CONNECTING

NONLINEAR INCREMENTAL LYAPUNOV STABILITY

WITH

THE LINEARIZATIONS LYAPUNOV STABILITY

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Outline

The main goals of the paper are to

- clarify links between *Incremental Lyapunov Stability* and Linearizations Lyapunov stability
- use *Weighted incremental framework* for handling Incremental Lyapunov stability

INCREMENTAL WEIGHTED FRAMEWORK:

BACKGROUND

$\mathcal{L}_\infty$ and its associated extended space $\mathcal{L}_\infty^e$

The space of almost everywhere bounded functions:

$$\mathcal{L}_\infty \triangleq \{f : [t_0, \infty) \to \mathbb{R}^n | \|f\|_{\mathcal{L}_\infty} < \infty\}$$

where

$$\|f\|_{\mathcal{L}_\infty} \triangleq \text{esssup}_{t \in [t_0, \infty)} \|f(t)\|.$$

The extended space associated to $\mathcal{L}_\infty$ defined by

$$\mathcal{L}_\infty^e \triangleq \{f : [t_0, \infty) \to \mathcal{R}^n | \forall T \in [t_0, \infty),\ \|f\|_{\mathcal{L}_\infty, T} < \infty\}$$

with $\|f\|_{\mathcal{L}_\infty, T} \triangleq \|P_T f\|_{\mathcal{L}_\infty}$ and where P_T is the *causal truncation* which is given by $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise.

Considered systems

$y = \Sigma_{t_0}(\xi)$ is defined from $\mathcal{W}$, an open (not empty) set of $\mathbb{R}^n$, into $\mathcal{L}_\infty^e$ and is given

$$y = \Sigma_{t_0}(\xi) \begin{cases} \dot{x}(t) &= f(t, x(t)) \\ y(t) &= x(t) \\ x(t_0) &= \xi \end{cases}$$

where $x(t) \in \mathbb{R}^n$. Σ_{t_0} is assumed well-defined from $\mathcal{W}$ into $\mathcal{L}_\infty^e$, that is, for any $T \in [t_0, \infty)$ and any $\xi \in \mathcal{W}$, the differential equation solution exists on $[t_0, T]$.

Assumption

f and $\dfrac{\partial f}{\partial x}$ are continuous functions of x uniformly for almost every $t \in [t_0, \infty)$ and are measurable functions on $[t_0, \infty)$ for every fixed value of $x \in \mathbb{R}^n$.

Gâteaux derivative on $\mathcal{L}^e_\infty$

Definition

If, for any $T \in [t_0, \infty)$ and for any $\nu \in \mathbb{R}^n$, there exists a continuous linear operator $D\Sigma_{t_0 G}[\xi]$ from $\mathbb{R}^n$ into $\mathcal{L}^e_\infty$ such that

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma_{t_0}(\xi + \lambda\nu) - \Sigma_{t_0}(\xi)}{\lambda} - D\Sigma_{t_0 G}[\xi](\nu) \right\|_{\mathcal{L}_\infty, T} = 0,$$

then

$D\Sigma_{t_0 G}[\xi](\nu)$ is the *Gâteaux derivative* (the *linearization*) of Σ_{t_0} at ξ on $\mathcal{L}^e_\infty$.

Gâteaux derivative of Σ_{t_0}

Proposition

For any $\xi \in \mathcal{W}$, Σ_{t_0} has a Gâteaux derivative that satisfies the following differential equations:

$$\bar{y} = D\Sigma_{t_0 G}[\xi](\nu) \begin{cases} \dot{\bar{x}}(t) &= A(t)\bar{x}(t) \\ \bar{y}(t) &= \bar{x}(t) \\ \bar{x}(t_0) &= \nu \end{cases}$$

with $A(t) = \dfrac{\partial f}{\partial x}(t, x(t))$ and $x(t)$ the state-trajectory of $\Sigma_{t_0}(\xi)$.

Mean value Theorem in norm

In the sequel $\mathcal{U}$ is a convex and closed subset of $\mathcal{W}$.

Theorem

For any $T \in [t_0, \infty)$, there exists $\eta_T > 0$ such that for any $\xi_1, \xi_2 \in \mathcal{U}$,

$$\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_2)\|_{\mathcal{L}_\infty, T} \leq \eta_T \|\xi_1 - \xi_2\|$$

if and only if for any $\xi \in \mathcal{U}$ and any $\nu \in \mathbb{R}^n$, we have

$$\|D\Sigma_{t_0 G}[\xi](\nu)\|_{\mathcal{L}_\infty, T} \leq \eta_T \|\nu\|.$$

From extended to non extended spaces

Proposition

Let $\eta > 0$ then

$$\|D\Sigma_{t_0 G}[\xi](\nu)\|_{\mathcal{L}_\infty, T} \leq \eta \|\nu\|$$

for any $T \in [t_0, \infty)$, any $\xi \in \mathcal{U}$ and any $\nu \in \mathbb{R}^n$ if and only if

$$\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_2)\|_{\mathcal{L}_\infty} \leq \eta \|\xi_1 - \xi_2\|$$

any $\xi_1, \xi_2 \in \mathcal{U}$.

INCREMENTAL ASYMPTOTIC STABILITY

AND

LINEARIZATIONS ASYMPTOTIC STABILITY

Incrementally Lyapunov stability definition

Definition

Σ_{t_0} is said to be *incrementally asymptotically Lyapunov stable on* $\mathcal{U}$ if there exists a class $\mathcal{KL}$ function β_{t_0} such that

$$\|\Sigma_{t_0}(\xi_1)(t) - \Sigma_{t_0}(\xi_2)(t)\| \leq \beta_{t_0}(t, \|\xi_1 - \xi_2\|)$$

for any $t \geq t_0$, any ξ_1, ξ_2 in $\mathcal{U}$.

Σ_{t_0} is said to be *incrementally exponentially Lyapunov stable on* $\mathcal{U}$ if there exists $a > 0$ and $b > 0$ such that

$$\|\Sigma_{t_0}(\xi_2)(t) - \Sigma_{t_0}(\xi_2)(t)\| \leq be^{-a(t-t_0)}\|\xi_2 - \xi_1\|$$

for any $t \geq t_0$ and any $\xi_1, \xi_2 \in \mathcal{U}$.

Strong asymptotical Lyapunov stable linearizations

Definition

The linearizations of Σ_{t_0} are said to be

– *Strongly asymptotically Lyapunov stable on* $\mathcal{U}$ if there exist a class $\mathcal{L}$ function σ_{t_0} and $b > 0$ such that

$$\|D\Sigma_{t_0 G}[\xi](\nu)(t)\| \leq b\|\nu\|\sigma_{t_0}(t - t_0)$$

for any $t \geq t_0$, any $\xi \in \mathcal{U}$ and any $\nu \in \mathbb{R}^n$.

– *Strongly exponentially stable on* $\mathcal{U}$ if there exists $a > 0$ and $b > 0$ such that

$$\|D\Sigma_{t_0 G}[\xi](\nu)(t)\| \leq be^{-a(t-t_0)}\|\nu\|$$

for any $t \geq t_0$ and any $\xi \in \mathcal{U}$.

Strongly Lyap. linearizations ⇒ Incremental Lyap. stability

Proposition

Σ_{t_0} is incrementally asymptotically Lyapunov stable on $\mathcal{U}$ if the linearizations of Σ_{t_0} are strongly asymptotically Lyapunov stable on $\mathcal{U}$.

Sketch of proof:

It is a routine to show that if the linearizations of Σ_{t_0} are strongly assymptotically Lyapunov stable then there exists a class $\mathcal{L}$ function σ_{t_0} and $b > 0$ such that

$$\|\sigma_{t_0}^{-1} D\Sigma_{t_0 G}[\xi](\nu)\|_\infty \leq b\|\nu\|$$

for $\xi \in \mathcal{U}$ and any $\nu \in \mathbb{R}^n$. The mean value theorem reveals that

$$\|\sigma_{t_0}^{-1} \left[\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_1)\right] \|_{\infty,T} \leq b\|\xi_1 - \xi_2\|.$$

We thus concluse that for almost all time, we have

$$\|\sigma_{t_0}(t - t_0)^{-1} \left[\Sigma_{t_0}(\xi_1)(t) - \Sigma_{t_0}(\xi_1)(t)\right] \| \leq b\|\xi_1 - \xi_2\|$$

and the announced result.

Strongly exp. linearizations ⇔ Incremental exp. Lyap. stability

Proposition

Σ_{t_0} is incrementally exponentially stable on $\mathcal{U}$ if and only if the linearizations of Σ_{t_0} are strongly exponentially stable on $\mathcal{U}$.

Sketch of proof :
Sufficienty: previous proposition.

Necessity: a consequence of the following result:

Lemma If there exist φ a C^1 class $\mathcal{K}$ function and σ a class $\mathcal{L}$ function such that for ξ_1, $\xi_2 \in \mathcal{U}$, any $t \in [t_0, \infty)$, we have

$$\|\Sigma_{t_0}(\xi_1)(t) - \Sigma_{t_0}(\xi_2)(t)\| \leq \varphi(\|\xi_1 - \xi_2\|)\sigma(t - t_0)$$

then we have

$$\|\sigma^{-1} D\Sigma_{t_0G}[\xi](\nu)\|_{\infty,T} \leq \varphi'(0)\|\nu\|$$

for any $T \in [t_0, \infty)$, any $\xi \in [\xi_1, \xi_2]$ and any $\nu \in \mathbb{R}^n$.

Incremental exponential stability versus exponential stability

Proposition

The following properties are equivalent:

(i) Σ_{t_0} is incrementally exponentially Lyapunov stable on $\mathcal{U}$;

(ii) Σ_{t_0} is exponentially Lyapunov stable on $\mathcal{U}$ at any $\xi \in \mathcal{U}$;

(iii) Σ_{t_0} is exponentially Lyapunov stable on $\mathcal{U}$ at a $\xi_0 \in \mathcal{U}$.

$\Rightarrow$ Related to a result given in Paper of D. Angeli (IEEE TAC 2002).

Sketch of proof :

$(i) \Rightarrow (ii)$ obvious. $(ii) \Rightarrow (iii)$ obvious.
$(iii) \Rightarrow (i)$: If Σ_{t_0} is exponentially Lyapunov stable on $\mathcal{U}$ at ξ_0 then there exist $a > 0$, $b > 0$ such that for any $\xi_{0p} \in \mathcal{U}$, we have

$$\|\Sigma_{t_0}(\xi_{0p}) - \Sigma_{t_0}(\xi_0)\| \leq be^{-a(t-t_0)}\|\xi_{0p} - \xi_0\|.$$

We thus deduce by the mean value theorem in norm and the convexity of $\mathcal{U}$ that $\|D\Sigma_{t_0G}[\xi](\nu)(t)\| \leq be^{-a(t-t_0)}\|\nu\|$ for any $\xi \in \mathcal{U}$ and thus the announced result.

CONNECTION WITH THE LENGTH APPROACH

AND

THE RELATED CONTRACTION ANALYSIS

The length of a curve and Tonelli's Theorem

The length approach and related results: D. C. Lewis (49), Z. Opial.(60), P. Hartman.(61), ..., W. Lohmiller and J. J. Slotine(98).

$\Rightarrow$ Characterize the time evolution of the length between two extreme trajectories *i.e.* for any fixed $t \in [t_0, \infty)$, we define

$$c_t(\alpha) \triangleq \Sigma_{t_0}(\xi_1 + \alpha(\xi_2 - \xi_1))(t)$$

with $\alpha \in [0,1]$ and $\xi_1, \xi_2 \in \mathcal{U}$. We thus compute the length between $\Sigma_{t_0}(\xi_1)(t)$ and $\Sigma_{t_0}(\xi_1)(t)$:

$$\begin{aligned} L(t) &\triangleq \int_0^1 \sqrt{\frac{dc_{t,1}}{d\alpha}^2(\alpha) + \cdots + \frac{dc_{t,n}}{d\alpha}^2(\alpha)} d\alpha \\ &= \int_0^1 \|D\Sigma_{t_0 G}[\xi_1 + \alpha(\xi_2 - \xi_1)](\xi_2 - \xi_1)(t)\| d\alpha. \end{aligned}$$

(Tonelli's Theorem)

Mean Value theorem vs lenght between two extreme trajectories

Proposition

For any $T \in [t_0, \infty)$, there exists $\eta_T > 0$ such that for any $\xi_1, \xi_2 \in \mathcal{U}$, we have

$$\sup_{t \in [t_0, T]} L(t) \triangleq \|L\|_{\mathcal{L}_\infty, T} \leq \eta_T \|\xi_2 - \xi_1\|$$

if and only if for any $\xi \in \mathcal{U}$ and any $\nu \in \mathbb{R}^n$, we have

$$\|D\Sigma_{t_0 G}[\xi](\nu)\|_{\mathcal{L}_\infty, T} \leq \eta_T \|\nu\|.$$

Moreover, since the length of a curve $c_t(\alpha)$ is necessarily greater than the length of the straight line between $c_t(0)$ and $c_t(1)$, we have

$$\boxed{\|\Sigma_{t_0}(\xi_1) - \Sigma_{t_0}(\xi_2)\|_{\mathcal{L}_\infty, T} \leq \|L\|_{\mathcal{L}_\infty} \leq \eta_T \|\xi_2 - \xi_1\|.}$$

$\Rightarrow$ Length approach and Mean value theorem approach are strongly related!

Conclusion

- Incremental exponential stability is equivalent to the exponential stability
- The length approach and the Mean Value Theorem approach are strongly related !

  ⇒ allows to clarify and obtain some insights about incremental Lyapunov stability, *contraction analysis* and infinitesimal type conditions (see paper).

We finally point out that incremental like properties have to consider the behaviors of nonlinear systems for input signals in order to lead non obvious properties.

⇒ Weighted incremental approach