TOPOLOGICAL SENSITIVITY ANALYSIS FOR THE MODIFIED HELMHOLTZ EQUATION UNDER AN IMPEDANCE CONDITION ON THE BOUNDARY OF A HOLE

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Abstract. The topological sensitivity analysis consists to provide an asymptotic expansion of a shape functional with respect to emerging of small holes in the interior of the domain occupied by the body. In this paper, such an expansion is obtained for the modified Helmholtz equation with an impedance condition prescribed on the boundary of a hole.

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1. Introduction

The aim of the topological sensitivity analysis is to provide an asymptotic expansion of a shape functional with respect to the size of a small inclusion inserted inside the domain. To present the main idea, let us consider a domain $\Omega \subset \mathbb{R}^d$ ($d \in \{2,3\}$) and a cost function $j(\Omega) = J(u_\Omega)$, where u_Ω is the solution to a given PDE defined over Ω. For a small parameter $\varepsilon > 0$, let Ω_ε be the domain obtained by removing a small part $\overline{x_0 + \varepsilon\omega}$ from Ω, where $x_0 \in \Omega$ and ω is a fixed bounded domain in R^d containing the origin, that is, $\Omega_\varepsilon = \Omega \backslash \overline{x_0 + \varepsilon\omega}$. In general, we have the following asymptotic expansion (as $\varepsilon \to 0^+$):

$$j(\Omega) - j(\Omega_\varepsilon) = f(\varepsilon)g(x_0) + o(f(\varepsilon)),$$

where $f(\varepsilon) > 0$ and $f(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. The function g is independent on ε and it is called topological gradient or topological derivative. To minimize the criterion j, one has to create holes at some points where the topological gradient is negative.

The topological sensitivity analysis has been studied for different kinds of topology optimization problems: the elasticity case [12], the Poisson equation [13], the Navier-Stokes equation [4], the Helmholtz equation [26], the heat equation [5] and the wave equation [5]. For other works on topological sensitivity analysis, we refer the reader to [3, 9, 14, 19, 22, 23, 24, 25, 27, 28].

In this paper, we apply the topological-shape sensitivity method to obtain the topological derivative for the modified Helmholtz equation under an impedance condition prescribed on the boundary of a hole.

The outline of this paper is as follows. The problem of interest is formulated in Section 2. In Section 3, we present some preliminaries including the adjoint method introduced in [22] and other useful results that will be used to establish our main result. The asymptotic analysis and the main result is presented in Section 4. Some numerical experiments are given in Section 5.

2. Problem formulation

Let Ω be a regular bounded domain in $\mathbb{R}^2$ with a regular boundary $\partial\Omega$. Let $u_\Omega \in H^1(\Omega)$ be the unique solution (weak solution) to the modified Helmholtz equation

$$-\Delta u_\Omega + a u_\Omega = 0 \text{ in } \Omega \tag{2.1}$$

with the Neumann boundary condition

$$\frac{\partial u_\Omega}{\partial n} = \zeta \text{ on } \partial\Omega, \tag{2.2}$$

where $a > 0$ is a constant and $\zeta \in H^{-1/2}(\partial\Omega)$.

Let $J : H^{1/2}(\partial\Omega) \to \mathbb{R}$ be a given differentiable mapping, and let

$$j(\Omega) := J(u_{\Omega|\partial\Omega}),$$

where $u_{\Omega|\partial\Omega}$ denotes the trace of $u_\Omega \in H^1(\Omega)$ on $\partial\Omega$. Let $p_\Omega \in H^1(\Omega)$ be the unique solution (weak solution) to the modified Helmholtz equation (2.1) with the boundary condition

$$\frac{\partial p_\Omega}{\partial n} = -DJ(u_{\Omega|\partial\Omega}) \text{ on } \partial\Omega, \tag{2.3}$$

where $DJ(u_{\Omega|\partial\Omega}) \in H^{-1/2}(\partial\Omega)$ denotes the derivative of J at the point $u_{\Omega|\partial\Omega}$.

For any sufficiently small parameter $\varepsilon > 0$, consider the perforated domain $\Omega_\varepsilon := \Omega \backslash \overline{B(x_0, \varepsilon)}$, where $x_0 \in \Omega$ and $\overline{B(x_0, \varepsilon)}$ is the closure of the open ball of center x_0 and radius ε. Possibly shifting the origin of the coordinate system, we assume for convenience that $x_0 = 0$. Let $u_{\Omega_\varepsilon} \in H^1(\Omega_\varepsilon)$ be the unique solution (weak solution) to the perturbed modified Helmholtz equation

$$-\Delta u_{\Omega_\varepsilon} + a u_{\Omega_\varepsilon} = 0 \text{ in } \Omega_\varepsilon \tag{2.4}$$

with an impedance condition on the boundary of the hole

$$u_{\Omega_\varepsilon} + \alpha \frac{\partial u_{\Omega_\varepsilon}}{\partial n} = 0 \text{ on } \partial B(x_0, \varepsilon) \tag{2.5}$$

and the Neumann boundary condition on $\partial\Omega$

$$\frac{\partial u_{\Omega_\varepsilon}}{\partial n} = \zeta \text{ on } \partial\Omega, \tag{2.6}$$

where $\alpha > 0$ is a constant.

For all $\varepsilon > 0$ (small enough), let $j(\Omega_\varepsilon) := J(u_{\Omega_\varepsilon|\partial\Omega})$. The main goal of this paper is to derive an asymptotic expansion of the variation $j(\Omega_\varepsilon) - j(\Omega)$ with respect to ε.

3. Preliminaries

3.1. **The adjoint method.** We recall briefly the adjoint technique introduced in [22]. Let H be a Hilbert space (over $\mathbb{R}$) endowed with the norm $\|\cdot\|_H$. Let $\{a_\varepsilon(\cdot,\cdot)\}_{\varepsilon \geq 0}$ be a family of bilinear and continuous forms on H. We suppose that there exists a constant $c > 0$ (independent on ε) such that

$$a_\varepsilon(u, u) \geq c\|u\|_H^2, \ \forall\, u \in H.$$

For all $\varepsilon \geq 0$, let $u_\varepsilon \in H$ be the unique solution to the variational problem

$$a_\varepsilon(u_\varepsilon, v) = \ell(v), \ \forall\, v \in H,$$

where $\ell \in H'$. Let $J : H \to \mathbb{R}$ be differentiable function, and let

$$F(\varepsilon) := J(u_\varepsilon), \ \forall\, \varepsilon \geq 0.$$

We suppose that the following condition holds: there exists a bilinear and continuous form δ_a on H such that

$$\|a_\varepsilon - a_0 - f(\varepsilon)\delta_a\|_{\mathcal{L}_2(H)} = o(f(\varepsilon)), \text{ as } \varepsilon \to 0^+,$$

where $f(\varepsilon) > 0$ for all $\varepsilon > 0$ (small enough) and $f(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. Here, $\|\cdot\|_{\mathcal{L}_2(H)}$ is the standard norm on $\mathcal{L}_2(H)$, the set of bilinear and continuous forms on H.

The following result provides us an asymptotic expansion of the variation $F(\varepsilon) - F(0)$ as $\varepsilon \to 0^+$.

Lemma 3.1. *Under the above conditions, we obtain that*

(i) $\|u_\varepsilon - u_0\|_H = O(f(\varepsilon))$;

(ii) $F(\varepsilon) - F(0) = f(\varepsilon)\delta_a(u_0, p_0) + o(f(\varepsilon))$, *where $p_0 \in H$ is the adjoint state, the unique solution to the adjoint problem*

$$a_0(v, p_0) = -\langle DJ(u_0), v\rangle_{H',H}, \ \forall\, v \in H.$$

Here, $\langle\cdot,\cdot\rangle_{H',H}$ denotes the duality product between H and H'.

3.2. **Some useful inequalities.** In this part, we present some inequalities involving modified Bessel functions of first and second kinds. Such inequalities will be useful to establish our main result.

Recall that the modified Bessel functions of the first kind $(I_n)_{n\in\mathbb{Z}}$ are defined by

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\, d\theta,\ \forall\, z \geq 0.$$

The modified Bessel functions of the second kind $(K_n)_{n\in\mathbb{Z}}$ are defined by

$$K_n(z) = \int_0^{\infty} e^{-z\cosh t}\cosh(nt)\, dt,\ \forall\, z \geq 0.$$

The following results exist in [16, 17].

Lemma 3.2. *We have*

$$e^{x-y}\left(\frac{x}{y}\right)^n < \frac{I_n(x)}{I_n(y)} < \left(\frac{x}{y}\right)^n < \frac{K_n(y)}{K_n(x)},$$

where $y > x > 0$ and $n \in \mathbb{N}$.

The following result exists in [6].

Lemma 3.3. *For all $u > 0$, we have*

$$0 \leq \frac{I_n'(u)}{I_n(u)} \leq \frac{\sqrt{u^2+n^2}}{u}; \qquad \frac{K_n'(u)}{K_n(u)} \leq -\frac{\sqrt{u^2+n^2}}{u},$$

where $n \in \mathbb{N}$.

The following result exists in [18].

Lemma 3.4. *For all $u > 0$, we have*

$$\frac{K_{n+1}(u)}{K_n(u)} < \frac{n+1+\sqrt{(n+1)^2+u^2}}{u},\ \forall\, n \in \mathbb{N}.$$

The following results exist in [1].

Lemma 3.5. *For all $n \in \mathbb{N}$ and $u > 0$, we have*

$$\Theta_n'(u) = \Theta_{n+1}(u) + \frac{n}{u}\Theta_n(u),$$

where Θ_n denotes I_n or $(-1)^n K_n$.

Lemma 3.6. *We have*

$$I_0(0) = 0, \quad I_1(0) = 0, \quad K_0(u) \sim -\ln u,\ \textit{as } u \to 0+.$$

Lemma 3.7. *As $u \to 0^+$, we have the following asymptotic expansions:*

$$I_0(u) = 1 + o(u),\ I_1(u) = \frac{u}{2} + o(u^2),\ K_0(u) = -\left(\ln(u/2) + \gamma\right) I_0(u) + o(u),\ k_1(u) = 1/u + O(1),$$

where γ is the Euler constant.

Lemma 3.8. *For all $n \in \mathbb{N}$ and $z > 0$, we have*

$$W(I_n(z), K_n(z)) := I_n(z)K_{n+1}(z) + I_{n+1}(z)K_n(z) = \frac{1}{z}.$$

Lemma 3.9. *For $u > 0$ and $n \to \infty$, we have*

$$I_n(u) \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{eu}{2n}\right)^n,\ K_n(u) \sim \sqrt{\frac{\pi}{2n}}\left(\frac{eu}{2n}\right)^{-n}.$$

4. Asymptotic analysis and main result

Let

$$F(\varepsilon) := \begin{cases} J(u_{\Omega_\varepsilon|\partial\Omega}) & \text{if} \quad \varepsilon > 0, \\ J(u_{\Omega|\partial\Omega}) & \text{if} \quad \varepsilon = 0. \end{cases} \tag{4.1}$$

Note that $u_{\Omega_\varepsilon} \in H^1(\Omega_\varepsilon)$, which is a functional space depending on ε. So, if we want to derive an asymptotic expansion of the variation $F(\varepsilon) - F(0)$, we cannot apply directly Lemma 3.1, which requires a fixed functional space (independent on ε). A truncation technique (see [22]) can be used to reformulate our problem in a fixed functional space.

4.1. Reformulation of the problem in a fixed functional space.

Let $R > 0$ be such the closed ball $\overline{B(x_0, R)} \subset \Omega$. It is supposed throughout this paper that ε remains small enough so that $\overline{B(x_0, \varepsilon)} \subset B(x_0, R)$. Let $\Omega_R := \Omega \backslash \overline{B(x_0, R)}$ be the truncated open subset. We denote by $\partial B(x_0, R)$ the boundary of the ball $B(x_0, R)$. For $\varepsilon > 0$, we denote by $D(\varepsilon, R)$ the open subset $B(x_0, R) \backslash \overline{B(x_0, \varepsilon)}$. For $\varepsilon > 0$ and $\varphi \in H^{1/2}(\partial B(x_0, R))$, let $u_\varepsilon^\varphi \in H^1(D(\varepsilon, R))$ be the unique solution to

$$\begin{cases} -\Delta u_\varepsilon^\varphi + a u_\varepsilon^\varphi &= 0 \text{ in } D(\varepsilon, R), \\ u_\varepsilon^\varphi &= \varphi \text{ on } \partial B(x_0, R), \\ u_\varepsilon^\varphi + \alpha \frac{\partial u_\varepsilon^\varphi}{\partial n} &= 0 \text{ on } \partial B(x_0, \varepsilon), \end{cases} \tag{4.2}$$

and $u_0^\varphi \in H^1(B(x_0, R))$ be the unique solution to

$$\begin{cases} -\Delta u_0^\varphi + a u_0^\varphi &= 0 \text{ in } B(x_0, R), \\ u_0^\varphi &= \varphi \text{ on } \partial B(x_0, R). \end{cases} \tag{4.3}$$

For all $\varepsilon \geq 0$, let $T_\varepsilon : H^{1/2}(\partial B(x_0, R)) \to H^{-1/2}(\partial B(x_0, R))$ be the Dirichlet-to-Neumann operator defined by

$$T_\varepsilon \varphi := \nabla u_\varepsilon^\varphi \cdot n_{|\partial B(x_0,R)}, \ \forall \, \varphi \in H^{1/2}(\partial B(x_0, R)), \tag{4.4}$$

where the unit normal $n_{|\partial B(x_0,R)}$ on $\partial B(x_0, R)$ is is chosen outward to $B(x_0, R)$.

For $\varepsilon \geq 0$, let $u_\varepsilon \in H^1(\Omega_R)$ be the unique solution (weak solution) to the truncated problem

$$\begin{cases} -\Delta u_\varepsilon + a u_\varepsilon &= 0 \text{ in } \Omega_R, \\ \dfrac{\partial u_\varepsilon}{\partial n} + T_\varepsilon u_\varepsilon &= 0 \text{ on } \partial B(x_0, R), \\ \dfrac{\partial u_\varepsilon}{\partial n} &= \zeta \text{ on } \partial\Omega. \end{cases} \tag{4.5}$$

Proposition 4.1. *For $\varepsilon > 0$, the restriction to Ω_R of the solution $u_{\Omega\varepsilon}$ to (2.4)-(2.6) is the solution u_ε to (4.5).*

Proof. We have to show that $w_\varepsilon := u_{\Omega\varepsilon|\Omega_R} \in H^1(\Omega_R)$ satisfies the variational problem

$$\int_{\Omega_R} (\nabla w_\varepsilon \cdot \nabla v + a w_\varepsilon v) \, dx + \int_{\partial B(x_0,R)} (T_\varepsilon w_\varepsilon) v \, d\partial B(x_0, R) = \int_{\partial\Omega} \zeta v \, d\partial\Omega, \tag{4.6}$$

for all $v \in H^1(\Omega_R)$.

Let $v \in H^1(\Omega_R)$. Let $\widetilde{v} \in H^1(\Omega_\varepsilon)$ such that $\widetilde{v}_{|\Omega_R} = v$. We have

$$\int_{\Omega_\varepsilon} (\nabla u_{\Omega\varepsilon} \cdot \nabla \widetilde{v} + a u_{\Omega\varepsilon} \widetilde{v}) \, dx + \frac{1}{\alpha} \int_{\partial B(x_0,\varepsilon)} u_{\Omega\varepsilon} \widetilde{v} \, d\partial B(x_0, \varepsilon) = \int_{\partial\Omega} \zeta \widetilde{v} \, d\partial\Omega.$$

This implies that

$$\int_{\Omega_R} (\nabla w_\varepsilon \cdot \nabla v + a w_\varepsilon v)\, dx + \left(\int_{D(\varepsilon,R)} (\nabla u_{\Omega_\varepsilon} \cdot \nabla \widetilde{v} + a u_{\Omega_\varepsilon} \widetilde{v})\, dx + \frac{1}{\alpha} \int_{\partial B(x_0,\varepsilon)} u_{\Omega_\varepsilon} \widetilde{v}\, d\partial B(x_0,\varepsilon) \right)$$

$$= \int_{\partial\Omega} \zeta v\, d\partial\Omega. \tag{4.7}$$

On the other hand, we have (in the sense of distributions) that

$$-\Delta u_{\Omega_\varepsilon} + a u_{\Omega_\varepsilon} = 0 \text{ in } D(\varepsilon, R),$$

which implies that

$$\int_{D(\varepsilon,R)} (\nabla u_{\Omega_\varepsilon} \cdot \nabla \widetilde{v} + a u_{\Omega_\varepsilon} \widetilde{v})\, dx + \frac{1}{\alpha} \int_{\partial B(x_0,\varepsilon)} u_{\Omega_\varepsilon} \widetilde{v}\, d\partial B(x_0,\varepsilon) = \int_{\partial B(x_0,R)} (T_\varepsilon w_\varepsilon) v\, d\partial B(x_0, R).$$

Then, we obtain the desired result by injecting the above expression in (4.7). □

Similarly, we can prove the following result.

Proposition 4.2. *The restriction to Ω_R of the solution u_Ω to (2.1)-(2.2) is the solution u_0 to (4.5) for $\varepsilon = 0$.*

We define now the truncated adjoint state $p_0 \in H^1(\Omega_R)$ solution (weak solution) to

$$\begin{cases} -\Delta p_0 + a p_0 &= 0 \text{ in } \Omega_R, \\ \dfrac{\partial p_0}{\partial n} + T_0 p_0 &= 0 \text{ on } \partial B(x_0, R), \\ \dfrac{\partial p_0}{\partial n} &= -DJ(u_{0|\partial\Omega}) \text{ on } \partial\Omega. \end{cases} \tag{4.8}$$

Similarly, we have the following result.

Proposition 4.3. *The restriction to Ω_R of the solution p_Ω to (2.1),(2.3) is the solution p_0 to (4.8).*

From the above propositions, the function $F(\varepsilon)$ defined by (4.1) can be writing as

$$F(\varepsilon) = \begin{cases} J(u_{\varepsilon|\partial\Omega}) & \text{if } \varepsilon > 0, \\ J(u_{0|\partial\Omega}) & \text{if } \varepsilon = 0. \end{cases}$$

On the other hand, for all $\varepsilon \geq 0$, $u_\varepsilon \in H^1(\Omega_R)$ is the unique solution to the variational problem

$$a_\varepsilon(u_\varepsilon, v) = \ell(v), \ \forall\, v \in H^1(\Omega_R),$$

where

$$a_\varepsilon(u, v) := \int_{\Omega_R} (\nabla u \cdot \nabla v + a u v)\, dx + \int_{\partial B(x_0,R)} (T_\varepsilon u) v\, d\partial B(x_0, R) \tag{4.9}$$

and

$$\ell(v) := \int_{\partial\Omega} \zeta v\, d\partial\Omega,$$

for all $u, v \in H^1(\Omega_R)$.

We now have at our disposal the fixed Hilbert space $H := H^1(\Omega_R)$ required by the adjoint method.

4.2. **Expressions of the Dirichlet-to-Neumann operators for $\varepsilon \geq 0$.** Thanks to the particular shape of the hole (circular hole), it is possible to obtain an explicit expression of $T_\varepsilon \varphi$ for every $\varepsilon \geq 0$ and $\varphi \in H^{1/2}(\partial B(x_0, R))$.

4.2.1. *The case $\varepsilon = 0$.* We have the following result.

Proposition 4.4. *For every $\varphi \in H^{1/2}(\partial B(x_0,R))$, the solution u_0^φ to (4.3) and the operator T_0 are given by the explicit expressions:*

$$u_0^\varphi(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{I_n(\sqrt{a}r)}{I_n(\sqrt{a}R)}\varphi_n e^{in\theta}$$

and

$$T_0\varphi(R,\theta) = \sum_{n\in\mathbb{Z}} \sqrt{a}\frac{I_n'(\sqrt{a}R)}{I_n(\sqrt{a}R)}\varphi_n e^{in\theta},$$

where (r,θ) are the polar coordinates in the plane and $(\varphi_n)_{n\in\mathbb{Z}}$ are the Fourier coefficients of φ.

Proof. We write the solution in the following form:

$$u_0^\varphi(r,\theta) = \sum_{n\in\mathbb{Z}} a_n(r)e^{in\theta}.$$

We obtain that

$$r^2 a_n''(r) + r a_n'(r) - (r^2 a + n^2)a_n(r) = 0,$$

for all $n \in \mathbb{Z}$. $I_n(\sqrt{a}r)$ and $K_n(\sqrt{a}r)$ are the two linearly independent solutions to the above modified Bessel's equation. So, we get that

$$a_n(r) = A_n I_n(\sqrt{a}r) + B_n K_n(\sqrt{a}r),$$

for all $n \in \mathbb{Z}$, where A_n and B_n are constants. Since $u_0^\varphi \in H^1(B(x_0,R))$, we have $B_n = 0$, for all $n \in \mathbb{Z}$. Thus, we have

$$u_0^\varphi(r,\theta) = \sum_{n\in\mathbb{Z}} A_n I_n(\sqrt{a}r)e^{in\theta}.$$

Using the boundary condition on $\partial B(x_0,R)$, we obtain the desired result. The expression of $T_\varepsilon\varphi$ follows by taking the normal derivative of u_0^φ on $\partial B(x_0,R)$. □

Proposition 4.5. *We have*

$$\int_{\partial B(x_0,R)} (T_0\varphi)\varphi \; d\partial B(x_0,R) \geq 0, \;\; \textit{for all } \varphi \in H^{1/2}(\partial B(x_0,R)).$$

Proof. From Proposition 4.4, we have

$$\int_{\partial B(x_0,R)} (T_0\varphi)\varphi \; d\partial B(x_0,R) = 2\pi R \sum_{n\in\mathbb{Z}} \sqrt{a}\frac{I_n'(\sqrt{a}R)}{I_n(\sqrt{a}R)}|\varphi_n|^2.$$

On the other hand, from Lemma 3.3, we have

$$\frac{I_n'(\sqrt{a}R)}{I_n(\sqrt{a}R)} \geq 0,$$

for all n. This makes end to the proof. □

We observe easily, that

Proposition 4.6.

$$\int_{\partial B(x_0,R)} (T_0\varphi)\delta \; d\partial B(x_0,R) = \int_{\partial B(x_0,R)} (T_0\delta)\varphi \; d\partial B(x_0,R),$$

for all $\varphi, \delta \in H^{1/2}(\partial B(x_0,R))$.

The following result follows immediately from Proposition 4.6.

Proposition 4.7. *For all $v \in H^1(\Omega_R)$, we have*

$$a_0(v,p_0) = < -DJ(u_{0|\partial\Omega}), v >_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)},$$

where $< \cdot,\cdot >_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}$ is the dual product between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

4.2.2. *The case $\varepsilon > 0$.* At first, we consider the following notations:

$$\begin{aligned}\mathcal{A}(n,\varepsilon) &:= K_n(\sqrt{a}\varepsilon) - \alpha\sqrt{a}K_n'(\sqrt{a}\varepsilon);\\ \mathcal{B}(n,\varepsilon) &:= I_n(\sqrt{a}\varepsilon) - \alpha\sqrt{a}I_n'(\sqrt{a}\varepsilon);\\ \chi(n,\varepsilon) &:= \frac{K_n(\sqrt{a}R)\mathcal{B}(n,\varepsilon)}{I_n(\sqrt{a}R)\mathcal{A}(n,\varepsilon)},\end{aligned}$$

where $n \in \mathbb{Z}$ and $\varepsilon > 0$.

The proof of the following result is similar to the proof of Proposition 4.4.

Proposition 4.8. *For all $\varepsilon > 0$, for every $\varphi \in H^{1/2}(\partial B(x_0, R))$, the solution u_ε^φ to (4.2) and the operator T_ε are given by the explicit expressions:*

$$u_\varepsilon^\varphi(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{\mathcal{A}(n,\varepsilon)I_n(\sqrt{a}r) - \mathcal{B}(n,\varepsilon)K_n(\sqrt{a}r)}{\mathcal{A}(n,\varepsilon)I_n(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n(\sqrt{a}R)}\varphi_n e^{in\theta}$$

and

$$T_\varepsilon\varphi(R,\theta) = \sum_{n\in\mathbb{Z}} \sqrt{a}\frac{\mathcal{A}(n,\varepsilon)I_n'(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n'(\sqrt{a}R)}{\mathcal{A}(n,\varepsilon)I_n(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n(\sqrt{a}R)}\varphi_n e^{in\theta}.$$

We have the following result.

Proposition 4.9. *For $\varepsilon > 0$ small enough, we have*

$$\int_{\partial B(x_0,R)} (T_\varepsilon\varphi)\varphi \; d\partial B(x_0,R) \geq 0, \;\textit{for all}\; \varphi \in H^{1/2}(\partial B(x_0,R)).$$

Proof. We have

$$\int_{\partial B(x_0,R)} (T_\varepsilon\varphi)\varphi \; d\partial B(x_0,R) = 2\pi R\sum_{n\in\mathbb{Z}} \sqrt{a}\frac{\mathcal{A}(n,\varepsilon)I_n'(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n'(\sqrt{a}R)}{\mathcal{A}(n,\varepsilon)I_n(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n(\sqrt{a}R)}|\varphi_n|^2.$$

We have to show that

$$\frac{\mathcal{A}(n,\varepsilon)I_n'(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n'(\sqrt{a}R)}{\mathcal{A}(n,\varepsilon)I_n(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n(\sqrt{a}R)} \geq 0, \;\forall n \in \mathbb{Z}. \tag{4.10}$$

We distinguish two cases.
Case 1. $n \neq 0$. We have

$$\frac{\mathcal{A}(n,\varepsilon)I_n'(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n'(\sqrt{a}R)}{\mathcal{A}(n,\varepsilon)I_n(\sqrt{a}R) - \mathcal{B}(n,\varepsilon)K_n(\sqrt{a}R)} = \frac{I_n'(\sqrt{a}R)}{I_n(\sqrt{a}R)}\frac{1}{1-\chi(n,\varepsilon)}\left[1 - \frac{K_n'(\sqrt{a}R)I_n(\sqrt{a}R)}{I_n'(\sqrt{a}R)K_n(\sqrt{a}R)}\chi(n,\varepsilon)\right]. \tag{4.11}$$

On the other hand, we can write that

$$\chi(n,\varepsilon) = \frac{K_n(\sqrt{a}R)}{I_n(\sqrt{a}R)}\;\frac{I_n(\sqrt{a}\varepsilon)}{K_n(\sqrt{a}\varepsilon)}\;\frac{1-\alpha\sqrt{a}\frac{I_n'}{I_n}(\sqrt{a}\varepsilon)}{1-\alpha\sqrt{a}\frac{K_n'}{K_n}(\sqrt{a}\varepsilon)}. \tag{4.12}$$

We claim that there exists a constant c (independent on n and ε) such that

$$|\chi(n,\varepsilon)| \leq c\varepsilon^2, \;\forall n \in \mathbb{Z}\backslash\{0\}. \tag{4.13}$$

From Lemma 3.3, we have

$$\frac{\sqrt{a\varepsilon^2+n^2}}{\sqrt{a}\varepsilon} \leq \left|\frac{K_n'}{K_n}(\sqrt{a}\varepsilon)\right|,$$

which implies that

$$\frac{\alpha\sqrt{a\varepsilon^2+n^2}}{\varepsilon} - 1 \leq \left|1 - \alpha\sqrt{a}\frac{K_n'}{K_n}(\sqrt{a}\varepsilon)\right| \tag{4.14}$$

Again, form Lemma 3.3, we have

$$0 \leq \frac{I_n'}{I_n}(\sqrt{a}\varepsilon) \leq \frac{\sqrt{a\varepsilon^2+n^2}}{\sqrt{a}\varepsilon},$$

which implies that

$$\left|1-\alpha\sqrt{a}\frac{I_n'}{I_n}(\sqrt{a}\varepsilon)\right|\leq 1+\frac{\alpha\sqrt{a\varepsilon^2+n^2}}{\varepsilon}. \tag{4.15}$$

Using (4.14) and (4.15) (for ε small enough and $n\neq 0$), we obtain that

$$\left|\frac{1-\alpha\sqrt{a}\frac{I_n'}{I_n}(\sqrt{a}\varepsilon)}{1-\alpha\sqrt{a}\frac{K_n'}{K_n}(\sqrt{a}\varepsilon)}\right|\leq\frac{1+\frac{\varepsilon}{\alpha\sqrt{a\varepsilon^2+n^2}}}{1-\frac{\varepsilon}{\alpha\sqrt{a\varepsilon^2+n^2}}}.$$

Since

$$\lim_{n\to\infty,\varepsilon\to 0}\frac{1+\frac{\varepsilon}{\alpha\sqrt{a\varepsilon^2+n^2}}}{1-\frac{\varepsilon}{\alpha\sqrt{a\varepsilon^2+n^2}}}=1,$$

there exists a constant $c>0$ (independent on n and ε) such that

$$\left|\frac{1-\alpha\sqrt{a}\frac{I_n'}{I_n}(\sqrt{a}\varepsilon)}{1-\alpha\sqrt{a}\frac{K_n'}{K_n}(\sqrt{a}\varepsilon)}\right|\leq c, \tag{4.16}$$

for all $n\neq 0$ and $\varepsilon>0$ (small enough). On the other hand, since $0<\varepsilon<R$, we have

$$0<\frac{K_n(\sqrt{a}R)}{K_n(\sqrt{a}\varepsilon)}<\frac{\varepsilon}{R}. \tag{4.17}$$

Using Lemma 3.2, we obtain that

$$0<\frac{I_n(\sqrt{a}\varepsilon)}{I_n(\sqrt{a}R)}<\frac{\varepsilon}{R}. \tag{4.18}$$

Now, our claim (4.13) follows immediately from (4.16), (4.17) and (4.18).

From (4.13), to obtain (4.10) (for $n\neq 0$), we have only to prove that for $\varepsilon>0$ (small enough),

$$1-\frac{K_n'(\sqrt{a}R)I_n(\sqrt{a}R)}{I_n'(\sqrt{a}R)K_n(\sqrt{a}R)}\chi(n,\varepsilon)\geq 0. \tag{4.19}$$

Using Lemma 3.4 and Lemma 3.5, we obtain that

$$0\leq\frac{-K_n'(\sqrt{a}R)}{K_n(\sqrt{a}R)}\leq\frac{1+\sqrt{(n+1)^2+aR^2}}{\sqrt{a}R}. \tag{4.20}$$

Using Lemma 3.3 and 3.5, we obtain that

$$0\leq\frac{I_n(\sqrt{a}R)}{I_n'(\sqrt{a}R)}=\frac{1}{\frac{I_{n+1}}{I_n}(\sqrt{a}R)+\frac{n}{\sqrt{a}R}}\leq\frac{\sqrt{a}R}{n}. \tag{4.21}$$

From (4.20) and (4.21), we have

$$0\leq-\frac{K_n'(\sqrt{a}R)I_n(\sqrt{a}R)}{I_n'(\sqrt{a}R)K_n(\sqrt{a}R)}\leq\frac{1+\sqrt{(n+1)^2+aR^2}}{n},$$

which implies that the positive sequence $\left\{-\frac{K_n'(\sqrt{a}R)I_n(\sqrt{a}R)}{I_n'(\sqrt{a}R)K_n(\sqrt{a}R)}\right\}_{n\neq 0}$ is bounded. Thus, from (4.13), we obtain (4.19).

Case 2. $n=0$. We have to show that

$$\frac{\mathcal{A}(0,\varepsilon)I_0'(\sqrt{a}R)-\mathcal{B}(0,\varepsilon)K_0'(\sqrt{a}R)}{\mathcal{A}(0,\varepsilon)I_0(\sqrt{a}R)-\mathcal{B}(0,\varepsilon)K_0(\sqrt{a}R)}\geq 0. \tag{4.22}$$

Using Lemma 3.6, we obtain that

$$\lim_{\varepsilon\to 0^+}\mathcal{A}(0,\varepsilon)=+\infty$$

and

$$\lim_{\varepsilon\to 0^+}\mathcal{B}(0,\varepsilon)=1.$$

Thus, we obtain that

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{A}(0,\varepsilon)I_0'(\sqrt{a}R)-\mathcal{B}(0,\varepsilon)K_0'(\sqrt{a}R)}{\mathcal{A}(0,\varepsilon)I_0(\sqrt{a}R)-\mathcal{B}(0,\varepsilon)K_0(\sqrt{a}R)} = \frac{I_0'}{I_0}(\sqrt{a}R) \geq 0.$$

Then, for ε small enough, we have (4.26). □

The following result follows immediately from Proposition 4.9.

Proposition 4.10. *There exists a constant $c>0$ (independent on ε) such that for $\varepsilon \geq 0$ small enough, we have*

$$a_\varepsilon(u,u) \geq c\|u\|^2_{H^1(\Omega_R)}, \ \forall\, u \in H^1(\Omega_R).$$

4.3. **Asymptotic expansion of $\mathbf{T}_\varepsilon - \mathbf{T_0}$ as $\varepsilon \to \mathbf{0}^+$.** We introduce the linear and continuous mapping $\delta_T : H^{1/2}(\partial B(x_0,R)) \to H^{-1/2}(\partial B(x_0,R))$ defined by

$$\delta_T\varphi := \frac{1}{\alpha R I_0^2(\sqrt{a}R)}\varphi_0,$$

for all $\varphi \in H^{1/2}(\partial B(x_0,R))$.

We have the following result.

Proposition 4.11. *We have*

$$\|T_\varepsilon - T_0 - \varepsilon\delta_T\|_{\mathcal{L}(H^{1/2}(\partial B(x_0,R)),H^{-1/2}(\partial B(x_0,R)))} = o(\varepsilon), \ \textit{as } \varepsilon \to 0^+,$$

where $\|\cdot\|_{\mathcal{L}(H^{1/2}(\partial B(x_0,R)),H^{-1/2}(\partial B(x_0,R)))}$ denotes the standard norm on the space of linear and continuous mappings from $H^{1/2}(\partial B(x_0,R))$ to $H^{-1/2}(\partial B(x_0,R))$.

Proof. For $\varphi \in H^s(\partial B(x_0,R))$, let

$$\|\varphi\|_{s,\partial B(x_0,R)} := \left(\sum_{n\in\mathbb{Z}} |\varphi|_n^2(1+|n|)^{2s}\right)^{1/2}.$$

It is well-known that $\|\cdot\|_{s,\partial B(x_0,R)}$ is a norm on $H^s(\partial B(x_0,R))$ that is equivalent to its usual norm.

Let $\varphi \in H^{1/2}(\partial B(x_0,R))$. From Propositions 4.4 and 4.8, and using Lemmas 3.8 and 3.5, we have

$$(T_\varepsilon - T_0)\varphi = \frac{\varphi_0}{RI_0(\sqrt{a}R)K_0(\sqrt{a}R)}\frac{1}{\frac{I_0(\sqrt{a}R)}{K_0(\sqrt{a}R)}\frac{\mathcal{A}(0,\varepsilon)}{\mathcal{B}(0,\varepsilon)}-1} + R_\varepsilon\varphi, \tag{4.23}$$

where

$$R_\varepsilon\varphi := \sum_{|n|\geq 1}\frac{\mathcal{B}(n,\varepsilon)\varphi_n e^{in\theta}}{RI_n(\sqrt{a}R)[\mathcal{A}(n,\varepsilon)I_n(\sqrt{a}R)-\mathcal{B}(n,\varepsilon)K_n(\sqrt{a}R)]}.$$

Using Lemma 3.7, we obtain that

$$\mathcal{A}(0,\varepsilon) = \frac{\alpha}{\varepsilon} - \ln\varepsilon + O(1), \ \varepsilon \to 0^+ \tag{4.24}$$

and

$$\mathcal{B}(0,\varepsilon) = 1 + o(\varepsilon), \ \varepsilon \to 0^+. \tag{4.25}$$

Thanks to (4.24) and (4.25), (4.23) becomes

$$(T_\varepsilon - T_0 - \varepsilon\delta_T)\varphi = O(\varepsilon^2\ln\varepsilon)\delta_T\varphi + R_\varepsilon\varphi, \ \varepsilon \to 0^+.$$

To obtain the desired result, we have only to show that

$$\|R_\varepsilon\varphi\|_{-1/2,\partial B(x_0,R)} = o(\varepsilon), \ \varepsilon \to 0^+. \tag{4.26}$$

We have

$$\|R_\varepsilon\varphi\|^2_{-1/2,\partial B(x_0,R)} = \sum_{|n|\geq 1}\mathcal{C}(n,\varepsilon)^2(1+|n|)|\varphi_n|^2,$$

where

$$\mathcal{C}(n,\varepsilon) := \frac{|\chi(n,\varepsilon)|}{(1+|n|)RI_n(\sqrt{a}R)K_n(\sqrt{a}R)|1-\chi(n,\varepsilon)|}.$$

Using Lemma 3.9, we obtain that

$$(1+|n|)RI_n(\sqrt{a}R)K_n(\sqrt{a}R) \to \frac{R}{2}, \text{ as } n \to +\infty. \tag{4.27}$$

On the other hand, from (4.13), there exists a constant $c > 0$ (independent on n and ε) such that

$$\frac{|\chi(n,\varepsilon)|}{|1-\chi(n,\varepsilon)|} \leq c\varepsilon^2, \tag{4.28}$$

for all $n \in \mathbb{Z}\backslash\{0\}$ and $\varepsilon > 0$ (small enough). Now, (4.26) follows from (4.27) and (4.28). □

4.4. **Asymptotic expansion of $\mathbf{a}_\varepsilon - \mathbf{a_0}$ as $\varepsilon \to \mathbf{0}^+$.** From (4.9), for all $\varepsilon > 0$, we have

$$(a_\varepsilon - a_0)(u,v) = \int_{\partial B(x_0,R)} [(T_\varepsilon - T_0)u]v \; d\partial B(x_0,R), \; \forall\, u,v \in H^1(\Omega_R).$$

Then, for all $u, v \in H^1(\Omega_R)$, we have

$$\begin{aligned}(a_\varepsilon - a_0)(u,v) &= \varepsilon \int_{\partial B(x_0,R)} (\delta_T u)v \; d\partial B(x_0,R) + \int_{\partial B(x_0,R)} [(T_\varepsilon - T_0 - \varepsilon\delta_T)u]v \; d\partial B(x_0,R) \\ &= \varepsilon\delta_a(u,v) + \int_{\partial B(x_0,R)} [(T_\varepsilon - T_0 - \varepsilon\delta_T)u]v \; d\partial B(x_0,R),\end{aligned}$$

where

$$\delta_a(u,v) = \int_{\partial B(x_0,R)} (\delta_T u)v \; d\partial B(x_0,R), \; \forall\, u,v \in H^1(\Omega_R). \tag{4.29}$$

Now, using Proposition 4.11, we obtain the following result.

Proposition 4.12. *We have*

$$\|a_\varepsilon - a_0 - \varepsilon\delta_a\|_{\mathcal{L}_2(H^1(\Omega_R))} = o(\varepsilon), \; \text{as } \varepsilon \to 0^+.$$

4.5. **Main result.** Now, we are ready to state and prove our main result.

Theorem 4.1. *We have the following asymptotic expansions:*

$$J(u_{\Omega_\varepsilon|\partial\Omega}) = J(u_{\Omega|\partial\Omega}) + \frac{2\pi\varepsilon}{\alpha} u_\Omega(x_0)p_\Omega(x_0) + o(\varepsilon), \; \text{as } \varepsilon \to 0^+.$$

Proof. Using Propositions 4.10, 4.1, 4.2 and 4.12, and Lemma 3.1, we obtain that

$$\begin{aligned}J(u_{\Omega_\varepsilon|\partial\Omega}) - J(u_{\Omega|\partial\Omega}) &= F(\varepsilon) - F(0) \\ &= \varepsilon\delta_a(u_0,p_0) + o(\varepsilon) \\ &= \varepsilon\delta_a(u_0,p_0) + o(\varepsilon) \\ &= \varepsilon\delta_a(u_{\Omega|\Omega_R}, p_{\Omega|\Omega_R}) + o(\varepsilon) \\ &= \varepsilon \int_{\partial B(x_0,R)} (\delta_T u_\Omega)p_\Omega \; d\partial B(x_0,R) + o(\varepsilon) \\ &= \frac{2\pi\varepsilon}{\alpha I_0(\sqrt{a}R)^2} {u_\Omega}^0 {p_\Omega}^0 + o(\varepsilon),\end{aligned}$$

where

$${u_\Omega}^0 = \frac{1}{2\pi}\int_0^{2\pi} u_\Omega(R,\theta)\; d\theta \quad \text{ and } \quad {p_\Omega}^0 = \frac{1}{2\pi}\int_0^{2\pi} p_\Omega(R,\theta)\; d\theta.$$

Since $-\Delta u_\Omega + a u_\Omega = 0$ in $B(x_0,R)$, we have

$${u_\Omega}^0 = I_0(\sqrt{a}R)u_\Omega(x_0).$$

Similarly, we have

$$p_{\Omega}{}^{0} = I_0(\sqrt{a}R)p_{\Omega}(x_0).$$

Then, the result follows. □

5. Numerical results

Recently, topological derivative has been used for solving inverse problems, and more precisely for imaging small inclusions, see [10, 8, 11, 24, 15] for instance. Let us describe the situation which will be under discussion here. Given a bounded domain Ω and an inclusion $\omega \subset \Omega$ (not necessarily connected), we denote by Ω_ω the perforated domain $\Omega \setminus \omega$. We consider the scattering of several plane waves g_ℓ $(\ell = 1,\dots,L)$ in Ω^ω:

$$\left\{\begin{array}{rcll} -\Delta u_\ell - k^2 u_\ell &=& 0 & \text{in } \Omega^\omega,\\ u_\ell + \alpha\dfrac{\partial u_\ell}{\partial n} &=& 0 & \text{on } \partial\omega,\\ \dfrac{\partial u_\ell}{\partial n} &=& \dfrac{\partial g_\ell}{\partial n} & \text{on } \partial\Omega. \end{array}\right. \tag{5.1}$$

We aim at retrieving ω from the knowledge of the scattered fields measured on the boundary : $\big(u_\ell|_{\partial\Omega}\big)_{\ell=1,\dots,L}$. To this end, we introduce the cost functional

$$j(\Omega_{x_0,\varepsilon}) = \sum_{\ell=1}^{L}\int_{\partial\Omega}|u^\ell_{x_0,\varepsilon} - u_\ell|^2\,\mathrm{d}\sigma_x,$$

where $\Omega_{x_0,\varepsilon}$ is obtained from Ω after having removed the ball $B(x_0,\varepsilon)$: $\Omega_{x_0,\varepsilon} = \Omega^{B(x_0,\varepsilon)}$, and $u^\ell_{x_0,\varepsilon}$ solves Problem (5.1) for this particular inclusion.

The results developed above in the paper easily extend to this case, and the topological derivative of j is given by

$$\mathrm{TD}(x) = \Re\left(\sum_{\ell=1}^{L} g_\ell(x)\overline{p_\ell(x)}\right), \tag{5.2}$$

where the adjoint state solves

$$\left\{\begin{array}{rcll} -\Delta p_\ell - k^2 p_\ell &=& 0 & \text{in } \Omega,\\ \dfrac{\partial p_\ell}{\partial n} &=& -2(g_\ell - u_\ell) & \text{on } \partial\Omega. \end{array}\right. \tag{5.3}$$

It is clear from Theorem 4.1 that negative values of the topological derivative $\mathrm{TD}(x)$ correspond – at first order – to a decrease of the quadratic misfit $j(\Omega_{x,\varepsilon})$. As a consequence, it is natural to expect $\mathrm{TD}(x)$ to have strong negative values for x near x_0. This empirical remark has led to the use of TD as an imaging tool for inclusions ω which are not only small balls, but with very few mathematical justification. Let us mention the recent works of [2] and [7], giving some tangible arguments.

In the following, we present numerical simulations. The considered geometries and sets of parameters are inspired by the existing literature, especially [20, 2, 7]. In the following, Ω is the unit disk, the parameter involved in the impedance condition $\alpha = 0.1$ (except for Figure 4 where $\alpha = 10$), and the wave number $k = 2\pi/\lambda$ with a wavelength $\lambda = 0.5$. The experiments have been done with the finite element library Mélina [21] with $\mathbb{P}_2$ triangular elements with mesh size $h = \lambda/10$. Rather than TD, its normalized version NTD is considered:

$$\mathrm{NTD}(x) = \frac{\mathrm{TD}(x)}{\max\limits_{x\in\Omega}|\mathrm{TD}(x)|}.$$

Last, we have chosen plane waves with equally distributed incident directions:

$$g_\ell(x) = e^{\mathrm{i}kx\cdot d_\ell},$$

with $d_\ell = \big(\cos(2\mathrm{i}\ell\pi/L), \sin(2\mathrm{i}\ell\pi/L)\big)$ for $\ell = 1,\dots,L$.

In Figure 1, we present the imaging of a small circular inclusion, centered at $(0.3, -0.5)$ and of radius 0.05. With L large enough, the reconstruction is quite satisfactory, since $\mathrm{NTD}(x)$ presents large negative values only in the vicinity of the inclusion (plotted in white on the pictures).

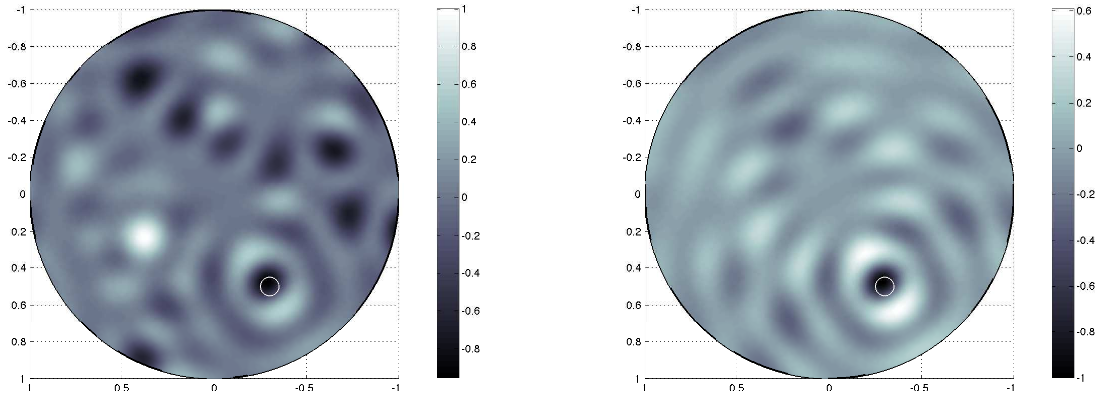

FIGURE 1. Map of $x \mapsto \mathrm{NTD}(x)$ for $L = 4$ *(left)* and $L = 32$ *(right)*.

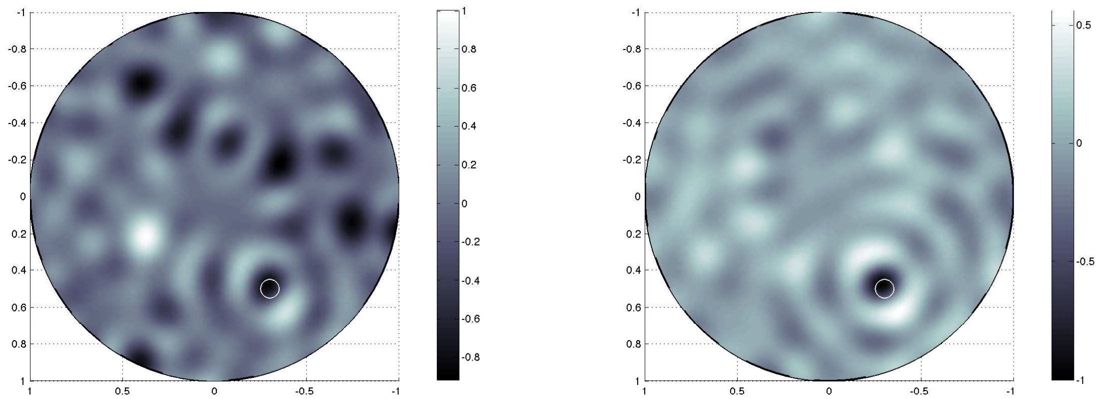

FIGURE 2. Map of $x \mapsto \mathrm{NTD}(x)$ for $L = 4$ *(left)* and $L = 32$ *(right)* for noisy data.

To investigate the robustness of the method, we add a noise to the measured field u_ℓ (spatially and ℓ-independent). The results are presented in Figure 2 for a uniform distribution and a signal to noise ratio equal to 2. It turns out that for L large enough the reconstruction is quite stable. Nevertheless, for a small number of incident directions, the results are not very accurate since many peaks of same amplitude appear at several locations away from the inclusion (see left picture of Figure 2). Let us mention that this remarkable robustness needs to be weakened by the fact that the chosen noise (which is the most usually encountered in the literature) is favorable. A global noise in the medium, for instance, would probably lead to larger differences.

We present in Figure 3 the reconstruction of multiple inclusions. The results are good in both cases.

The originality of the present work lies in the choice of an impedance boundary condition of Robin type on the inclusion. It is worth noticing that small values of the parameter α correspond to a penalization of the Dirichlet condition on the inclusion. Let us emphasize that the expression of the topological derivative coincide for Dirichlet and Robin conditions. On the contrary, large values of α can be seen as a penalization of the Neumann boundary condition.

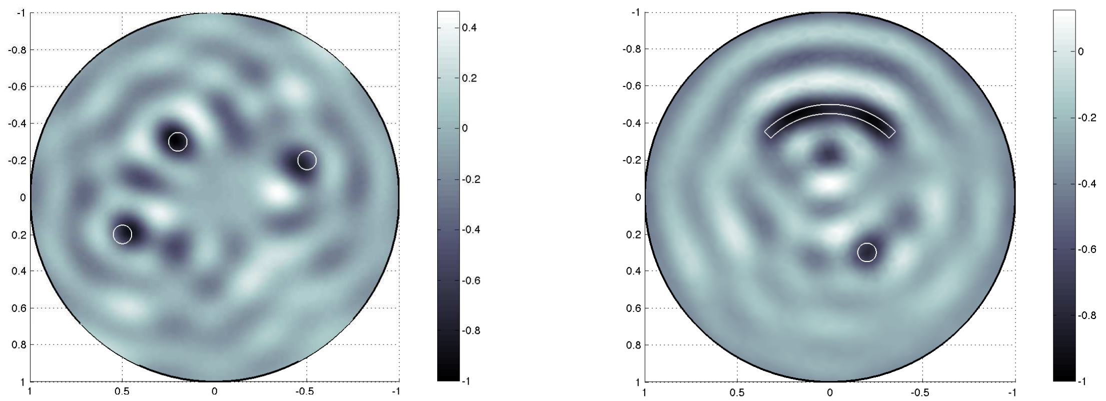

FIGURE 3. Map of $x \mapsto \mathrm{NTD}(x)$ for $L = 32$ in the case of multiple inclusions.

Let us recall the expression of the topological derivative for Neumann problems – see e.g. [11]

$$\mathrm{TD_N}(x) = \Re\left(\sum_{\ell=1}^{L} k^2 g_\ell(x)\overline{p_\ell(x)} - \nabla g_\ell(x)\cdot\overline{\nabla p_\ell(x)}\right), \tag{5.4}$$

with the obvious notation $\mathrm{NTD_N}$ for its normalized counterpart. In Figure 4, we have plotted the topological derivatives NTD and $\mathrm{NTD_N}$ for the value $\alpha = 10$ (and the geometry with a single circular inclusion already used in Figures 1 and 2). The reconstruction does not fit the exact inclusion in the case of a Dirichlet/Robin expression for the topological derivative, but the Neumann expression leads to a good result, which is quite natural.

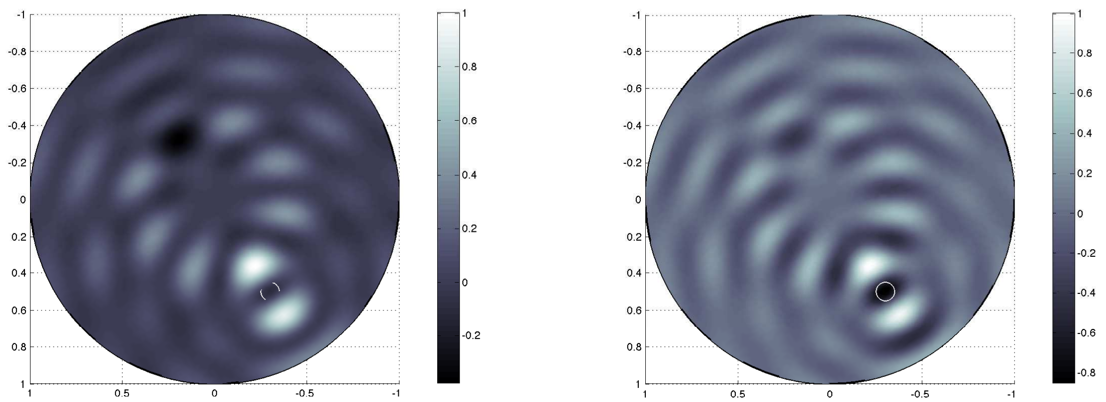

FIGURE 4. Map of $x \mapsto \mathrm{NTD}(x)$ *(left)* and $x \mapsto \mathrm{NTD_N}(x)$ *(right)* for $L = 32$ and $\alpha = 10$.

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