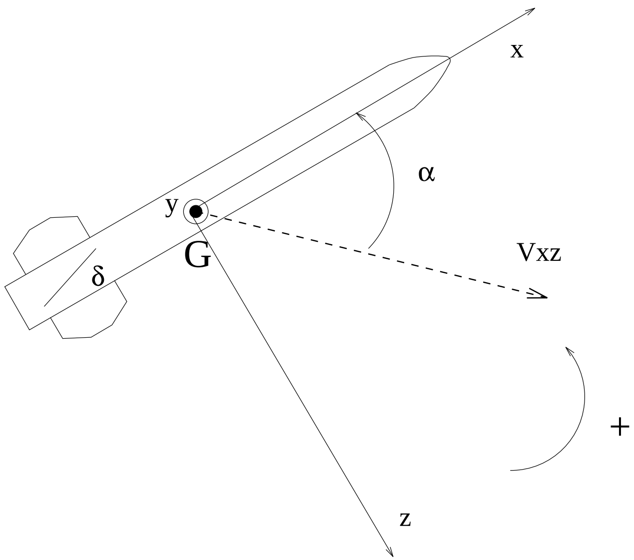
A New Tool For Analyzing Nonlinear Systems:

The Weighted Incremental Norm

Vincent Fromion
INRA - France
fromion@ensam.inra.fr

ILLUSTRATIVE EXAMPLES

Missile Control Problem



Highly Non Linear Model (Reichert)

$$\begin{aligned} \dot{\alpha} &= cos(\alpha) K_\alpha M C_n(\alpha, \delta) + q \\ \dot{q} &= K_q M^2 C_m(\alpha, \delta) \end{aligned}$$

with Uncertain Stability Derivatives

$$C_n(\alpha, \delta) = a_n \alpha^3 + b_n |\alpha| \alpha + c_n (2 - \tfrac{M}{3}) \alpha + d_n \delta$$
$$C_m(\alpha, \delta) = a_m \alpha^3 + b_m |\alpha| \alpha + c_m (-7 + \tfrac{8M}{3}) \alpha + d_m \delta$$

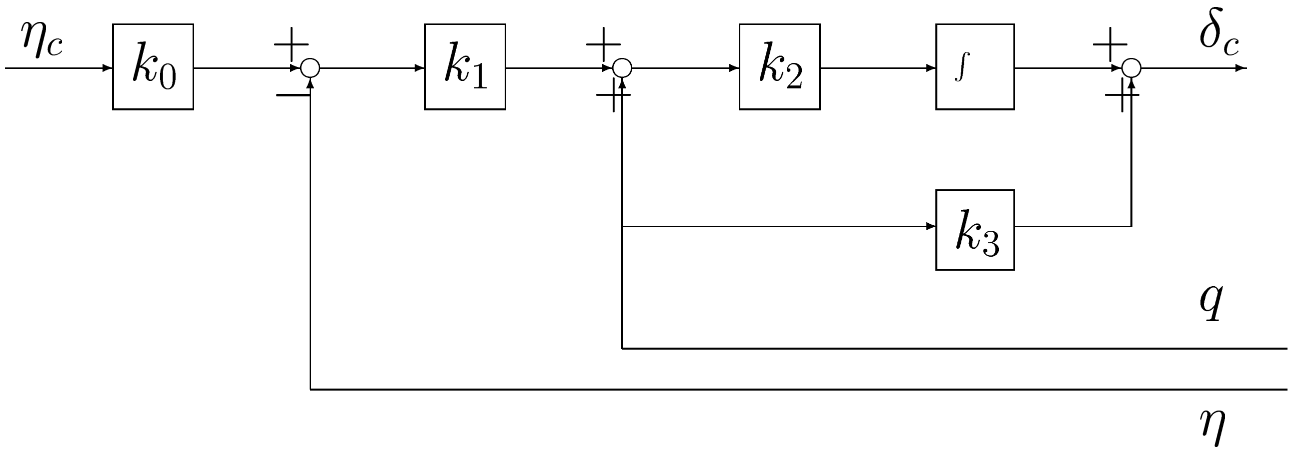
Output: $\eta = \frac{K_z}{grav} M^2 C_n(\alpha, \delta, M)$

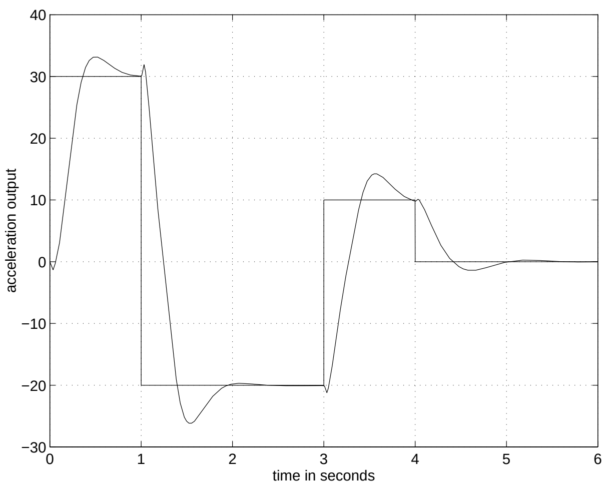
Actuator: $\delta(s) = \dfrac{\omega_a^2}{s^2 + 2\xi_a \omega_a s + \omega_a^2} \delta_c(s)$

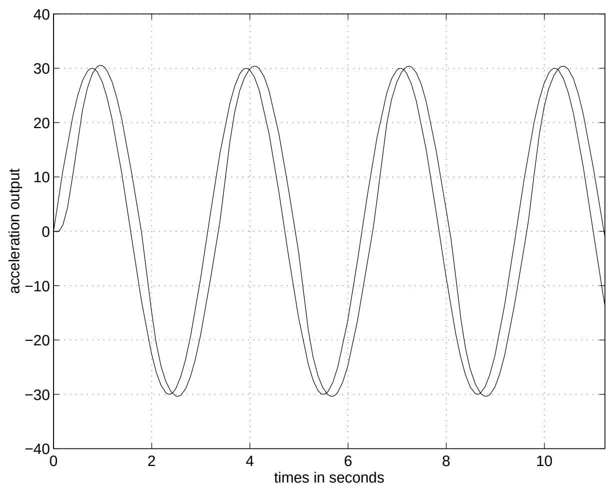
Control objectives

- step tracking, time constant ≤ 0.35 s, low steady state error, overshot $\leq 20\%$.
- sinusoid tracking

PI Controller: classical design







Nonlinear PI(D) controllers

Typical applications:

- Speed–up delayed systems
- Adaptation of the step tracking
- Nonlinear "filtering effects"

Methods

- Gain scheduling rules
- Fuzzy rules (fuzzy PID controllers).
- etc.

Control objectives

- step tracking, time constant, low steady state error, overshot.
- rejection of perturbation, sinusoid tracking.

NECESSITY OF A THEORETICAL FRAMEWORK

Two problems

1. How to reduce performance/robustness to a mathematical criterium ?

2. How to test the corresponding conditions in an efficient way ?

INCREMENTAL NORM AS A POSSIBLE FRAMEWORK

From linear control to nonlinear H_∞ control

Main fact: In linear context the weighted H_∞ approach

$\Rightarrow$ Is it possible to extend the weighted H_∞ approach to the nonlinear context?

Two possible solutions

1– $\mathcal{L}_2$ gain.

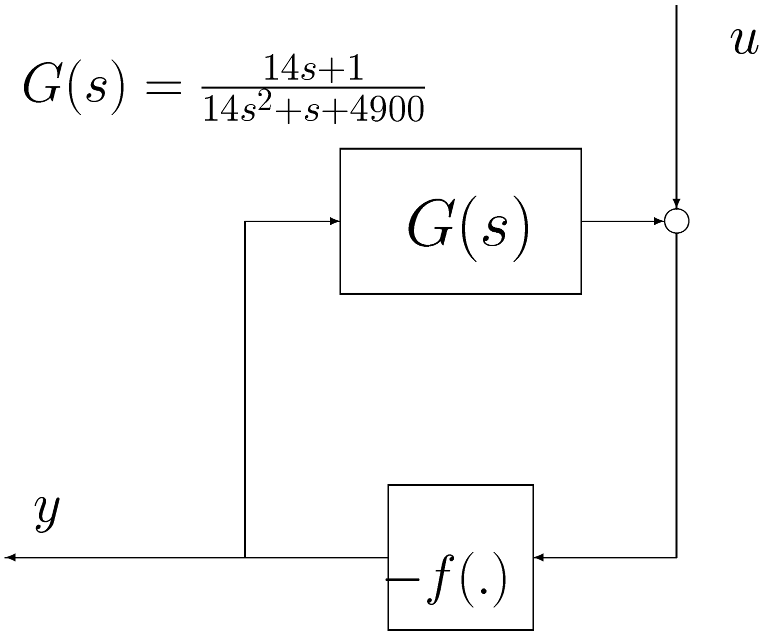
2– Incremental gain (Lipschitz constant).

$\Rightarrow$ Only the incremental framework allows to take into account performance requirements !

Limits of $\mathcal{L}_2$ gain stability

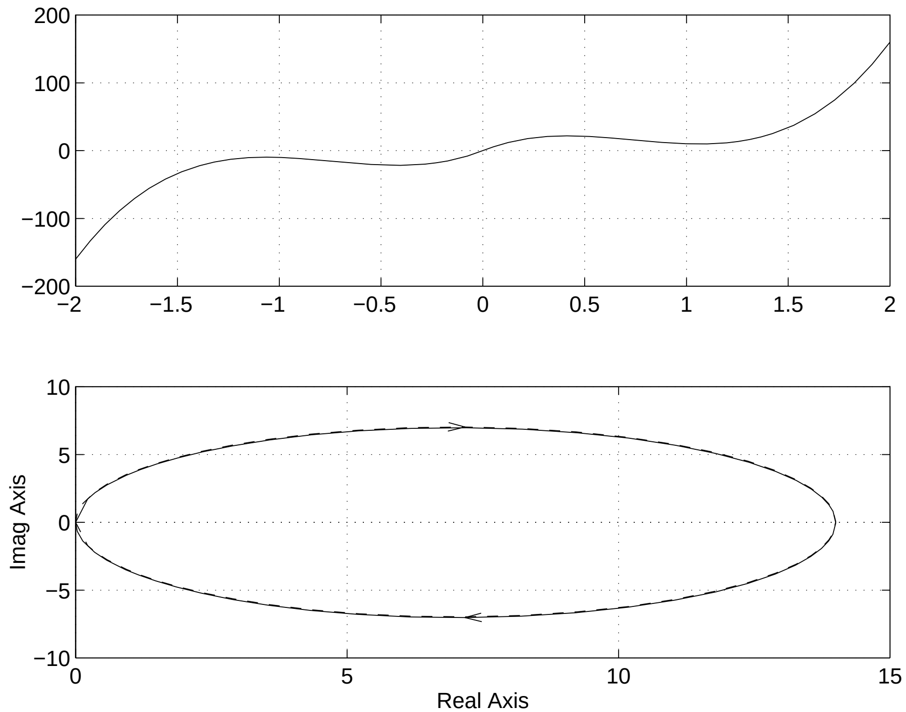
Considered system

$$G(s) = \frac{14s+1}{14s^2+s+4900}$$



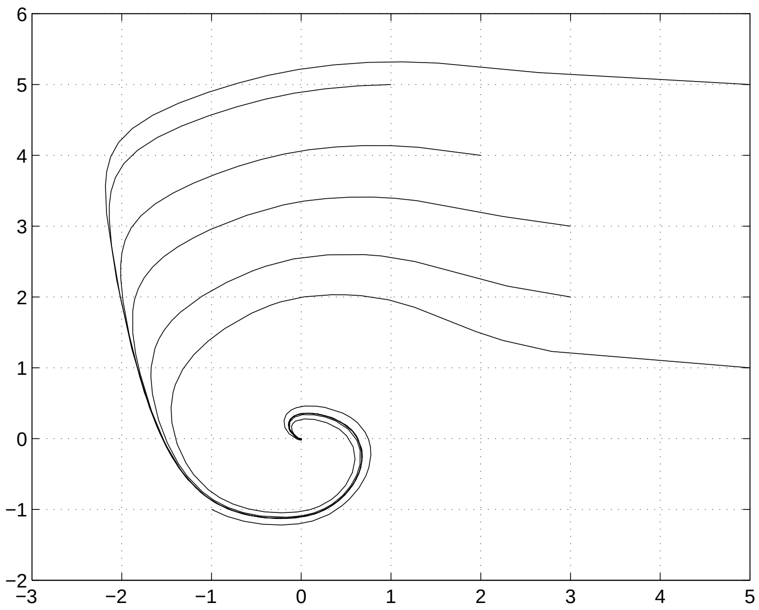
$$f(p) = 90p^3 - 200p|p| + 120p$$

Stability analysis via circle criterium

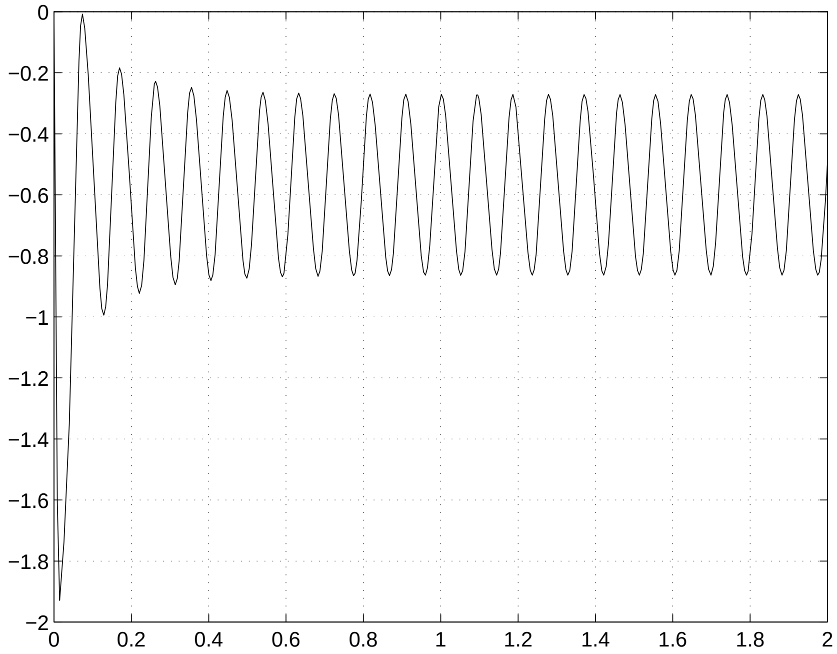


Limits of $\mathcal{L}_2$ gain stability (II)

Response to initial conditions



Step response



Incremental Norm: performance and robustness requirements

Incremental norm allows to take into account

Qualitative specifications

- Unique steady state properties
- Constant/periodic output asymptotically obtained from constant/periodic input
- Lyapunov stability of the unperturbed motions

Quantitative specifications

- Tracking error norm versus specific type of inputs
- Robustness with respect to uncertainties
- *Desensitivity* with respect to unmeasured perturbations and uncertainties

Considered systems

In the sequel, we consider systems with the differential representation

$$\Sigma \begin{cases} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \\ x(t_0) &= x_0 \end{cases} \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $y(t) \in \mathcal{R}^m$, and $u(t) \in \mathcal{R}^p$.
f and h are C^2, uniformly Lipschitz and such that $f(x_0, 0) = 0$ and $h(x_0, 0) = 0$.

Definition Σ is $\mathcal{L}_2$ gain stable if there exists $\gamma \geq 0$ such that for any $u \in \mathcal{L}_2$, one has $\|\Sigma(u)\|_2 \leq \gamma \|u\|_2$.

Definition Σ is incrementally stable on $\mathcal{L}_2$ if there exists $\eta \geq 0$ such that for any $u_1, u_2 \in \mathcal{L}_2$, one has $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta \|u_1 - u_2\|_2$.

CHARACTERIZATION OF INCREMENTAL STABILITY

Exactly testing incremental stability is a difficult problem

Let us characterize the incremental norm of this affine nonlinear system:

$$\begin{cases} \dot{x}(t) & = & f(x(t)) + g(x(t))u(t) \\ y(t) & = & h(x(t)) \\ x(t_0) & = & x_0 \end{cases} \qquad (2)$$

Theorem Let η a positive constant. The dynamical system (2) has a incremental gain less or equal to η if there exists a C^1 function, S defined from $\mathcal{R}^n \times \mathcal{R}^n$ into $\mathcal{R}$, which satisfies for all $x_1, x_2 \in \mathcal{R}^n$ the following conditions:

$$S(x_0, x_0) = 0$$

$$S(x_1, x_2) \geq 0$$

$$\frac{\partial S}{\partial x_1}(x_1, x_2) f(x_1) + \frac{\partial S}{\partial x_2}(x_1, x_2) f(x_2) + \frac{1}{4}\eta^{-2} \frac{\partial S}{\partial x_1}(x_1, x_2) g(x_1) g^T(x_1) \frac{\partial S^T}{\partial x_1}(x_1, x_2) + (h(x_1) - h(x_2))^T (h(x_1) - h(x_2)) \leq 0$$

$$\frac{\partial S}{\partial x_1}(x_1, x_2) g(x_1) + \frac{\partial S}{\partial x_2}(x_1, x_2) g(x_2) = 0$$

HOW TO TEST THE CORRESPONDING CONDITIONS IN AN EFFICIENT WAY ?

Quadratic incremental stability

$\Rightarrow$ Quadratic type solutions allows to obtain a good compromise between complexity and conservatism.

More generally, system (1) is said to be *quadratically incrementally stable* if there exist $P = P^T > 0$, $\epsilon > 0$, $\sigma_{f_u} > 0$ such that for any $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^p$, one has

$$(i) \quad P\,\frac{\partial f}{\partial x}(x,u) + \frac{\partial f}{\partial x}^T(x,u)\,P \leq -\epsilon I_n$$

$$(ii) \quad \left(\frac{\partial f}{\partial u}\right)^T \left(\frac{\partial f}{\partial u}\right) \leq \sigma_{f_u} I$$

which implies that there exits η_x such that for any $u_1, u_2 \in \mathcal{L}_2$, one has

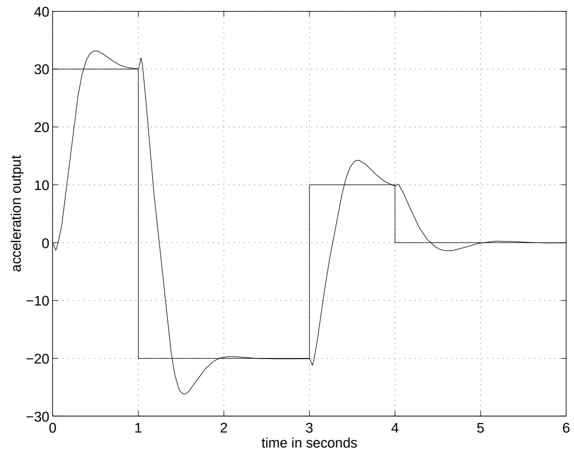
$$\|x_1 - x_2\|_2 \leq \eta_x \|u_1 - u_2\|_2.$$

Stronger notion but ...
(i) can be ensured by *Convex Optimization* over LMI constraints

QUALITATIVE PROPERTIES OF QUADRATICALLY INCREMENTALLY STABLE SYSTEMS

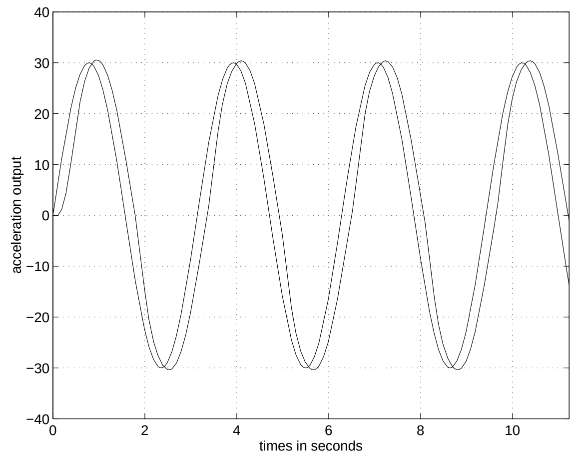
For a constant input...

$$x(t) \to \text{constant}$$



For T periodic input...

$$x(t) \to T \text{ periodic output}$$



+ Many properties with respect to initial condition and perturbations.

Testing incremental stability: LMI approaches

Condition to verify:

$$(i)\;\; P\,\frac{\partial f}{\partial x}(t,x,u) + \frac{\partial f}{\partial x}^T(t,x,u)\,P \le -\epsilon I_n$$

Interpretation: quadratic stability of: $\dot{\overline{\delta x}} = \frac{\partial f}{\partial x}(t,x,u)\delta x$

Two possible modeling:

1. Polytopic model

$$\left\{\frac{\partial f}{\partial x}(t,x,u)\right\} \subset \left\{F(\theta(t)) = \sum_{i=1}^{r} \lambda_i(t) A_i, \;\begin{cases} \Sigma_{i=1}^{r} \lambda_i(t) = 1 \\ \lambda_i(t) \ge 0 \end{cases}\right\}$$

2. so-called "Linear Fractional Transformation" model

$$\left\{\frac{\partial f}{\partial x}(t,x,u)\right\} \subset \left\{F(\theta(t)) = A + B_p\Delta(\theta(t))\left(I - D_{qp}\Delta(\theta(t))\right)C_q\right\}$$

with

$$\Delta(\theta(t)) = \mathbf{diag}\left(\theta_1(t)I_{n_1},\ldots,\theta_r(t)I_{n_r}\right), \quad \theta_i(t) \in [a_i,\, b_i]$$

ILLUSTRATIVE EXAMPLES

PI controlled missile incremental stability: circle criterion

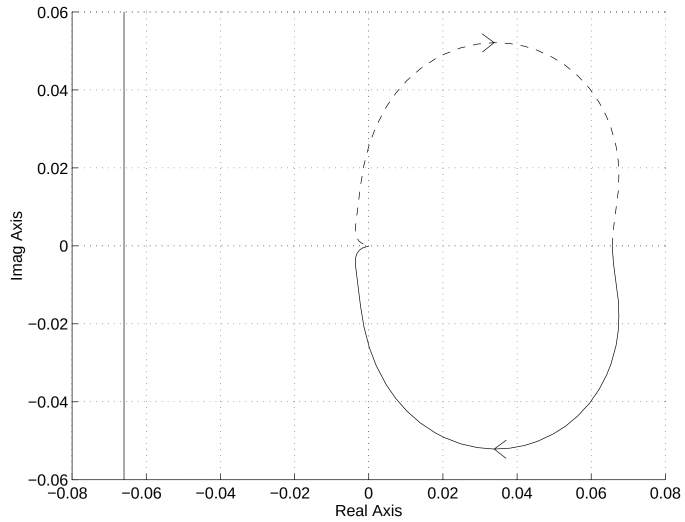
$h(s) = c_q(sI - A)^{-1}b_p$ connected with $k(t)$

$$A = \begin{bmatrix} K_\alpha M c_n(2 - \frac{M}{3}) & 1 & K_\alpha M d_n & 0 & 0 \\ K_q M^2 c_m(-7 + 8\frac{M}{3}) & 0 & K_q M^2 d_m & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -k_1 k_2 \frac{K_z}{g} M^2 c_n(2 - \frac{M}{3}) & k_2 & -k_1 k_2 \frac{K_z}{g} M^2 d_n & 0 & 0 \\ 0 & k_3 w_a^2 & -w_a^2 & -2\xi_a w_a & w_a^2 \end{bmatrix}$$

$$b_p = \begin{bmatrix} K_\alpha M \\ 2K_q M^2 \\ 0 \\ 0 \end{bmatrix} \quad c_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad k(t) \in [-15,\ 0]$$

Nyquist plot of $h(s)$

(Circle Criterion)



PI controlled missile incremental stability: an LMI approach

- Polytopic modeling

$$\frac{\partial f}{\partial x}(t,x,u) = A + k(t)bc = \lambda(t)A + (1-\lambda(t))\,(A - 15bc)$$

- Associated optimization problem: $P > 0$

$$A^T P + PA < 0 \text{and} (A - 15bc)^T P + P(A - 15bc) < 0$$

- Readily solved by `LMITOOL` :

$$P = \begin{bmatrix} 6.5479 & 0.0974 & -0.0927 & -0.0009 & -0.6899 \\ 0.0974 & 0.2843 & -0.3851 & -0.0033 & 0.6318 \\ -0.0927 & -0.3851 & 0.7955 & 0.0056 & -0.7863 \\ -0.0009 & -0.0033 & 0.0056 & 0.0001 & -0.0069 \\ -0.6899 & 0.6318 & -0.7863 & -0.0069 & 2.3722 \end{bmatrix}$$

Nonlinear PI(D) controller incremental stability: circle criterion

At the end of the talk !

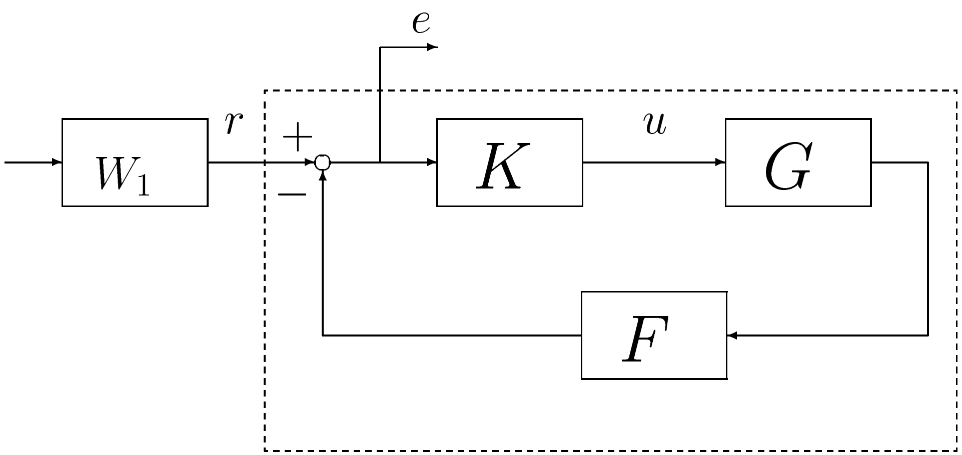
GUARANTEEING DESIGN SPECIFICATIONS: QUANTITATIVE ASPECTS

Quantitative analysis: from nominal performance...

Nominal performance = tracking of input signals in R_d^e for a nonlinear system

R_d^e defined by W_1 such that

$\|P_T W_1^{-1}(r)\|_2 \ll \|P_T r\|_2 \;\; \forall r \in R_d^e$



Tracking if $\exists T_0 \geq 0, \forall T \geq T_0$:

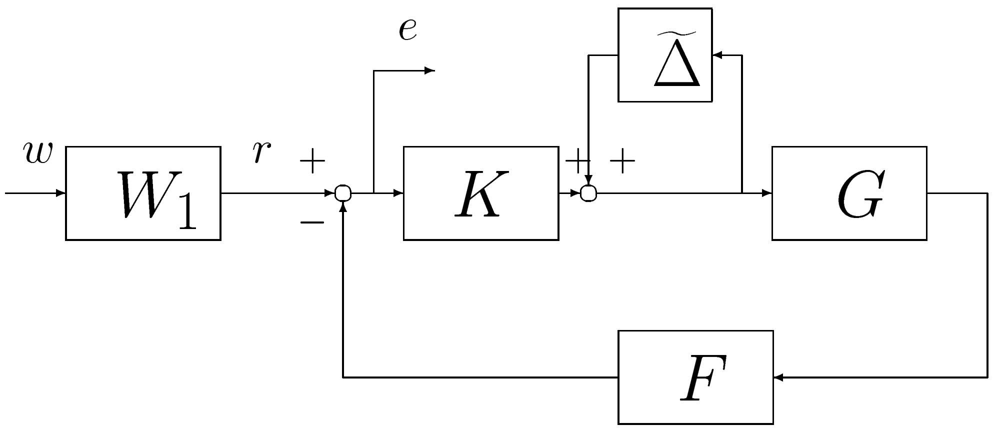
$\|P_T e\|_2 \ll \|P_T r\|_2 \;\; \forall r \in R_d^e$

The signal in R_d^e are tracked if

$$\|(I + FGK)^{-1} W_1\|_\Delta \leq 1$$

In the linear case, H_∞ performance is recovered.

Quantitative analysis: ...to robust performance



In the extension of H_∞ framework:

- Robust incremental stability analysis
- Robust incremental performance analysis

HOW TO (PRACTICALLY) CHECK THE PREVIOUS CONDITIONS?

Scaling and small gain type condition

Theorem The previous interconnected system is incrementally stable, for any Δ whose incremental norm is less than 1, if there exists a scaling $D(s)$ satisfying:

$$\text{for all } \omega, \qquad M_{qp}(j\omega)^* D(j\omega) M_{qp}(j\omega) - D(j\omega) < 0$$

with

$$D(\Delta) \triangleq \mathbf{diag}\,(d_1 I, \ldots, d_i I, \ldots, d_r I)$$

where the scalars $d_i = \lambda_i^2$ possibly depend on the frequency, if Δ_i is LTI

Multipliers and passive type condition

Theorem The previous interconnected system is incrementally stable, for any Δ which is incrementally passive, if there exists a multiplier $M_{mul} \in \mathcal{M}_{mul}(\Delta)$ satisfying:

$$\text{for all } \omega, \qquad M_{mul}(j\omega)M_{qp}(j\omega) + (M_{mul}(j\omega)M_{qp}(j\omega))^* < 0.$$

where

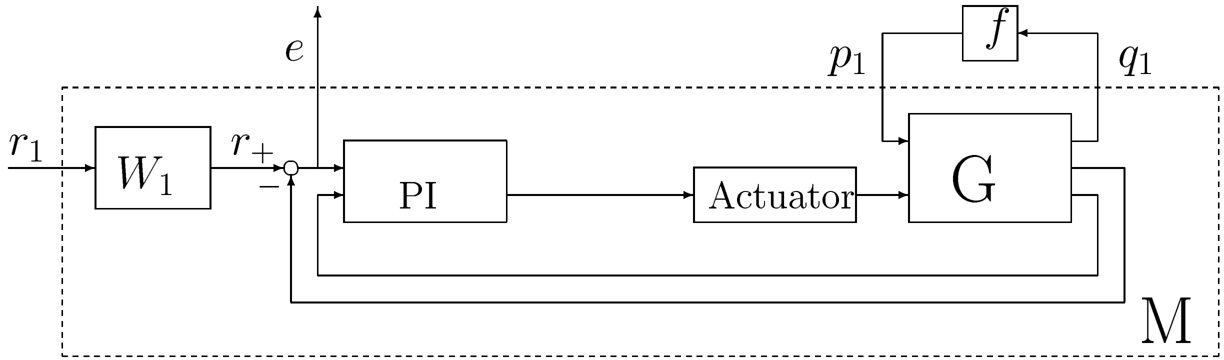
$$M_{mul} \triangleq \mathbf{diag}\,(m_1 I, \ldots, m_i I, \ldots, m_r I)$$

where m_i is possibly frequency dependent if Δ_i is LTI.

ILLUSTRATIVE EXAMPLES

PI controlled missile: nominal performance

Proved for $W_1(s) = 0.05\frac{s+2.5}{s+0.25}$ by convex optimization

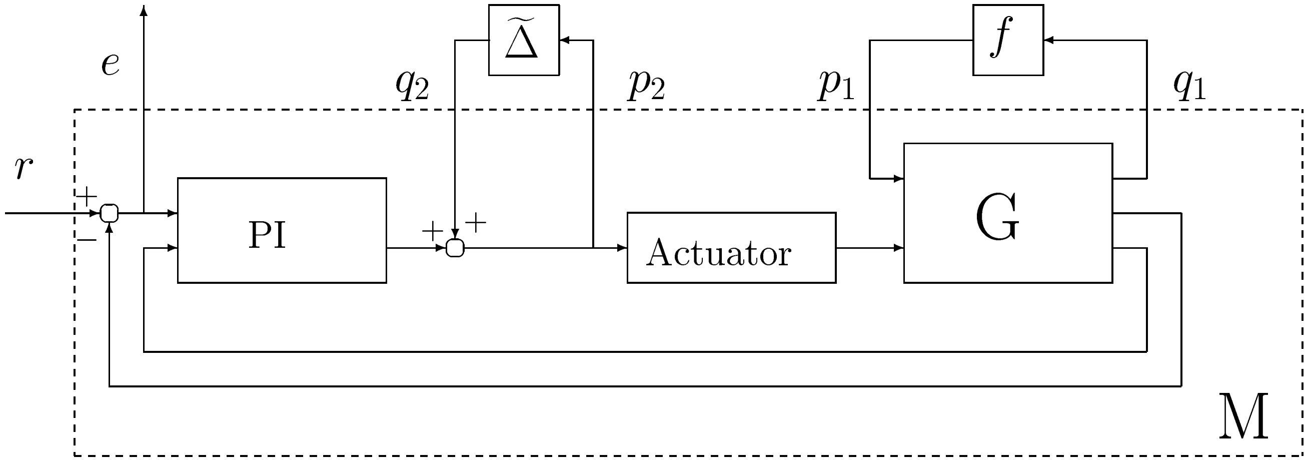


$$M \sim \begin{bmatrix} A_M & B_M \\ C_M & D_M \end{bmatrix}$$

Find P, $\beta > 0$ and $\lambda > 0$ such that:

$$\begin{bmatrix} A_M & B_M \\ C_M & D_M \\ 1 & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 & -15\lambda & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ \hline P & 0 & 0 & 0 & 0 & 0 \\ 0 & -15\lambda & 0 & 0 & -2\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta I \end{array}\right] \begin{bmatrix} A_M & B_M \\ C_M & D_M \\ 1 & 0 \\ 0 & I \end{bmatrix} < 0$$

PI controlled missile: Robust performance



Performance : $W_1(s) = 0.05\frac{s+2.5}{s+0.25}$

Robustness : 6dB input margin

Find $D(s) \triangleq \mathbf{diag}\,(d_1, d_2(s), d_3)$ such that

$$M(j\omega)^* D(j\omega) M(j\omega) - D(j\omega) < 0$$

Nonlinear PI Controller: an example