \mathfrak{R}^{\lambda,\lambda-1} + \mathfrak{R}^{\lambda,\lambda-2} + \cdots + \mathfrak{R}^{\lambda,\lambda-p}\right) + \mathfrak{Z}_p^\lambda$$

is a solution of Problem (P^∞). *The first term satisfies $\mathfrak{R}^{\lambda,\lambda} = \mathfrak{s}_*^\lambda$.*

Proof:

• *Step 1. Formal calculus.* Consider a formal series $\sigma = \sum_{\ell\geq 0} \mathfrak{K}^{\lambda-\ell}$ where $\mathfrak{K}^{\lambda-\ell} \in S^{\lambda-\ell}(Q)$. We assume that σ formally solves Problem (P^∞), and we identify terms with same power of R, using the expression of the following operators involved in Q_-:

$$\begin{aligned}
\Delta^2 &= \partial_R^4 + 2\partial_R^2\partial_\Theta^2 + \partial_\Theta^4,\\
M_- &= D_-\left[\nu_-\partial_R^2 + \partial_\Theta^2\right],\\
T_- &= D_-\left[(2-\nu_-)\,\partial_\Theta\partial_R^2 + \partial_\Theta^3\right].
\end{aligned}$$

Likewise, the operators Δ^2, M_+ and T_+, involved in Q_+ are expressed in polar coordinates (their expressions are homogeneous in R). We get the following two problems

$$\left\{\begin{array}{rcll}
\Delta^2\mathfrak{K}^{\lambda-\ell} &=& 0 & in\ Q_+\\
M_+\mathfrak{K}_+^{\lambda-\ell} &=& \mathfrak{g}_+ & \theta=\pi\\
T_+\mathfrak{K}_+^{\lambda-\ell} &=& \mathfrak{h}_+ & \theta=\pi\\
\mathfrak{K}_+^{\lambda-\ell} &=& 0 & \theta=0\\
\partial_\theta\mathfrak{K}_+^{\lambda-\ell} &=& 0 & \theta=0,
\end{array}\right. \tag{13}$$

with $\mathfrak{g}_+ = D_-\left[\partial^2_\Theta \mathfrak{K}^{\lambda-\ell-2}_- + \nu_- \partial^2_R \mathfrak{K}^{\lambda-\ell}_-\right]$ and $\mathfrak{h}_+ = D_-\left[\partial^3_\Theta \mathfrak{K}^{\lambda-\ell-3}_- + (2-\nu_-)\,\partial_\Theta \partial^2_R \mathfrak{K}^{\lambda-\ell-1}_-\right]$, and

$$\left\{\begin{array}{rcll} \partial^4_\Theta \mathfrak{K}^{\lambda-\ell}_- &=& \mathfrak{f}_- & \Theta \in (\pi, \pi+1) \\ \partial^3_\Theta \mathfrak{K}^{\lambda-\ell}_- &=& \tilde{\mathfrak{g}}_- & \Theta = \pi+1 \\ \partial^2_\Theta \mathfrak{K}^{\lambda-\ell}_- &=& \tilde{\mathfrak{h}}_- & \Theta = \pi+1 \\ \partial_\Theta \mathfrak{K}^{\lambda-\ell}_- &=& \mathfrak{g}_- & \Theta = \pi \\ \mathfrak{K}^{\lambda-\ell}_- &=& \mathfrak{h}_- & \Theta = \pi \end{array}\right. \tag{14}$$

with $\mathfrak{f} = -2\partial^2_\Theta \partial^2_R \mathfrak{K}^{\lambda-\ell+2}_- - \partial^4_R \mathfrak{K}^{\lambda-\ell+4}_-$, $\tilde{\mathfrak{g}}_- = -(2-\nu_-)\,\partial_\Theta \partial^2_R \mathfrak{K}^{\lambda-\ell+2}_-$, $\tilde{\mathfrak{h}}_- = -\nu_- \partial^2_R \mathfrak{K}^{\lambda-\ell+2}_-$, $\mathfrak{g}_- = \frac{1}{R}\partial_\Theta \mathfrak{K}^{\lambda-\ell+1}_+$, and $\mathfrak{h}_- = \mathfrak{K}^{\lambda-\ell}_+$ (where we have used the convention $\mathfrak{K}^\mu = 0$ for $\mu > \lambda$).

We give existence results in spaces $S^\mu(Q)$ for Problems (13) and (14) in lemmas 4.4 and 4.5 below.

• *Step 2. Effective construction of the terms.* We consider Problem (14) for $\ell = 0$. Since $\mathfrak{f}$, $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{h}}$, and $\mathfrak{g}_-$ all vanish in this case, we end up with $\mathfrak{K}^\lambda_-(R,\Theta) = \mathfrak{K}^\lambda_+|_{G^0}(R)$ (independent of Θ), the term $\mathfrak{K}^\lambda_+$ is still being unknown at this stage. We continue with Problem (14) for $\ell = 1$, leading to

$$\mathfrak{K}^{\lambda-1}_-(R,\Theta) = \frac{\Theta - \pi}{R}\partial_\Theta \mathfrak{K}^\lambda_+|_{G^0}(R) + \mathfrak{K}^{\lambda-1}_+|_{G^0}(R).$$

(once again, $\mathfrak{K}^{\lambda-1}_+$ will be determined, later). Proceeding in the same way for $\ell = 2$ and $\ell = 3$, we obtain more complicated formulæ, we will not detail here. We just indicate here the relations

$$\partial^2_\Theta \mathfrak{K}^{\lambda-2}_-(R,\pi) = -\nu_- \partial^2_R \mathfrak{K}^\lambda_-(R,\pi) \quad \text{and} \quad \partial^3_\Theta \mathfrak{K}^{\lambda-3}_-(R,\pi) = (\nu_- - 2)\partial_\theta \partial^2_R \mathfrak{K}^{\lambda-1}_-(R,\pi). \tag{15}$$

We now come back to $\ell = 0$ and consider Problem (13). Thanks to equations (15), the right hand-sides $\mathfrak{g}_+$ and $\mathfrak{h}_+$ vanish for $\ell = 0$. It is then natural to set $\mathfrak{K}^\lambda_+ = \mathfrak{s}^\lambda$. Since $\mathfrak{K}^\lambda_-(R,\Theta) = \mathfrak{K}^\lambda_+|_{G^0}(R)$, we get $\mathfrak{K}^\lambda = \mathfrak{s}^\lambda_*$ – compare with (3).

It turns out that the computation of the right-hand sides $\mathfrak{g}_+$ and $\mathfrak{h}_+$ is possible at any order, even if the resolution of the involved term $\mathfrak{K}^{\lambda-\ell-k}$ is only partial (up to an affine function of Θ). Furthermore, since $\partial_R \mathfrak{K} \in S^{\mu-1}(Q)$ if $\mathfrak{K} \in S^\mu(Q)$, we have $\mathfrak{g}_+ \in S^{\lambda-\ell-2}(G^0)$ and $\mathfrak{h}_+ \in S^{\lambda-\ell-3}(G^0)$. Lemma 4.4 below ensures existence of $\mathfrak{K}^{\lambda-\ell}$ in $S^{\lambda-\ell}(Q_+)$. The interior term $\mathfrak{K}^{\lambda-\ell}$ being known, Problem (14) is well defined with right-hand side in S^λ. It admits a solution in $S^{\lambda-\ell}(Q_-)$ thanks to Lemma 4.5.

We are hence able to build the sequence of terms $(\mathfrak{K}^{\lambda-\ell})_{\ell\geq 0}$ so that the series σ formally solves Problem (P^∞).

• *Step 3.Construction of a solution to problem* (P^∞). Beyond the previous formal derivation of terms, we need the variational framework to show existence of a solution. By construction, the sum

$$\mathfrak{R} = -\mathscr{E}\left(\psi(\mathfrak{K}^\lambda + \mathfrak{K}^{\lambda-1} + \mathfrak{K}^{\lambda-2} + \cdots + \mathfrak{K}^{\lambda-p})\right)$$

solves Problem (12) for a smooth right-hand side with strong decreasing at infinity: for any $F \in \{f_+, f_-, g_1, g_2, h_1, h_2\}$, there exists κ with compact support such that:

$$F - \kappa \in \bigoplus_{\ell=1}^{4} S^{\lambda-p-\ell}(Q). \tag{16}$$

The term κ comes from the cut-off. Hence, for p large enough, i.e. $p > \mathrm{Re}(\lambda) + 3/2$, Lemma 4.1 ensures existence and uniqueness of $\mathfrak{Z}^\lambda_p \in \mathfrak{V}$ solution of the same problem. By linearity, $\mathfrak{R} + \mathfrak{Z}^\lambda_p$ is a solution of

Problem (P^∞). To get the stated result, we just have to remark that for $\varphi \in S^\lambda(Q)$, $\mathscr{E}(\varphi) = \varphi + q$ with $q \in S^{\lambda-1}(Q)$: for $\ell \geq 1$, the term $\mathfrak{R}^{\lambda,\lambda-\ell}$ is given by

$$\mathfrak{R}^{\lambda,\lambda-\ell} = \mathfrak{K}^{\lambda-\ell} + \mathscr{E}(\mathfrak{K}^{\lambda-\ell+1}) - \mathfrak{K}^{\lambda-\ell+1}.$$

■

Lemma 4.4 *If* $\mathfrak{g}_+ \in S^{\mu-2}(G^0)$ *and* $\mathfrak{h}_+ \in S^{\mu-3}(G^0)$*, there exists* $\mathfrak{R}_+ \in S^\mu(Q_+)$ *solution of* (13). *The solution is unique if* $\mu \notin \mathfrak{S}$.

Proof: This can be proved by elementary arguments using polar coordinates and an explicit form for elements in $S^\mu(Q_+)$. It is also possible to derive this result by Mellin transform, see [Dau88, Chap. 4]. ■

Lemma 4.5 *If* $\mathfrak{f}_- \in S^\mu(Q_-)$*,* $\tilde{\mathfrak{g}}_-, \tilde{\mathfrak{h}}_- \in S^\mu(G)$*, and* $\mathfrak{g}_-, \mathfrak{h}_- \in S^\mu(G^0)$*, there exists a unique solution* $\mathfrak{R}_- \in S^\mu(Q_-)$ *of* (14).

Proof: This is elementary since the resolution in Θ may be written explicitly. ■

4.3 Behavior at infinity: subvariational expansion

For a function ϕ_+ defined in Q_+, we define its Mellin transform $\hat{\phi}_+$ for $\Lambda \in \mathbb{C}$ as

$$\hat{\phi}_+(\Lambda, \Theta) = \int_0^{+\infty} R^{-\mu} \phi_+(R, \Theta) \frac{\mathrm{d}R}{R}.$$

If $\Lambda = \xi + i\eta$, this correspond to a Fourier transform in η of the function $e^{t\xi}\phi_+(e^t, \Theta)$, with $t = \log R$ and Θ being a parameter. Likewise, the Mellin transform of a function ϕ_- defined in Q_- is given by the formula

$$\hat{\phi}_-(\Lambda, \Theta) = \int_0^{+\infty} R^{-\mu} \phi_-(R, \Theta) \frac{\mathrm{d}R}{R}.$$

The use of the Mellin transform allows to transform Problem (12) into a one-dimensional boundary value problem in the variable Θ, the dual variable Λ becoming a parameter.

We will apply the Mellin transform to the function ϕ defined by

$$\phi(R, \Theta) = \chi(R)\mathfrak{Z}_p^\lambda(R, \Theta), \tag{17}$$

where $\mathfrak{Z}_p^\lambda$ is the variational solution involved in Proposition 4.3, and χ a radial cut-off function vanishing near 0, and equal to 1 near away from a neighborhood of 0.

Proposition 4.6 *The Mellin transform* $\hat{\phi}$ *of* ϕ *is holomorphic for* $\mathrm{Re}(\Lambda) > 3/2$*, and admits a meromorphic extension to the whole complex plane. The poles are contained in the set* $\mathfrak{S}^\lambda$*, where* $\mathfrak{S}^\lambda = \bigcup_{P>0} \mathfrak{S}_P^\lambda$*, see* (4).

Proof: By definition of the variational space $\mathfrak{B}$, the function $R \mapsto R^{-2}\phi(R, \Theta)$ belongs to $\mathrm{L}^2(0, +\infty)$. Hence, $\hat{\phi}_+$ is defined a an L^2 function – in the variable $\mathrm{Im}(\Lambda)$ – for $\mathrm{Re}(\Lambda) \geq 1$, and $\hat{\phi}_-$ for $\mathrm{Re}(\Lambda) \geq 3/2$, both being holomorphic in the corresponding complex open sets. Applying the Mellin transformation to Problem (12), we get equations (18) and (19).

$$\left\{\begin{array}{rcll} D_+\big[(\Lambda-2)^2+\partial^2_\Theta\big]\big[\Lambda^2+\partial^2_\Theta\big]\hat\phi_+(\Lambda) & = & \mathfrak{f}_+(\Lambda-4) & \Theta\in(0,\pi)\,, \\ D_+\big[-\partial^3_\Theta\hat\phi_+(\Lambda)+\big((\nu-2)\big(\Lambda^2-\Lambda\big) & & & \\ +(1-2\nu)\Lambda-2(1-\nu)\big)\partial_\Theta\hat\phi_+(\Lambda)\big] & = & \mathfrak{g}_2(\Lambda-3)+D_-\big[\partial^3_\Theta\hat\phi_-(\Lambda-3) & \\ & & +(2-\nu)\big(\Lambda^2-3\Lambda+2\big)\partial_\Theta\hat\phi_-(\Lambda-1)\big] & \Theta=\pi, \\ D_+\left[\partial^2_\Theta+\nu\Lambda^2+(1-\nu)\Lambda\right]\hat\phi_+(\Lambda) & = & \mathfrak{g}_1(\Lambda-2)+D_-\partial^2_\Theta\hat\phi_-(\Lambda-2) & \\ & & +\nu D_-(\Lambda^2-\Lambda)\hat\phi_-(\Lambda) & \Theta=\pi, \end{array}\right. \tag{18}$$

$$\left\{\begin{array}{rcll} D_-\partial^4_\Theta\hat\phi_-(\Lambda) & = & \mathfrak{f}_-(\Lambda)-D_-\big[2\big(\Lambda^2-\Lambda\big)\partial^2_\Theta\hat\phi_-(\Lambda+2) & \\ & & +\Lambda(\Lambda-1)(\Lambda^2-5\Lambda+6)\hat\phi_-(\Lambda+4)\big] & \Theta\in(\pi,\pi+1)\,, \\ \hat\phi_-(\Lambda) & = & \hat\phi_+(\Lambda) & \Theta=\pi, \\ \partial_\Theta\hat\phi_-(\Lambda) & = & \partial_\Theta\hat\phi_+(\Lambda+1) & \Theta=\pi, \\ D_-\partial^2_\Theta\hat\phi_-(\Lambda) & = & \mathfrak{h}_1(\Lambda)-\nu D_-\big((\Lambda+2)^2 & \\ & & -(\Lambda+2)\big)\hat\phi_-(\Lambda+2) & \Theta=\pi+1, \\ D_-\partial^3_\Theta\hat\phi_-(\Lambda) & = & \mathfrak{h}_2(\Lambda)-D_-(2-\nu)\big[(\Lambda+2)^2 & \\ & & -(\Lambda+2)\partial_\Theta\hat\phi_-(\Lambda+2)\big] & \Theta=\pi+1, \end{array}\right. \tag{19}$$

where the data $\mathfrak{f}_\pm$, $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ come from the Mellin transform of the terms defined by the right hand-side in the problem solved by the variational part $\mathfrak{Z}^\lambda_p$ (see proof of proposition 4.3) and from the truncation. Using (16), we can easily show that these terms admit a meromorphic extension to $\mathbb{C}$, holomorphic in $\mathbb{C}\setminus\{\lambda-p-1,\lambda-p-2,\lambda-p-3,\lambda-p-4\}$.

We will first build the meromorphic extension of $\hat\phi(\Lambda)$ for $\mathrm{Re}(\Lambda)>1/2$. Let us consider Problem (18), which has been written for $\mathrm{Re}(\Lambda)>3/2$, and show that its right hand-side may be defined for $\mathrm{Re}(\Lambda)>1/2$. From fourth equation of Problem (19), $\partial^2_\Theta\hat\phi_-(\Lambda)$ is obviously meromorphic for $\mathrm{Re}(\Lambda)>-1/2$. Inserting this information into the first equation of (19), we obtain that $\partial^4_\Theta\hat\phi_-(\Lambda)$ is meromorphic for $\mathrm{Re}(\Lambda)>-5/2$. If we denote by $\Phi(\Lambda,\Theta)$ the quantity

$$\Phi(\Lambda,\Theta)=\partial^3_\Theta\hat\phi_-(\Lambda-3)+(2-\nu)\big(\Lambda^2-3\Lambda+2\big)\partial_\Theta\hat\phi_-(\Lambda-1),$$

and write

$$\Phi(\Lambda,\pi)=\Phi(\Lambda,\pi+1)-\int_\pi^{\pi+1}\partial_\Theta\Phi(\Lambda,\vartheta)\vartheta,$$

it appears that $\Phi(\Lambda,\pi)$ is meromorphic for $\mathrm{Re}(\Lambda)>1/2$. The same kind of arguments can be applied to the other right hand-side of Problem (18). We can hence define a solution to that problem, meromorphic for $\mathrm{Re}(\Lambda)>1/2$. Let us mention that new poles may appear, which were not present in the right-hand-sides. They come directly from the operator, and are in the set $\mathfrak{S}$. It is clear that the obtained solution is an extension of $\hat\phi$. The same argument can be repeated to get an extension to the whole complex plane. ∎

The decreasing at infinity being closely linked to the real part of the dual variable μ, we deduce from the previous result an asymptotic expansion at infinity of the variational term $\mathfrak{Z}_p^\lambda$.

Proposition 4.7 *The variational term $\mathfrak{Z}_p^\lambda$ (defined in proposition 4.3) admits the following asymptotic expansion at infinity for any integer P:*

$$\mathfrak{Z}_p^\lambda = \sum_{\substack{\mu\in\mathfrak{S}^\lambda \\ -P<\mathrm{Re}(\mu)<3/2}} \mathfrak{R}_p^{\lambda,\mu} + \mathcal{O}(R^{-P}), \tag{20}$$

with $\mathfrak{R}_p^{\lambda,\mu} \in S^\mu(Q)$.

Proof: We denote by $\mathfrak{m}(\Lambda,\Theta)$ the meromorphic extension obtained in proposition 4.6, and by $\mathscr{M}_\xi^{-1}$ the inverse Mellin transformation along the line $\mathrm{Re}(\Lambda)=\xi$:

$$\mathscr{M}_\xi^{-1}\mathfrak{m}(R,\Theta) = \int_{\mathbb{R}} R^{\xi+i\eta}\mathfrak{m}(\xi+i\eta,\Theta)\mathrm{d}\eta.$$

It is clear that $\mathscr{M}_\xi^{-1}\mathfrak{m} = \phi$ if $\xi > 3/2$ – the function ϕ is defined from the variational solution by (17). Let us fix $a < b$ such that the lines $\mathrm{Re}(\Lambda) = a$ and $\mathrm{Re}(\Lambda) = b$ do not meet the set $\mathfrak{S}^\lambda$. Integrating $R^{\xi+i\eta}\mathfrak{m}(\xi+i\eta,\Theta)$ along the boundary of the rectangle

$$a<\xi<b, \qquad -c<\eta<c,$$

we can show that the horizontal sides have a contribution vanishing as c goes to infinity, and get, thanks to the Cauchy integral formula:

$$\mathscr{M}_b^{-1}\mathfrak{m}(R,\Theta) - \mathscr{M}_a^{-1}\mathfrak{m}(R,\Theta) = \sum_{a<\mathrm{Re}(\mu)<b} \mathrm{Res}_{\lambda=\mu}\left(R^\lambda\mathfrak{m}(\lambda,\Theta)\right), \tag{21}$$

the sum being extended to the poles $\mu\in\mathfrak{S}^\lambda$ of $\mathfrak{m}$. A simple computation shows that the residue at $\lambda=\mu$ belong to the space $S^\mu(Q)$. It remains to see that, for a small, the inverse transform $\mathscr{M}_a^{-1}\mathfrak{m}(R,\Theta)$ is strongly decreasing as $R\to\infty$. This is a classical result on the Mellin transformation, which follows from the Plancherel identity. Taking $b>3/2$ leads to the stated result. ∎

Combining Propositions 4.3 and 4.7, we get the stated result, see Theorem 2.4.

Remark 4.8 *In Proposition 4.7, the sub-variational terms $\mathfrak{R}_p^{\lambda,\mu}$ are unique. It is not the case for super-variational ones defined in Proposition 4.3 since kernel elements of the interior problems may appear. We emphasize that in our expansions μ is of the form $\lambda-\ell$, and hence is an element of the singular set $\mathfrak{S}$ (this phenomenon is called* resonence*).*

Remark 4.9 *It can be easily checked that our construction is consistent with the complex conjugation. We precisely have $\mathfrak{R}^{\bar{\lambda},\mu} = \overline{\mathfrak{R}^{\lambda,\bar{\mu}}}$, up to kernel elements for super-variational terms.*

5 Concluding remarks

In this work, we have been able to build an asymptotic expansion for the solution of the problem of a plate described by the Kirchhoff-Love model, surrounded by a thin stiffener on a portion of its boundary. The expansion involves profiles which account for the presence of singularities due to the change of boundary conditions in the limit problem. The presented results can be used to investigate the performance of approximate boundary conditions which are usually set on the boundary Γ (i.e. in the smooth case, where the stiffener lies on the whole boundary) to replace the effect of the thin layer in a numerical computation, cf. [Via05].

We conclude by mentioning several extensions which are more or less straightforward. Other external boundary conditions may be treated in the same way, clamped boundary conditions on Γ^δ leading to a less technical algorithmic construction between Ω_+ and Ω_-^δ. Likewise, the case where the right hand-sides of Problem (P^δ) are not flat near the "corner" points O_1 and O_2 is similar, adding extra profiles arising from their Taylor expansions at O_1 and O_2. Let us mention that the situation of a curved boundary near the stiffener is not a direct adaptation of the present work, since singular functions are more delicate to describe. More interesting is the case of a layer of stiffness δ^{-1}: the profiles depend on δ and, beyond their expansion at infinity, an expansion in δ is needed.

References

[AB86] Emilio Acerbi and Giuseppe Buttazzo. Reinforcement problems in the calculus of variations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 3(4):273–284, 1986.

[AHH00] Habib Ammari, Laurence Halpern, and Kamel Hamdache. Asymptotic behaviour of thin ferromagnetic films. *Asymptot. Anal.*, 24(3-4):277–294, 2000.

[AKS06] Habib Ammari, Hyeonbae Kang, and Fadil Santosa. Scattering of electromagnetic waves by thin dielectric planar structures. *SIAM J. Math. Anal.*, 38(4):1329–1342 (electronic), 2006.

[ALG98] H. Ammari and C. Latiri-Grouz. Approximate boundary conditions for thin periodic coatings. In *Mathematical and numerical aspects of wave propagation (Golden, CO, 1998)*, pages 297–301. SIAM, Philadelphia, PA, 1998.

[AMM06] Abdelaziz Ait Moussa and Jamal Messaho. Limit behavior of an oscillating thin layer. In *Proceedings of the 2005 Oujda International Conference on Nonlinear Analysis*, volume 14 of *Electron. J. Differ. Equ. Conf.*, pages 21–33 (electronic), San Marcos, TX, 2006. Southwest Texas State Univ.

[BL96] A. Bendali and K. Lemrabet. The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. *SIAM J. Appl. Math.*, 56(6):1664–1693, 1996.

[BL08] A. Bendali and K. Lemrabet. Asymptotic analysis of the scattering of a time-harmonic electromagnetic wave by a perfectly conducting metal coated with a thin dielectric shell. *Asymptot. Anal.*, 57(3-4):199–227, 2008.

[BMNP01] Maryse Bourlard, Abderrahman Maghnouji, Serge Nicaise, and Luc Paquet. Asymptotic expansion of the solution of a mixed Dirichlet-Ventcel problem with a small parameter. *Asymptot. Anal.*, 28(3-4):241–278, 2001.

[BR80] H. Blum and R. Rannacher. On the boundary value problem of the biharmonic operator on domains with angular corners. *Math. Methods Appl. Sci.*, 2(4):556–581, 1980.

[CCDV06] Gabriel Caloz, Martin Costabel, Monique Dauge, and Grégory Vial. Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer. *Asymptotic Analysis*, 50(1, 2):121–173, 2006.

[CD96] M. Costabel and M. Dauge. A singularly perturbed mixed boundary value problem. *Comm. Partial Differential Equations*, 21:1919–1949, 1996.

[Dau88] Monique Dauge. *Elliptic Boundary Value Problems in Corner Domains – Smoothness and Asymptotics of Solutions*. Lecture Notes in Mathematics, Vol. 1341. Springer-Verlag, Berlin, 1988.

[DTV09] Monique Dauge, Sébastien Tordeux, and Grégory Vial. Self-similar perturbation near a corner: matching versus multiscale expansions for a model problem. *To appear*, 2009.

[EN93] B. Engquist and J. C. Nedelec. Effective boundary conditions for electromagnetic scattering in thin layers. Rapport interne 278, CMAP, 1993.

[GKL99] Giuseppe Geymonat, Françoise Krasucki, and Stefano Lenci. Mathematical analysis of a bonded joint with a soft thin adhesive. *Math. Mech. Solids*, 4(2):201–225, 1999.

[Gri85] P. Grisvard. *Boundary value problems in non-smooth domains*. Pitman, London, 1985.

[HJ01] H. Haddar and P. Joly. Effective boundary conditions for thin ferromagnetic coatings. Asymptotic analysis of the 1D model. *Asymptot. Anal.*, 27(2):127–160, 2001.

[Il'92] A.M. Il'lin. Matching of asymptotic expansions of solutions of boundary value problems. *Translations of Mathematical Monographs*, 1992.

[KMO04] Francoise Krasucki, Arnaud Münch, and Yves Ousset. Mathematical analysis of nonlinear bonded joint models. *Math. Models Methods Appl. Sci.*, 14(4):535–556, 2004.

[Kon67] V. A. Kondrat′ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.*, 16:227–313, 1967.

[LL88] J. Lagnese and J.-L. Lions. *Modelling analysis and control of thin plates*, volume 6 of *Research in Applied Mathematics*. Masson, Paris, 1988.

[LT92] K. Lemrabet and D. Teniou. Vibrations d'une plaque mince avec raidisseur sur le bord. *Maghreb Math. Rev.*, 2(1):27–41, 1992.

[MNP00] Vladimir G. Maz'ya, Sergey A. Nazarov, and Boris A. Plamenevskij. *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains*. Birkhäuser, Berlin, 2000.

[MS03] Mongi Mabrouk and Hassan Samadi. Linear and semi-linear reinforcement problems by thin layers. *Z. Angew. Math. Phys.*, 54(2):349–375, 2003.

[Nic93] Serge Nicaise. *Polygonal interface problems*, volume 39 of *Methoden und Verfahren der Mathematischen Physik [Methods and Procedures in Mathematical Physics]*. Verlag Peter D. Lang, Frankfurt am Main, 1993.

[Poi08] Clair Poignard. Asymptotics for steady-state voltage potentials in a bidimensional highly contrasted medium with thin layer. *Math. Methods Appl. Sci.*, 31(4):443–479, 2008.

[Poi09] Clair Poignard. Approximate transmission conditions through a weakly oscillating thin layer. *Math. Methods Appl. Sci.*, 32(4):435–453, 2009.

[Rah04] Leila Rahmani. Ventcels boundary conditions for a dynamic nonlinear plate. *Asymptot. Anal.*, 38:319–337, 2004.

[Rah09] L. Rahmani. Modelling of the effect of a thin stiffener on the boundary of a nonlinear thermoelastic plate. *Math. Model. Anal.*, 14(3):353–368, 2009.

[RV08] Leila Rahmani and Grégory Vial. Reinforcement of a thin plate by a thin layer. *Math. Methods Appl. Sci.*, 31(3):315–338, 2008.

[SP74] Enrique Sánchez-Palencia. Problèmes de perturbations liés aux phénomènes de conduction à travers des couches minces de grande résistivité. *J. Math. Pures Appl. (9)*, 53:251–269, 1974.

[Via03] Grégory Vial. Analyse multi-échelle et conditions aux limites approchées pour un problème de couche mince dans un domaine à coin. Thèse de doctorat 2840, Université de Rennes I, IRMAR, 2003.

[Via05] Grégory Vial. Efficiency of approximate boundary conditions for corner domains coated with thin layers. *CR. Acad. Sci.*, 340:215–220, January 2005.