Filter design under magnitude constraints is a finite dimensional convex optimization problem

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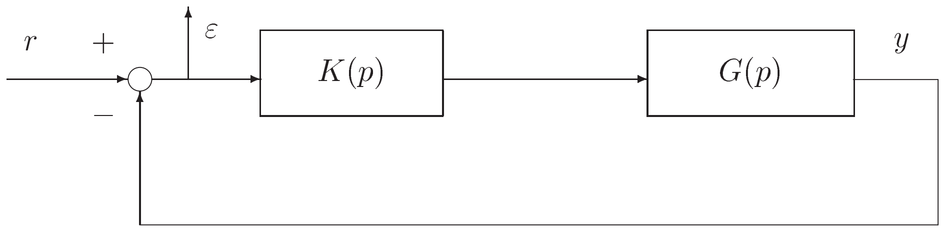
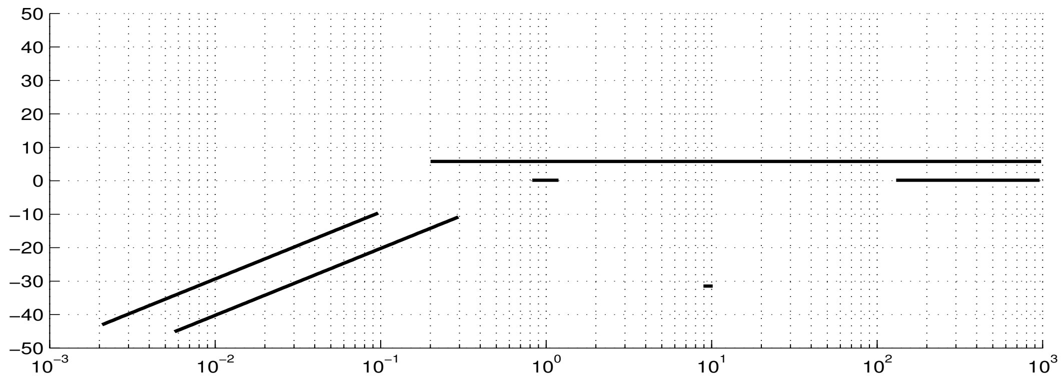
Weighting function for H_∞ control



Step and sinusoidal reference signals r tracking $\Longleftrightarrow$ Magnitude of $T_{r\to\varepsilon}$ bounded by a weighting function magnitude satisfying the considered set of constraints

Purpose $\Longrightarrow$

Weighting function design



$\boxed{\text{General Problem Statement}}$

1. N_u upper bounds $U_i(\omega)$ defined on sub intervals Ω_i^u
2. N_l upper bounds $L_j(\omega)$ defined on sub intervals Ω_j^l
3. a positive integer n

find if there exists a minimum phase proper stable filter F of order bounded by n such that :

$$\begin{cases} \text{for all } \ i \in \{1, \cdots, N_u\}, \ \text{ for all } \omega \in \Omega_i^u, \ |F(j\omega)| < U_i(\omega) \\ \text{for all } \ j \in \{1, \cdots, N_l\}, \ \text{ for all } \omega \in \Omega_j^l, \ L_j(\omega) < |F(j\omega)| \end{cases}$$

and **compute** it.

Problem difficulties

1. Infinite number of decision variables

2. Infinite number of inequalities

Main result : General Problem Solution

Find $X_N = X_N^T,\ X_D = X_D^T,\ P_N = P_N^T,\ P_D = P_D^T,\ P_i^u,\ P_j^l$

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} \mathbf{X_N} \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T \mathbf{P_N} + \mathbf{P_N} A & \mathbf{P_N} B_b \\ B_b^T \mathbf{P_N} & 0 \end{bmatrix} > 0 \tag{1}$$

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} \mathbf{X_D} \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T \mathbf{P_D} + \mathbf{P_D} A & \mathbf{P_D} B_b \\ B_b^T \mathbf{P_D} & 0 \end{bmatrix} > 0 \tag{2}$$

$$\forall i, \mathbf{P_i^u} + \mathbf{P_i^{u*}} > 0, \quad \begin{bmatrix} \overline{C}_i^* \\ \overline{D}_i^* \end{bmatrix} \begin{bmatrix} -\mathbf{X_D} & 0 \\ 0 & \mathbf{X_N} \end{bmatrix} \begin{bmatrix} \overline{C}_i & \overline{D}_i \end{bmatrix} + \begin{bmatrix} \overline{A}_i^* \mathbf{P_i^{u*}} + \mathbf{P_i^u} \overline{A}_i & \mathbf{P_i^u} \overline{B}_i \\ \overline{B}_i^* \mathbf{P_i^{u*}} & 0 \end{bmatrix} < 0 \tag{3}$$

$$\forall j, \mathbf{P_j^l} + \mathbf{P_j^{l*}} > 0, \quad \begin{bmatrix} \underline{C}_j^* \\ \underline{D}_j^* \end{bmatrix} \begin{bmatrix} \mathbf{X_D} & 0 \\ 0 & -\mathbf{X_N} \end{bmatrix} \begin{bmatrix} \underline{C}_j & \underline{D}_j \end{bmatrix} + \begin{bmatrix} \underline{A}_j^* \mathbf{P_j^{l*}} + \mathbf{P_j^l} \underline{A}_j & \mathbf{P_j^l} \underline{B}_j \\ \underline{B}_j^* \mathbf{P_j^{l*}} & 0 \end{bmatrix} < 0 \tag{4}$$

where $A_b, B_b, C_b, D_b, \overline{A}_i, \overline{B}_i, \overline{C}_i, \overline{D}_i, \underline{A}_i, \underline{B}_i, \underline{C}_i, \underline{D}_i$ are defined from the problem data.

$\Longrightarrow$ **An LMI optimization Problem**

$\rightarrow$ The solution is

$$F(j\omega) = \frac{N(j\omega)}{D(j\omega)} \quad \text{with} \quad \left\{ \begin{array}{rcl} N(j\omega)^* N(j\omega) & = & B(j\omega)^* X_N B(j\omega) \\ D(j\omega)^* D(j\omega & = & B(j\omega)^* X_D B(j\omega) \end{array} \right.$$

Infinite dimensional problem formulation
$\Downarrow$
LMI finite dimensional convex optimization problem
$\Downarrow$
Efficient Algorithm

Sketching the proof in the case of Simplified Problem

$\rightarrow$ One upper bound constraint $U(\omega)$ on $\Omega = (\underline{\omega}; \overline{\omega})$

$(\blacklozenge)$

Find $F(j\omega)$ such that :

for all $\omega \in \Omega,\ |F(j\omega)| < U(\omega)$

and **compute** it

Infinite dimensional formulation

1. Infinite number of decision variables

2. Infinite number of inequalities

Semi infinite dimensional formulation (I)

- **Change of variable** $\quad F(j\omega) \to |F(j\omega)|$

$\to$ *Constraint affine in* $|F|$ (infinite dimension variable)

$|F(j\omega)| \to F(j\omega)$: Classic Spectral Factorization.

- $F(j\omega) = \dfrac{N(j\omega)}{D(j\omega)}, \qquad (\blacklozenge) \Longleftrightarrow |N(j\omega)| < |D(j\omega)| U(\omega)$

$\to$ *Constraint affine in* $|N|$ *an* $|D|$

- **Finite dimensional parametrization**

$$\begin{aligned} |N(j\omega)|^2 &= B(j\omega)^* X_N B(j\omega) \\ |D(j\omega)|^2 &= B(j\omega)^* X_D B(j\omega) \end{aligned}$$

with

$$B(j\omega) = \left[\begin{array}{ccccc} (j\omega)^n & (j\omega)^{n-1} & \cdots & j\omega & 1 \end{array} \right]^T$$

$\to$ *Constraint affine in* X_N *and* X_D (infinite dimension variable)

Semi infinite dimensional formulation (II)

$$(\blacklozenge) \quad \Longleftrightarrow$$

C.1 for all $\omega \in [0, +\infty)$, $B(j\omega)^* \mathbf{X_N} B(j\omega) > 0$

C.2 for all $\omega \in [0, +\infty)$, $B(j\omega)^* \mathbf{X_D} B(j\omega) > 0$

C.3 for all $\omega \in \Omega$, $\begin{bmatrix} B(j\omega)U(\omega) \\ B(j\omega) \end{bmatrix}^* \begin{bmatrix} -\mathbf{X_D} & 0 \\ 0 & \mathbf{X_N} \end{bmatrix} \begin{bmatrix} B(j\omega)U(\omega) \\ B(j\omega) \end{bmatrix} < 0$

Problem : Infinite number of inequalities

Extension of Kalman Yakubovic Popov Lemma

- **KYP** Consider $\Phi(j\omega) = D + C(j\omega I - A)^{-1}B$

$\forall \omega \in [0,\ +\infty), \qquad \Phi^*(j\omega)M\Phi(j\omega) < 0$ $\rightarrow$ *infinite number of inequalities*

$$\Longleftrightarrow$$

$$\exists P = P^T,\ \begin{bmatrix} C^* \\ D^* \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^*P^* + PA & PB \\ B^*P^* & 0 \end{bmatrix} < 0$$

$\rightarrow$ *finite dimensional optimization problem*

$\longrightarrow$ Using ROBUST THEORY, we obtain

- **Extended KYP** Consider $\Phi(\delta) = D + C\delta(I - \delta A)^{-1}B$

$\forall \delta \in (0,\ +\infty), \qquad \Phi^*(\delta)M\Phi(\delta) < 0$ $\rightarrow$ *infinite number of inequalities*

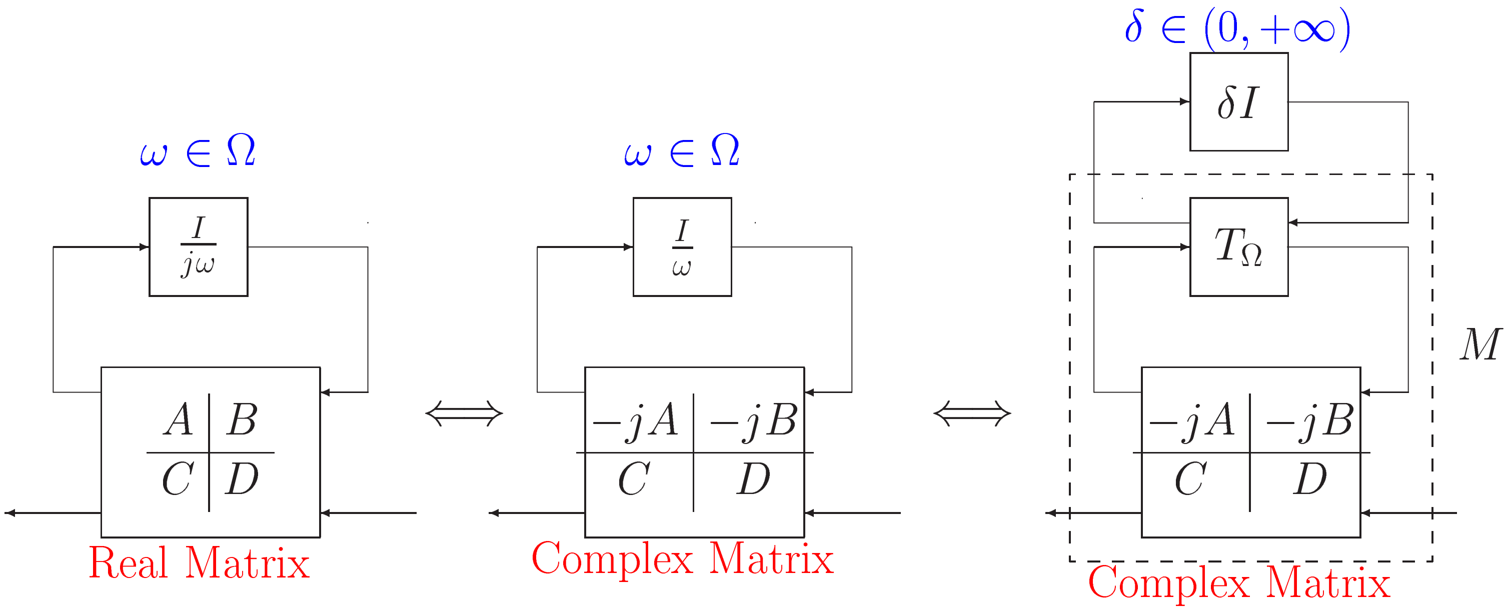
$$\Longleftrightarrow$$

$$\exists P,\ \ P + P^* > 0,\ \begin{bmatrix} C^* \\ D^* \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^*P^* + PA & PB \\ B^*P^* & 0 \end{bmatrix} < 0$$

$\rightarrow$ *finite dimensional optimization problem*

Use of this KYP lemma Extension

Loop-shifting



Equivalent finite dimensional formulation

$$(\blacklozenge) \quad \Longleftrightarrow$$

$$\text{Find } X_N = X_N^T,\ X_D = X_D^T,\ P_N = P_N^T,\ P_D = P_D^T,\ P + P^* > 0,$$

$$\mathbf{C.1} \overset{KYP}{\Longleftrightarrow} \begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_N \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_N + P_N A & P_N B_b \\ B_b^T P_N & 0 \end{bmatrix} > 0 \tag{5}$$

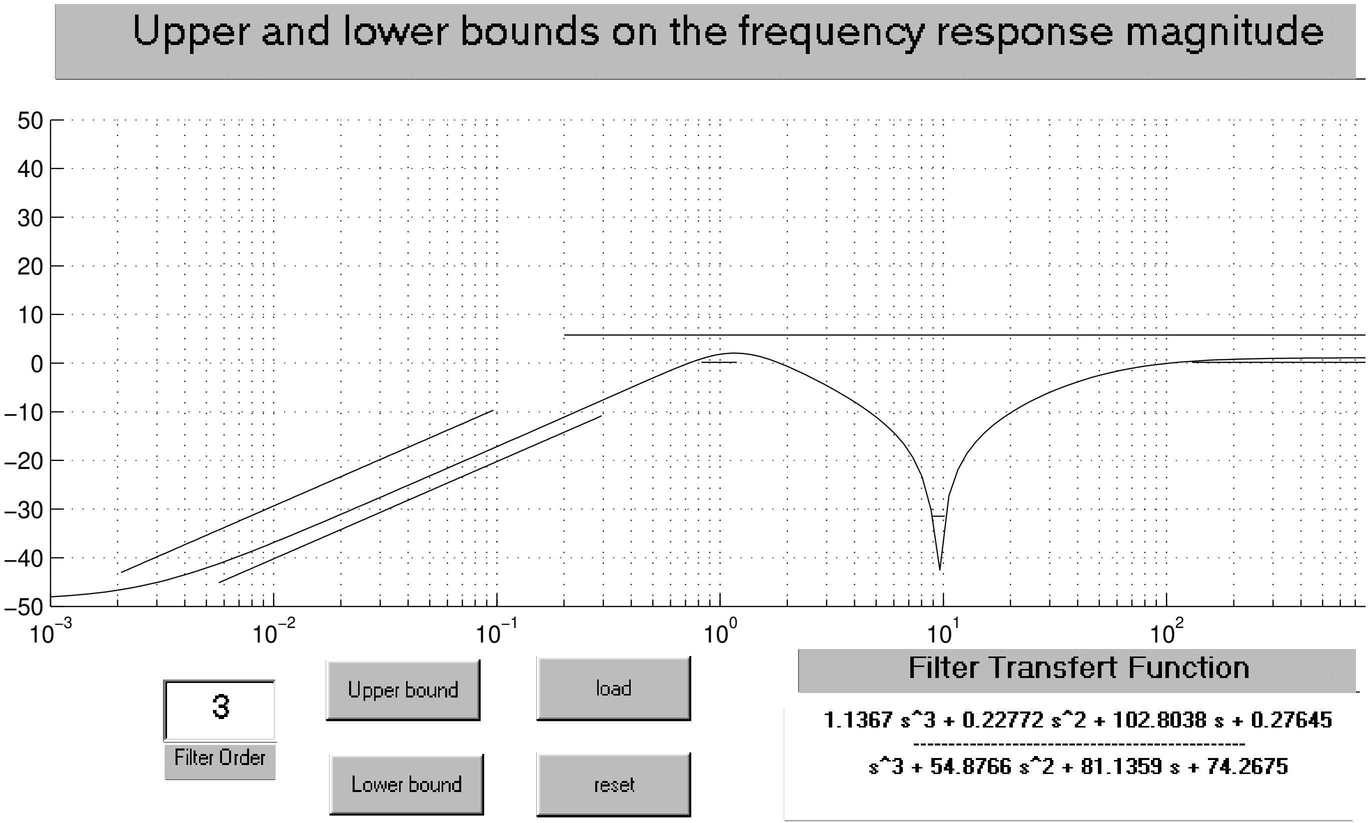
$$\mathbf{C.2} \overset{KYP}{\Longleftrightarrow} \begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_D \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_D + P_D A & P_D B_b \\ B_b^T P_D & 0 \end{bmatrix} > 0 \tag{6}$$

$$\mathbf{C.3} \overset{EKYP}{\Longleftrightarrow} \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^* P^* + PA & PB \\ B^* P^* & 0 \end{bmatrix} < 0 \tag{7}$$

$\rightarrow$ The solution is

$$F(j\omega) = \frac{N(j\omega)}{D(j\omega)} \quad \text{with} \quad \left\{ \begin{array}{rcl} N(j\omega)^* N(j\omega) & = & B(j\omega)^* X_N B(j\omega) \\ D(j\omega)^* D(j\omega & = & B(j\omega)^* X_D B(j\omega) \end{array} \right.$$

Application : H_∞ weighting function design



Conclusion

1. *finite dimensional convex* optimization solution involving Linear Matrix Inequality constraints for an a priori *infinite dimensional problem*

2. convex = efficient solution

3. *flexibility* of the proposed formulation, allowing to solve problems using a natural form

4. *no extra assumption* of the obtained filter (as the filter poles are fixed or the phase filter is linear)

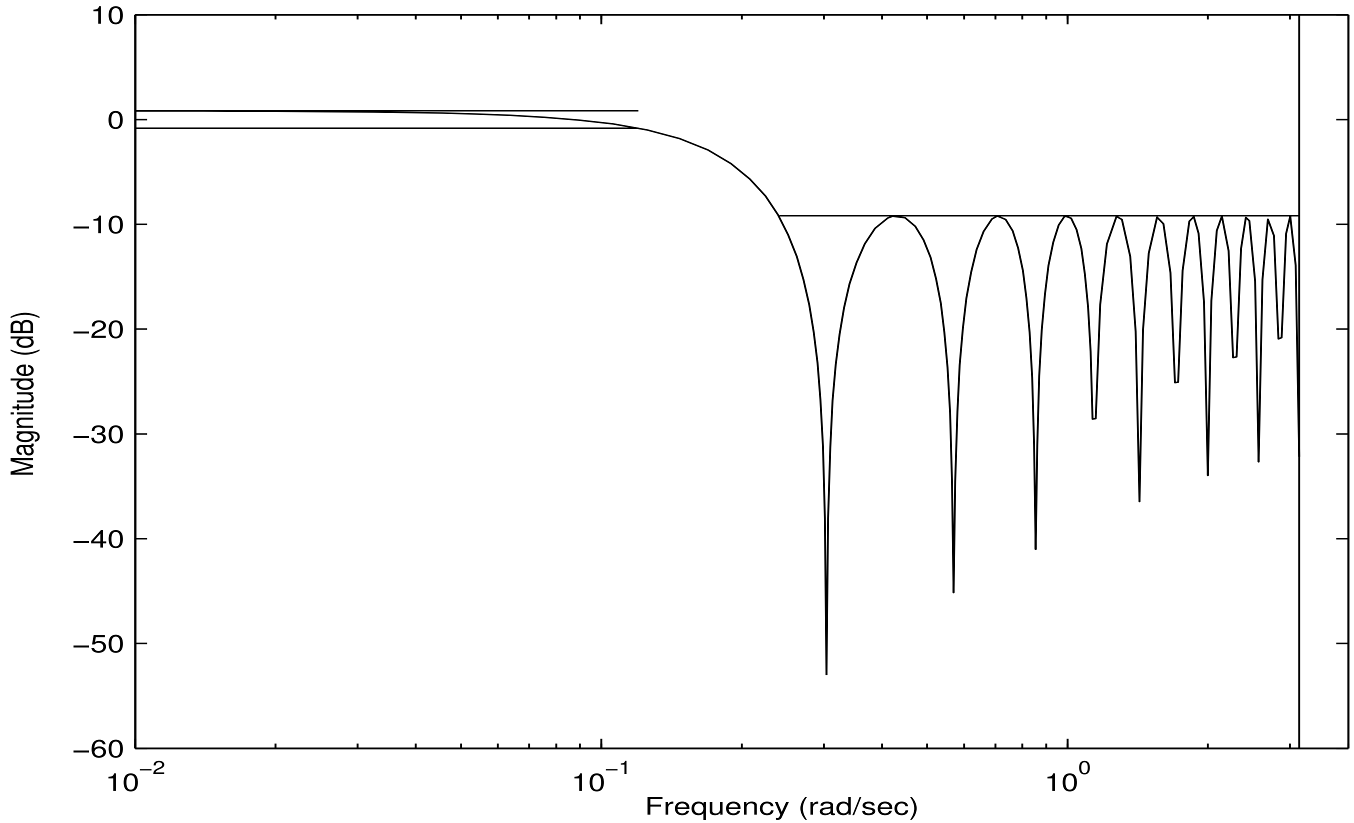
5. For a given order, if it exists, obtention of a filter satisfying upper and lower magnitude bounds

Related Problems

Similar approach can be applied to

1. Digital filter design (FIR,IIR) under magnitude and/or phase constraints
2. Lead-lag Network design satisfying phase constraints
3. Model reduction by Phase Matching on given frequency range
4. Model reduction by Magnitude Approximation on frequency range

Example of FIR design



Loop-shifting operation

$$G(j\omega) = \left(\frac{I}{\omega} \star \left[\begin{array}{c|c} -jA & -jB \\ \hline C & D \end{array} \right] \right)$$

Change of variable :

$$\forall \omega \in]\underline{\omega}\ \bar{\omega}[,\ \frac{I}{\omega} = \frac{I}{\underline{\omega}} + \delta I (I + \delta \bar{\omega})^{-1} (1 - \frac{\bar{\omega}}{\underline{\omega}}) = \delta I \star T_{\Omega}, \quad \delta \in]0,\ +\infty[$$

$$avec \qquad T_{\Omega} = \left[\begin{array}{c|c} -\overline{\omega} I & (1 - \frac{\overline{\omega}}{\underline{\omega}}) I \\ \hline I & (\frac{1}{\underline{\omega}}) I \end{array} \right]$$

$$\forall \omega \in]\underline{\omega}\ \bar{\omega}[, G(j\omega) = \left(\delta I \star T_{\Omega} \star \left[\begin{array}{c|c} -jA & -jB \\ \hline C & D \end{array} \right] \right)$$