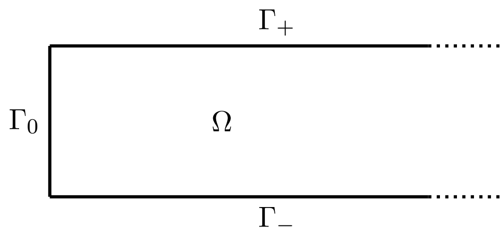
MOS 5.3 – Physical problems in unbounded domains

Activity 4. Resolution of the Helmholtz equation in a half-waveguide

We consider $\Omega = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \; ; \; x > 0 \text{ and } 0 < y < 1\}$, and the problem

$$\begin{cases} \Delta u + k^2 u = 0, \text{in } \Omega, \\ u = g, \;\text{ on } \Gamma_0 \quad (\text{i.e. for } x = 0), \\ u = 0, \;\text{ on } \Gamma_\pm \quad (\text{i.e. for } y = 0 \text{ and } y = 1). \end{cases} \tag{P}$$



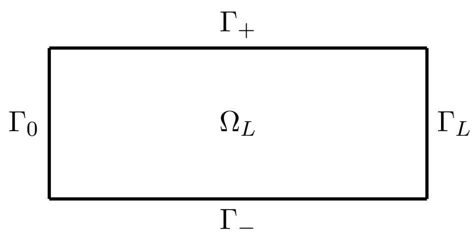
1. We consider the value $k = 4$, and the boundary datum

$$g(y) = 2\sin(\pi y) + \sin(2\pi y).$$

$\boxed{\mathcal{Q}}$ Give the expression of the solution u, satisfying the outgoing radiation condition.

2. We consider the truncated domain (with $L > 0$) :

$$\Omega_L = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \; ; \; 0 < x < L \text{ and } 0 < y < 1\},$$



and the problem (for $k = 4$)

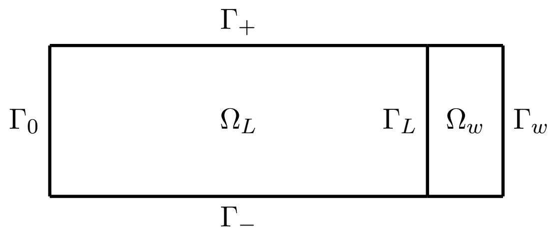
$$\begin{cases} \Delta u + k^2 u = 0, \text{in } \Omega_L, \\ u = g, \;\text{ on } \Gamma_0 \quad (\text{i.e. for } x = 0), \\ u = 0, \;\text{ on } \Gamma_\pm \quad (\text{i.e. for } y = 0 \text{ and } y = 1), \\ \partial_n u = i\sqrt{k^2 - \pi^2}\, u, \text{on } \Gamma_L \quad (\text{i.e. for } x = L). \end{cases} \tag{P$_{\text{LF}}$}$$

The condition on Γ_L is called *low-frequency approximate boundary condition.*

$\boxed{\mathcal{Q}}$ With `Freefem++`, solve Problem (P_{LF}), and compare the obtained solution to the exact solution (for $L = 3$).

3. We consider the truncated domain with a layer (of thickness w) :

$$\Omega_{L,w} = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \ ; \ 0 < x < L + w \text{ and } 0 < y < 1\},$$



and the problem with Perfectly Matched Layer (again for $k = 4$) :

$$\begin{cases} \Delta u + k^2 u = 0, \text{in } \Omega_L, \\ \alpha^2 \partial_x^2 u + \partial_y^2 u + k^2 u = 0, \text{in } \Omega_w, \\ u = g, \ \text{on } \Gamma_0 \quad \text{(i.e. for } x = 0), \\ u = 0, \ \text{on } \Gamma_\pm \quad \text{(i.e. for } y = 0 \text{ and } y = 1), \\ u = 0, \ \text{on } \Gamma_w \quad \text{(i.e. for } x = L), \\ u(L^-, y) = u(L^+, y), \ \text{for } 0 < y < 1, \\ \partial_n u(L^-, y) = \alpha \partial_n u(L^+, y), \ \text{for } 0 < y < 1. \end{cases} \tag{P$_{\text{PML}}$}$$

For the numerical tests, we shall take $L = 3$, $w = 1$, and $\alpha = \frac{1-i}{10}$.

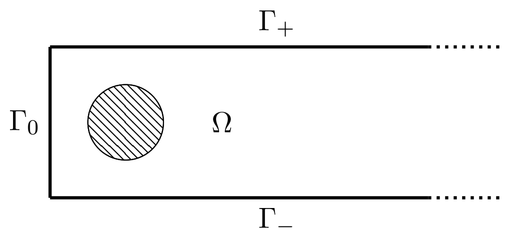
$\boxed{\mathcal{Q}}$ Write a variational formulation for the problem (P_{PML}).

$\boxed{\mathcal{Q}}$ Use `Freefem++` to solve Problem (P_{PML}), and compare the obtained solution to the exact solution.

$\boxed{\mathcal{Q}}$ Comment the results obtained in the case where $\alpha = \frac{1+i}{10}$.

4. We consider now $k = 7$. Repeat the previous steps (question 1., 2. , 3.). Comment the results.

5. We consider the following geometry (the disc is not contained in Ω) :



Adapt the previous methods to solve Problem (P) in this context for $k = 5$.