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Filter design under magnitude constraints is a finite dimensional convex optimization problem

L. Rossignol[△], G. Scorletti[△], V. Fromion[‡]

△ LAP, ISMRA, 6 bd du Maréchal Juin, F14050 Caen cedex, France. {scorletti@greyc.ismra.fr}

‡ LASB, INRA–Montpellier, 2 place Viala, 34060 Montpellier, France. {fromion@ensam.inra.fr}

Abstract

We consider the design of filters satisfying upper and lower bounds on the frequency response magnitude, in the continuous and discrete time domains. The paper contribution is to prove that such a problem is equivalent to a *finite* dimensional *convex* optimization program involving Linear Matrix Inequality contraints. At now, such optimization problems can be efficiently solved. Note that this filter design problem is usually reduced to a semi infinite dimensional Linear Programming optimization problem under the additional assumption that the filter poles are fixed (for instance, when considering FIR design). Furthermore, the semi infinite dimensional optimization is practically solved, using a gridding approach on the frequency. In addition to be finite dimensional, our formulation allows to set or not the filter poles. Numerical applications emphasize the interest of the proposed results.

1 Introduction

Problem interest The design of filters, in the continuous time or in the discrete time is an important issue in signal processing (see *e.g.* [RG75a, GPR99]). The design specifications are formulated in the frequency domain. These problems were largely considered and at now some of them are still open.

Nevertheless, in this context, the filter design is of interest in other fields of engineering sciences, for instance, in automatic control. During the last twenty years, dramatic advances were accomplished in the design of control laws using the frequency domain approach. A new field, referred to as Robust control or H_∞ control [Zam81, DGKF89, ZDG95] is strongly developing. In these approaches, the specifications of the control law are first translated as constraints on the closed-loop transfer function magnitude in the frequency domain [Zam81]. These constraints are defined by the use of filters whose magnitude constraints the closed-loop transfer function magnitude. These filters are referred to as weighting functions. At now, these approaches were successfully applied to industrial control problems (see e.g. [FD94, FFDM96]) and, in some aspects, extended to the case of non linear systems [Fro95, FSF99].

Nevertheless, the practical design of the weighting transfer functions was hardly considered. To our best knowledge, the few existing approaches [Fon95] considers filters of specific structure. As a consequence, a general tool, allowing an easy translation of the control design specifications in weighting transfer functions is missing. Such a tool will allow a wider application of these control design methods. In addition to respect arbitrary upper and lower bounds on the frequency response magnitude, the weighting function order is required to be small. Actually, the H_∞ control law order equals the weighting function orders plus the plant order. As a low order control law is desirable, low order weighting functions are of interest. Note that the weighting function phase is not constrained.

Review of some previous filter design methods A first classical approach of the design of analog filter satisfying magnitude bounds is based on prototype filters such as Chebyshev filters, Butterworth filters, elliptic filters, etc [RG75a] for achieving given standard types (lowpass, highpass, etc...). One the one hand, note that the obtained filter order is high and that the underlying formulation is not flexible. For instance, in this solution is not adapted to the H_∞ weighting transfer function design. On the other hand, the obtained solution is not optimal. Note that this approach, by the use of the bilinear transformation [RG75a, DV75] allows to design digital filters.

More recently, a lot of work was dedicated to the design of Finite Impulse Response (FIR) filters, especially in the case of linear phase FIR filters. Note that FIR filter design can be interpreted as the design of a filter whose poles are fixed. For instance, using the Chebychev approximation, several algorithms were developed whose most famous is the Remes exchange algorithm (see *e.g.* [PC72, PC74, RG75a]). Note that these approaches heavily rely on the linear phase constraint of the designed FIR. Furthermore, the general problem of designing a FIR filter satisfying arbitrary upper and lower bounds on its magnitude is not solved in a clear way.

An alternative approach to FIR filter design is to recast the problem as an linear programming optimization problem. Such a formulation is much more flexible than the previous one. In [SPK92], this approach was applied in order to design linear phase FIR filters. Nevertheless, note that the underlying optimization problem is semi infinite (so called SIP, Semi Infinite Problem) [GPR99]: in [SPK92], this problem was solved in an approximative way, using a frequency gridding in order to obtain a finite dimensional optimization problem. The advantage is that there exist efficient algorithms solving LP optimization problem. But, in general, there is no guarantee that the semi infinite constraints are satisfied between frequency samples. The purpose of this paper is to show that magnitude filter design problem can be equivalently formulated as a convex finite optimisation problem.

The case of the FIR design satisfying arbitrary magnitude bounds, without the linear phase constraint, using optimization, was considered in [GPR99, WBV96]. Note that, in the context of H_∞ weighting transfer function design, the linear phase overconstraints the problem. In [WBV96, Wu97], this design problem is formulated as a semi infinite optimization problem. Some of the constraints can be equivalently replaced by finite dimensional constraints. Actually, their approach is based on a (partial) gridding of the convex optimization problem leading to a large-scale optimization problem involving Linear Matrix Inequality[1] contraints. As a matter of fact, if at now large scale Linear Programming problems can be efficiently solved, it is not the case for LMI programming problems. Furthermore, under strong assumptions on the magnitude bounds, the whole problem can be equivalently formulated as a finite dimensional optimization [Wu97]. As a consequence, the proposed solution is not satisfying enough with respect to our objectives.

In this paper, we actually prove that the filter design problem can be formulated as a finite dimensional optimization without these strong assumptions. Our result holds in the case of FIR filters and in the case of IIR filters. Using IIR (Infinite Impulse Response) filter rather that FIR filters generally allows to obtain filter satisfying a given magnitude specification with a less important order. Thus, for instance, FIR filters are not adapted for the H_∞ weighting transfer function design.

Proposed approach and contributions Our approach is closely related to the Wu's one [WBV96] but bypasses her disadvantages. Actually, under mild assumptions on the arbitrary lower and upper bounds on the filter magnitude, we prove that the filter design is a finite dimensional convex optimization problem. This result still holds when the FIR filter design is considered. Furthermore, this proof is constructive and of practical

[1]Linear Matrix Inequalities are briefly defined in the sequel.

interest, that is, we propose a finite dimensional convex optimization problem involving Linear Matrix Inequality constraints. An LMI constraint is a matrix inequality of the form $F(\xi) \triangleq F_0 + \sum_{i=1}^{m} \xi_i F_i > 0$, where $\xi \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n \times n}$, $i = 0, \ldots, m$ are given symmetric matrices. Several algorithms that have practical efficiency have been devised for solving convex problems involving these constraints see *e.g.* [VB96]. The LMI problems can be solved using the free ware code SP [VB96] and its Matlab/Scilab interface `LMITOOL` [END95], or the available commercial Matlab Toolbox, LMI Control Toolbox [GNLC95].

In fact, the Wu and al results are derived from an interesting application of the positive real lemma [Pop73] in the discrete time domain. In contrast, our approach is based on an extension of the Kalman Yakubovitch Popov lemma [Pop73] which is strongly related to the LMI based μ analysis upper bound [FTD91, LSC94].

Our approach is mainly presented for the continuous time (that is, analog) filter design which is of strong interest *e.g.* in the case of H_∞ weighting function design. Of course, in the field of signal processing, a more important problem is the design of discrete time (that is, digital) filters.

Nevertheless, using our approach, a direct solution can be proposed for the digital filter design. In fact, the solution is a little bit more elegant in the continuous time case than in the discrete time case. In fact, when considering continuous time filters, the corresponding transfer function is naturally a Linear Fractional Transformation (LFT)[2] in ω (or $\frac{1}{\omega}$), that is in this special case, a rational function of ω (or $\frac{1}{\omega}$). The analysis of the so-called LFT system was intensively investigated in the field of robust control (see *e.g.* [ZDG95, LZD96, SG98]. In contrast with the continuous time case, discrete time transfer functions can not be directly written as an LFT (that is a rational function) of ω (or $\frac{1}{\omega}$). This straightforward remark is in fact the fundamental motivation of our choice.

Nevertheless, using the classical bilinear transformation (see *e.g.* [DV75, RG75b]) between both continuous time formulation and discrete time formulation, our presented approach allows to design discrete time filters: an application to the FIR filter design is developed in section 5.2.

Paper outline. The paper is organized as follows. The problem under consideration is presented in the section 2. For the sake of clearness, a simplified magnitude constraint filter design is considered. In section 3, a simplified version of the main result is presented: this problem can be recast as a finite dimensional convex optimization problem involving LMI constraints. The general case is then considered in section 4. Numerical applications are developed in the section 5: design of an H_∞ weighting function and design of a FIR filter. Finally, some concluding remarks are made about the approach.

[2]LFT is formally defined in the sequel.

Notations I_r and 0_r denotes the identity matrix and the zero matrices of $\mathbf{R}^{r\times r}$, with I_0 (or 0_0) empty. The subscript is omitted when it is evident from the context.
For a real symmetric (complex hermitian) matrix P, $P > 0$ denotes that P is definite positive.

We now introduce the notation currently used in μ analysis, but not so used in other fields. This notation allows to dramatically simplify problem formulation.

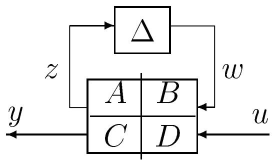

Figure 1: Linear Fractional Transformation

Linear Fractional Transformation A rational function in δ can be expressed as a Linear Fractional Transformation (LFT), that is, as $D + C\Delta(I - A\Delta)^{-1}B$ with $\Delta = \delta I$ and A, B, C, D are matrices of convenient size. An LFT can be denoted:

$$\Delta \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where $\star$ denote the Redheffer star product which is defined in the sequel [ZDG95]. Note that, in the case of a continuous time transfer function (that is a function which is rational in $\frac{1}{s}$), the LFT boils down to the usual state space representation (A, B, C, D):

$$D + C(sI - A)^{-1}B \;=\; \tfrac{1}{s}I \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

A similar remark can be made in the case of discrete time transfer function:

$$D + C(zI - A)^{-1}B \;=\; z^{-1}I \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Redheffer Star Product The Redheffer star product can be defined as

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \star \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{c|c} A_1 + B_1A_2(I - D_1A_2)^{-1}C_1 & B_1(I - A_2D_1)^{-1}B_2 \\ \hline C_2(I - D_1A_2)^{-1}C_1 & D_2 + C_2D_1(I - A_2D_1)^{-1}B_2 \end{array}\right]$$

This operation is well-defined if $(I - D_1A_2)$ and $(I - A_2D_1)$ are invertible.

LFT Product

$$\Delta \star \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \times \Delta \star \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \star \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$$

LFT stacking

$$\begin{bmatrix} \Delta \star \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \\ \\ \Delta \star \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \star \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{array}\right]$$

2 Problem statement

Our purpose is to design a fixed order filter whose magnitude is upper (respectively lower) bounded on different N_u (resp. N_l) frequency sub open intervals by different frequency dependent rational functions.

Let us denote the considered N_u open sub interval as $\Omega_1^u, \cdots, \Omega_{N_u}^u$. Let us introduce the positive frequency dependent functions $U_i(\omega)$ which are defined on the sub interval Ω_i^u. In the same way, let us denote the N_l open sub interval as $\Omega_1^l, \cdots, \Omega_{N_l}^l$. Let us introduce the positive frequency dependent functions $L_j(\omega)$ defined on the sub interval Ω_j^l. Furthermore, for all i and j such that $\Omega_i^u \bigcap \Omega_j^l \neq \emptyset$, for all $\omega \in \Omega_i^u \bigcap \Omega_j^l$, $L_j(\omega) < U_i(\omega)$, that is, lower and upper constraints are consistent.

In the sequel, the upper and lower bound functions $L_j(\omega)$ and $U_i(\omega)$ are assumed to be rational (including polynomial functions). This assumption is rather mild due to the special interest of this class of functions in the approximation theory. The important interest of rational functions is that they can be written as a Linear Fractional Transformation [ZDG95] of the variable ω or of the variable $\frac{1}{\omega}$, that is, there exist matrices A_i^u, B_i^u, C_i^u, D_i^u, A_j^l, B_j^l, C_j^l et D_j^l such that:

$$\begin{cases} \forall \omega \in \Omega_i^u, \quad U_i(\omega) = D_i^u + C_i^u \Delta_i(\omega)(I - A_i^u \Delta_i(\omega))^{-1} B_i^u \\ \\ \forall \omega \in \Omega_j^l, \quad L_j(\omega) = D_j^l + C_j^l \Delta_j(\omega)(I - A_j^l \Delta_j(\omega))^{-1} B_j^l \end{cases} \tag{1}$$

with $\Delta_i(\omega) = \omega I$ or $\Delta_i(\omega) = \frac{1}{\omega} I$ and $\Delta_j(\omega) = \omega I$ or $\Delta_j(\omega) = \frac{1}{\omega} I$. Furthermore, this rational functions are assumed well-posed, that is, $\forall \omega \in \Omega_i^u$, the matrix $I - A_i^u \Delta_i(\omega)$ is invertible and $\forall \omega \in \Omega_i^l$, the matrix $I - A_i^l \Delta_i(\omega)$ is invertible.

We now present the problem under consideration.

General problem formulation *Given*

1. N_u upper bounds $U_i(\omega)$ defined on sub intervals Ω_i^u defined as above;
2. N_l upper bounds $L_j(\omega)$ defined on sub intervals Ω_j^l defined as above;
3. a positive integer n

find if there exists a minimum phase proper stable filter F of order bounded by n such that :

$$\left\{\begin{array}{ll} \text{for all } i \in \{1, \cdots, N_u\}, \text{ for all } \omega \in \Omega_i^u, & |F(j\omega)| < U_i(\omega) \\ \text{for all } j \in \{1, \cdots, N_l\}, \text{ for all } \omega \in \Omega_j^l, & L_j(\omega) < |F(j\omega)| \end{array}\right. \tag{2}$$

and *compute* it.

Remark For the sake of simplicity, we assume that the numerator order is smaller than the denominator one. Nevertheless, in your approach, if necessary, the numerator order of the filter F can be fixed to any arbitrary value.

Special functions of interest are power functions, that is, functions of the form:

$$U_i(\omega) = E_i^u \omega^{K_i^u} \quad \text{and} \quad L_j(\omega) = E_j^l \omega^{K_j^l} \tag{3}$$

where E_i^u, E_j^l are real positive and K_i^u, K_j^l are integer. In this special case, when magnitudes are expressed in dB, condition (2) boils down to:

$$\left\{\begin{array}{lll} \text{for all } & i \in \{1, \cdots, N_u\}, \text{ for all } & \omega \in \Omega_i^u, \quad |F(j\omega)|_{dB} < 20\log_{10}(E_i^u) + 20K_i^u \log_{10}(\omega) \\ \text{for all } & j \in \{1, \cdots, N_l\}, \text{ for all } & \omega \in \Omega_j^l, \quad 20\log_{10}(E_j^l) + 20K_j^l \log_{10}(\omega) < |F(j\omega)|_{dB} \end{array}\right. \tag{4}$$

Remark In this case, we can be formulated the bound functions as an LFT in ω or in $\frac{1}{\omega}$, indeed with $K > 0$

$$E\omega^K = \omega I_K \star \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \\ \hline E & 0 & \dots & 0 & 0 \end{array}\right] \quad \textit{or} \quad E\omega^{-K} = \frac{1}{\omega} I_K \star \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \\ \hline E & 0 & \dots & 0 & 0 \end{array}\right]$$

As an example, a set of constraints are given in the table 1 (see also figure 2).

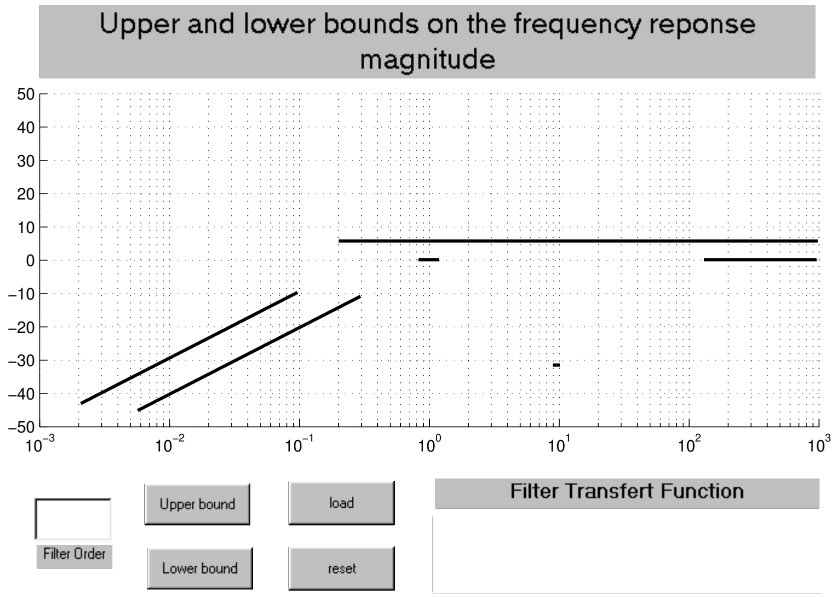

Figure 2: Example of constraint

Upper bound functions	**Lower bound functions**
$U_1(\omega) = 3.4\omega$ defined on $(0.002,\ 0.0965)$	$L_1(\omega) = 0.97\omega$ defined on $(0.01,\ 0.1)$
$U_2(\omega) = 0.0267$ defined on $(8.9189,\ 10.14)$	$L_2(\omega) = 1.0212$ defined on $(0.8244,\ 1.19)$
$U_3(\omega) = 1.9440$ defined on $(0.2,\ 978)$	$L_3(\omega) = 1.0212$ defined on $(130,\ 958)$

Table 1: An example of set of upper and lower bound functions

Remark As a matter of fact, conditions (4) allow to recover classical (constant) constraint specifications (see *e.g.* [OS75], see also section 5.2) and by a priori setting the filter F poles. This point is of interest in some cases [OS75]. Actually, our approach can be simplified in order to handle this case.

Particular problem formulation *Given*

1. N_u upper bounds $U_i(\omega)$ defined on sub intervals Ω_i^u defined as above;
2. N_l upper bounds $L_j(\omega)$ defined on sub intervals Ω_j^l defined as above;
3. a set of n stable poles p_i

find if there exists a proper stable minimum phase filter F with the fixed poles such that :

$$\begin{cases} \text{for all } i \in \{1, \cdots, N_u\}, \text{ for all } \omega \in \Omega_i^u, & |F(j\omega)| < U_i(\omega) \\ \text{for all } j \in \{1, \cdots, N_l\}, \text{ for all } \omega \in \Omega_j^l, & L_j(\omega) < |F(j\omega)| \end{cases} \tag{5}$$

and *compute* it.

The FIR filter design can be formulated as a special case of the PARTICULAR PROBLEM FORMULATION. For details, see subsection 5.2.

Discussion In the problem formulation (see *e.g.* conditions 2), the decision variable is the filter transfer function $F(j\omega)$. Note that, using spectral factorization, an equivalent problem is to consider as decision variable $|F(j\omega)|$. With this change of variable, conditions (2) are linear thus convex constraints in the decision variable. Under this suitable change of variable, solving the GENERAL PROBLEM FORMULATION is a convex optimization problem but infinite dimensional. As the constraints depend on ω, the number of constraints is infinite. Furthermore, as the decision variable is a real rational function, the decision variable is in an infinite dimensional space. In the PARTICULAR PROBLEM FORMULATION, the problem is only a semi infinite (so called SIP) since in this case the number of decision variables is finite. This case correspond for instance to the FIR design.

In the sequel, we prove that both problems can be equivalently formulated as *finite* dimensional optimization problems involving Linear Matrix Inequalities constraints. In this formulation, the number of variables and the number of constraints are finite, that is, the solution can be readily computed using standard LMI optimization toolboxes [GNLC95], without any approximation.

For the sake of clearness, we first present our solution in a special case.

3 Main result: simplified formulation

One upper bound constraint ($N_u = 1$) on the frequency domain interval $\Omega = (\underline{\omega}; \overline{\omega})$ is considered and denoted in the sequel $U(\omega)$. Note that the subscripts are dropped. There is no lower bound constraint ($N_l = 0$). To sum up, find $F(j\omega)$ such that:

$$\text{for all } \omega \in \Omega, \quad |F(j\omega)| < U(\omega) \tag{6}$$

In the two following subsections, our solution is presented. In the section 4, the proposed solution is straightforwardly extended to the general case (that is, several upper and lower constraints on different frequency intervals).

3.1 Semi infinite dimensional formulation

Note that the filter F has to satisfy a set of constraints involving the magnitude of F, $|F(j\omega)|$. As a consequence, the real variable is $|F(j\omega)|$. We first propose a process which allows to compute (if it exists) $|F(j\omega)|$ satisfying constraints (2). Using standard results on spectral factorization [Fra87], a stable minimum phase transfer F can be then computed.

In the sequel, the filter F is factorized as follows:

$$F(j\omega) = \frac{N(j\omega)}{D(j\omega)}.$$

Several factorizations are possible. For instance, $N(j\omega)$ and $D(j\omega)$ can be selected as: (coprime) polynomial factors or coprime (rational) factors. In the sequel, without loss of generality, we focus on the first case. With this factorization, condition (6) is equivalent to the following condition:

$$|N(j\omega)| < |D(j\omega)|U(\omega) \tag{7}$$

Note that equation (6) is affine in $|F(j\omega)|$ but to get a finite number of decision variables, we need to set the poles of F, which is conservative. With the previous factorization and with the new variables $|N(j\omega)|$ and $|D(j\omega)|$, the formulation (7) which is equivalent to (6), is still affine in the decision variables $|N(j\omega)|$ and $|D(j\omega)|$. By introducing a suitable parameterization of N and D, we are able to generate all the possible transfers F of order n. Using these ideas, we derive the following lemma.

Lemma 3.1 *Let $U(\omega)$ be an upper bound magnitude specification defined on Ω and let n be a positive integer. Let us introduced the following polynomial column base*

$$B(j\omega) = \begin{bmatrix} (j\omega)^n & (j\omega)^{n-1} & \cdots & j\omega & 1 \end{bmatrix}^T \tag{8}$$

The two following propositions are equivalent:

(*i*) *There exists a stable, minimum phase, $F(j\omega)$ filter of order bounded by n such that:*

$$\textit{for all } \omega \in \Omega, \quad |F(j\omega)| < U(\omega) \tag{9}$$

(*ii*) *There exist X_N and X_D symmetric matrices in $R^{(n+1)\times(n+1)}$ such that:*

$$\textit{for all } \omega \in \Omega, \qquad \begin{bmatrix} B(j\omega)U(\omega) \\ B(j\omega) \end{bmatrix}^* \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} B(j\omega)U(\omega) \\ B(j\omega) \end{bmatrix} < 0 \tag{10}$$

$$\textit{for all } \omega \in [0,\ +\infty), \qquad B(j\omega)^* X_N B(j\omega) > 0 \quad \textit{and} \quad B(j\omega)^* X_D B(j\omega) > 0 \tag{11}$$

Moreover, if there exist X_N and X_D satisfying conditions (10) and (11), the solution F satisfying condition (9) is given by $F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ with

$$N(j\omega)^* N(j\omega) = B(j\omega)^* X_N B(j\omega) \quad \textit{and} \quad D(j\omega)^* D(j\omega) = B(j\omega)^* X_D B(j\omega) \tag{12}$$

Proof See the proof in appendix, section A.

Sketch of the proof The underlying idea relies on the (finite dimensional) parameterization of (all) the squared magnitude functions $|N(j\omega)|^2$ and $|D(j\omega)|^2$ given by conditions (12). Note that the constraint (7) can be equivalently rewritten as: $N(j\omega)^* N(j\omega) < U(\omega)^* D(j\omega)^* D(j\omega) U(\omega)$. Condition (10) is thus obtained by replacing $|N(j\omega)|$ and $|D(j\omega)|$ by their parameterization. Furthermore, conditions (11) ensure that the obtained squared magnitudes are positive.

Remark Independently of the denominator order (polynomial $D(p)$ order), we can enforce the numerator order (polynomial $N(p)$ order) by setting to zero suitable lines and suitable columns of X_N.

In this section, we present a parameterization of the to-be-obtained filter which is linear and finite dimensional. Nevertheless, the obtained conditions in Lemma 3.1 must be satisfied for frequency intervals, that is, they are an infinite number of inequalities. In the next section, we propose an equivalent formulation where the number of (Linear Matrix) inequalities is finite.

3.2 Equivalent finite dimensional formulation

The previously considered polynomial vector

$$B(j\omega) = \begin{bmatrix} (j\omega)^n & (j\omega)^{n-1} & \cdots & j\omega & 1 \end{bmatrix}^T$$

can be easily formulated as an LFT in $j\omega$, that is, $B(j\omega) = D_b + C_b j\omega (I - j\omega A_b)^{-1} B_b$ with

$$A_b = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \cdots & 0 & \vdots \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad C_b = \begin{bmatrix} 0 & \cdots & \cdots & 1 \\ 0 & \cdots & 1 & 0 \\ 1 & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad D_b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Theorem 3.1 *Let $U(\omega) = D_u + C_u \Delta(\omega)(I - \Delta(\omega) A_u)^{-1} B_u$, with $\Delta(\omega) = \omega I$ or $\Delta(\omega) = \frac{1}{\omega} I$ be an upper bound magnitude specification defined on Ω and let n be a positive integer. Let us* *introduced the following polynomial column base defined by (8). Let be (A,B,C,D) complex matrices defined as:*

$$\forall\delta \in (0,\infty),\quad \delta \star \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \delta \star T_\Omega \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \times \delta \star \widetilde{T}_\Omega \star \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right] \\ \delta \star T_\Omega \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

with

$$T_\Omega = \left[\begin{array}{c|c} -\overline{\omega} I & (\underline{\omega} - \overline{\omega}) I \\ \hline \overline{\omega} I & \overline{\omega} I \end{array}\right]$$

and

$$\text{if} \quad \Delta(\omega) = \omega I, \qquad \widetilde{T}_\Omega = T_\Omega$$

$$\text{if} \quad \Delta(\omega) = \tfrac{1}{\omega} I, \qquad \widetilde{T}_\Omega = \left[\begin{array}{c|c} -\underline{\omega} I & (1 - \frac{\underline{\omega}}{\overline{\omega}}) I \\ \hline I & (\frac{1}{\overline{\omega}}) I \end{array}\right]$$

The two following propositions are equivalent:

(i) There exists a stable, minimum phase, $F(j\omega)$ filter of order bounded by n such that:

$$\text{for all } \omega \in \Omega, \quad |F(j\omega)| < U(\omega) \tag{13}$$

(ii) There exist X_N and X_D symmetric matrices in $R^{(n+1)\times(n+1)}$ such that there exist two symmetric matrices P_N and P_D and a complex matrix P such that:

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_N \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_N + P_N A & P_N B_b \\ B_b^T P_N & 0 \end{bmatrix} > 0 \tag{14}$$

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_D \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_D + P_D A & P_D B_b \\ B_b^T P_D & 0 \end{bmatrix} > 0 \tag{15}$$

$$P + P^* > 0 \qquad \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^* P^* + PA & PB \\ B^* P^* & 0 \end{bmatrix} < 0 \tag{16}$$

Moreover, if there exist X_N and X_D satisfying conditions (14), (15) and (16), the solution F satisfying condition (13) is given by $F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ with

$$N(j\omega)^* N(j\omega) = B(j\omega)^* X_N B(j\omega) \quad \text{and} \quad D(j\omega)^* D(j\omega) = B(j\omega)^* X_D B(j\omega) \tag{17}$$

Proof See the proof in appendix, section C.

Computation Note that satisfying proposition (ii) conditions of Theorem 3.1 reduces to solving a *convex, finite-dimensional optimization problem* involving complex Linear Matrix Inequality (LMI) constraints. It is equivalent to an optimization problem over twice as big real LMI constraints [BE93]. As a matter of fact, this latter is a standard LMI optimization problem (actually a feasibility problem, see e.g. [BEFB94]) which can be efficiently solved[3] using standard (Matlab) solvers asthe free ware code SP [VB96] and its Matlab/Scilab interface `LMITOOL` [END95], or the available commercial Matlab Toolbox, LMI Control Toolbox [GNLC95]. See sections 5.1 or 5.2 for numerical applications.

Interpretation The most sticking fact is that the infinite number of inequalities (10), (11) of Lemma 3.1 are replaced by a finite set of inequalities. To this purpose, Theorem 3.1 is based on an extension of the Kalman Yakubovich Popov (KYP) lemma [Pop73]. Remember that, given a real rational transfer function $H(j\omega) = D + C(j\omega I - A)^{-1}B$, this lemma claims that testing that $\forall\omega$, $H(j\omega) + H(j\omega)^* > 0$ is equivalent to finding a symmetric matrix P such that :

$$\left[\begin{array}{cc} A^TP + PA & PB + C \\ B^TP + C^T & D + D^T \end{array}\right] > 0.$$

Actually, this classical lemma allows to replace an infinite number of inequalities by one (matrix) inequality by introducing an additional decision variable P. Note that the transfer function $H(j\omega)$ is an LFT of $\frac{1}{j\omega}$ and that the frequency dependent inequality has to be satisfied for any ω. In our problem, the frequency dependent functions can be expressed as LFTs of ω or $\frac{1}{\omega}$ with ω in a bounded interval. Thus the KYP lemma can not be applied. But a suitable change of variable, called loop-shifting [DV75], these functions are expressed as LFT of δ, with $\delta \in (0, +\infty)$. Actually, the KYP lemma can be extended to the case of LFTs of a real (positive) parameter δ. We thus obtain the following lemma.

Lemma 3.2 *Let us consider $\phi(\delta) = D + C\delta(I - \delta A)^{-1}B$ a well-posed rational function where A, B, C and D are complex matrices and let M be an hermitian matrix. The two following propositions are then equivalent.*

(i)

$$\forall\delta \in (0, +\infty), \qquad \Phi^*(\delta)M\Phi(\delta) < 0 \tag{18}$$

(ii) *There exists a complex matrix P such that $P + P^* > 0$ and:*

$$\left[\begin{array}{c} C^* \\ D^* \end{array}\right] M \left[\begin{array}{cc} C & D \end{array}\right] + \left[\begin{array}{cc} A^*P^* + PA & PB \\ B^*P^* & 0 \end{array}\right] < 0 \tag{19}$$

[3] in polynomial time

Proof: See the proof in appendix, section B.

A direct application of this theorem allows to get conditions (15), (14) and (16). Indeed, conditions (15) and (14) are equivalent to conditions (11). In addition, condition (16) is equivalent to condition (10).

4 Main result: general formulation

We consider the general problem formulation with N_u upper bound constraint and N_l lower bound constraint. We first present the solution when the filter poles are free. This solution is then specialized to the case when the filter poles are set.

4.1 General problem formulation

Theorem 4.1 *Let us defined an upper and lower bound magnitude specification set defined as*

$$\begin{cases} \textit{for all}\, i \in \{1, \cdots, N_u\}, & \forall \omega \in \Omega_i^u, \quad U_i(\omega) = D_i^u + C_i^u \Delta_i^u(\omega)(I - A_i^u \Delta_i^u(\omega))^{-1} B_i^u \\ \textit{for all}\, j \in \{1, \cdots, N_l\}, & \forall \omega \in \Omega_j^l, \quad L_j(\omega) = D_j^l + C_j^l \Delta_j^l(\omega)(I - A_j^l \Delta_j^l(\omega))^{-1} B_j^l \end{cases}$$

with $\Delta_i^u(\omega) = \omega I$ *or* $\Delta_i^u(\omega) = \frac{1}{\omega} I$ *and* $\Delta_j^l(\omega) = \omega I$ *or* $\Delta_j^l(\omega) = \frac{1}{\omega} I$. *Let* n *be a positive integer. Let us introduced the following polynomial column base defined by (8). Let be* $(\overline{A}_i, \overline{B}_i, \overline{C}_i, \overline{D}_i)$ *complex matrices defined as:*

for all $i \in \{1, \cdots, N_u\}$,

$$\forall \delta \in (0, +\infty), \quad \delta \star \begin{bmatrix} \overline{A}_i & \overline{B}_i \\ \overline{C}_i & \overline{D}_i \end{bmatrix} = \begin{bmatrix} \delta \star T_{\Omega_i^u} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \times \delta \star \widetilde{T}_{\Omega_i^u} \star \left[\begin{array}{c|c} A_i^u & B_i^u \\ \hline C_i^u & D_i^u \end{array}\right] \\ \delta \star T_{\Omega_i^u} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

with $\Omega_i^u = (\underline{\omega}_i, \overline{\omega}_i)$

$$T_{\Omega_i^u} = \left[\begin{array}{c|c} -\overline{\omega}_i I & (\underline{\omega}_i - \overline{\omega}_i) I \\ \hline \overline{\omega}_i I & \overline{\omega}_i I \end{array}\right]$$

and

$$\textit{if} \quad \Delta_i^u(\omega) = \omega I, \qquad \widetilde{T}_{\Omega_i^u} = T_{\Omega_i^u}$$

$$\textit{if} \quad \Delta_i^u = \frac{1}{\omega} I, \qquad \widetilde{T}_{\Omega_i^u} = \left[\begin{array}{c|c} -\underline{\omega}_i I & (1 - \frac{\underline{\omega}_i}{\overline{\omega}_i}) I \\ \hline I & (\frac{1}{\overline{\omega}_i}) I \end{array}\right]$$

In addition, let be $(\underline{A}_j, \underline{B}_j, \underline{C}_j, \underline{D}_j)$ *complex matrices defined as:*

for all $j \in \{1, \cdots, N_l\}$,

$$\forall \delta \in (0, +\infty), \quad \delta \star \begin{bmatrix} \underline{A}_j & \underline{B}_j \\ \underline{C}_j & \underline{D}_j \end{bmatrix} = \begin{bmatrix} \delta \star T_{\Omega_j^l} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \times \delta \star \widetilde{T}_{\Omega_j^l} \star \left[\begin{array}{c|c} A_j^l & B_j^l \\ \hline C_j^l & D_j^l \end{array}\right] \\ \delta \star T_{\Omega_j^l} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

with $\Omega_j^l = (\underline{\omega}_j, \ \overline{\omega}_j)$

$$T_{\Omega_j^l} = \left[\begin{array}{c|c} -\overline{\omega}_j I & (\underline{\omega}_j - \overline{\omega}_j) I \\ \hline \overline{\omega}_j I & \overline{\omega}_j I \end{array}\right]$$

and

$$\text{if} \quad \Delta_j^l(\omega) = \omega I, \qquad \widetilde{T}_{\Omega_j^l} = T_{\Omega_j^l}$$

$$\text{if} \quad \Delta_j^l = \tfrac{1}{\omega} I, \qquad \widetilde{T}_{\Omega_j^l} = \left[\begin{array}{c|c} -\underline{\omega}_j I & (1 - \frac{\underline{\omega}_j}{\overline{\omega}_j}) I \\ \hline I & (\frac{1}{\overline{\omega}_j}) I \end{array}\right]$$

The two following propositions are then equivalent.

(i) *There exists a minimum phase proper stable filter* F *of order bounded by* n *such that:*

$$\begin{cases} \text{for all } i \in \{1, \cdots, N_u\}, \ \text{for all } \omega \in \Omega_i^u, & |F(j\omega)| < U_i(\omega) \\ \text{for all } j \in \{1, \cdots, N_l\}, \ \text{for all } \omega \in \Omega_j^l, & L_j(\omega) < |F(j\omega)| \end{cases} \tag{20}$$

(ii) *There exist* X_N *and* X_D *symmetric matrices in* $R^{(n+1)\times(n+1)}$ *such that there exist two symmetric matrices* P_N *and* P_D, N_u *complex matrices* P_i^u *and* N_l *complex matrices* P_j^l *such that:*

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_N \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_N + P_N A & P_N B_b \\ B_b^T P_N & 0 \end{bmatrix} > 0 \tag{21}$$

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_D \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_D + P_D A & P_D B_b \\ B_b^T P_D & 0 \end{bmatrix} > 0 \tag{22}$$

$$\forall i, P_i^u + P_i^{u*} > 0, \quad \begin{bmatrix} \overline{C}_i^* \\ \overline{D}_i^* \end{bmatrix} \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} \overline{C}_i & \overline{D}_i \end{bmatrix} + \begin{bmatrix} \overline{A}_i^* P_i^{u*} + P_i^u \overline{A}_i & P_i^u \overline{B}_i \\ \overline{B}_i^* P_i^{u*} & 0 \end{bmatrix} < 0 \tag{23}$$

$$\forall j, P_j^l + P_j^{l*} > 0, \quad \begin{bmatrix} \underline{C}_j^* \\ \underline{D}_j^* \end{bmatrix} \begin{bmatrix} X_D & 0 \\ 0 & -X_N \end{bmatrix} \begin{bmatrix} \underline{C}_j & \underline{D}_j \end{bmatrix} + \begin{bmatrix} \underline{A}_j^* P_j^{l*} + P_j^l \underline{A}_j & P_j^l \underline{B}_j \\ \underline{B}_j^* P_j^{l*} & 0 \end{bmatrix} < 0 \tag{24}$$

Moreover, if there exist X_N and X_D satisfying conditions (21), (22), (23) and (24), the solution F of the General problem formulation *is given by $F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ with*

$$N(j\omega)^* N(j\omega) = B(j\omega)^* X_N B(j\omega) \quad \text{and} \quad D(j\omega)^* D(j\omega) = B(j\omega)^* X_D B(j\omega) \tag{25}$$

Proof Note that when several upper and lower bounds are considered, the corresponding problem formulation is obtained by adding the corresponding conditions. As a consequence, this Theorem can be readily proved by a slight modification of Theorem 3.1 proof.

Computation The comments on the Theorem 3.1 condition computation still hold for Theorem 4.1.

4.2 Particular problem formulation

In the case when the filter poles are a priori set, Theorem 4.1 has to be slightly adapted. First, we consider a different vector $B(j\omega)$:

$$B(j\omega) = \begin{bmatrix} \frac{1}{(j\omega)-p_1} & \frac{1}{(j\omega)-p_2} & \cdots & \frac{1}{(j\omega)-p_n} & 1 \end{bmatrix}^T.$$

It can be easily formulated as an LFT in $\frac{1}{j\omega}$, that is, $B(j\omega) = D_b + C_b j\omega(I - j\omega A_b)^{-1} B_b$ with

$$A_b = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_n \end{bmatrix}, \quad B_b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad C_b = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad D_b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Theorem 4.2 *Let us defined an upper and lower bound magnitude specification set defined as*

$$\begin{cases} \text{for all } i \in \{1, \cdots, N_u\}, & \forall \omega \in \Omega_i^u, \quad U_i(\omega) = D_i^u + C_i^u \Delta_i^u(\omega)(I - A_i^u \Delta_i^u(\omega))^{-1} B_i^u \\ \text{for all } j \in \{1, \cdots, N_l\}, & \forall \omega \in \Omega_j^l, \quad L_j(\omega) = D_j^l + C_j^l \Delta_j^l(\omega)(I - A_j^l \Delta_j^l(\omega))^{-1} B_j^l \end{cases}$$

with $\Delta_i^u(\omega) = \omega I$ or $\Delta_i^u(\omega) = \frac{1}{\omega} I$ and $\Delta_j^l(\omega) = \omega I$ or $\Delta_j^l(\omega) = \frac{1}{\omega} I$. Let n be a positive integer. Let us introduced the following polynomial column base defined by (8). Let be $(\overline{A}_i, \overline{B}_i, \overline{C}_i, \overline{D}_i)$ complex matrices defined as:

$$\text{for all } i \in \{1, \cdots, N_u\},$$

$$\forall \delta \in (0, +\infty), \quad \delta \star \begin{bmatrix} \overline{A}_i & \overline{B}_i \\ \overline{C}_i & \overline{D}_i \end{bmatrix} = \begin{bmatrix} \delta \star \widetilde{T}_{\Omega_i^u} \star \left[\begin{array}{c|c} A_i^u & B_i^u \\ \hline C_i^u & D_i^u \end{array}\right] \\ \delta \star T_{\Omega_i^u} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

with $\Omega_i^u = (\underline{\omega}_i,\ \overline{\omega}_i)$

$$T_{\Omega_i^u} = \left[\begin{array}{c|c} -\underline{\omega}_i I & (1-\frac{\underline{\omega}_i}{\overline{\omega}_i})I \\ \hline I & (\frac{1}{\overline{\omega}_i})I \end{array}\right]$$

and

$$\begin{array}{lll} \text{if} \quad \Delta_i^u(\omega) = \omega I, & & \widetilde{T}_{\Omega_i^u} = \left[\begin{array}{c|c} -\overline{\omega}_i I & (\underline{\omega}_i - \overline{\omega}_i)I \\ \hline \overline{\omega}_i I & \overline{\omega}_i I \end{array}\right] \\ \text{if} \quad \Delta_i^u = \frac{1}{\omega} I, & & \widetilde{T}_{\Omega_i^u} = T_{\Omega_i^u} \end{array}$$

In addition, let be $(\underline{A}_j, \underline{B}_j, \underline{C}_j, \underline{D}_j)$ *complex matrices defined as:*

$$\text{for all } \ j \in \{1,\ \cdots, N_l\},$$

$$\forall \delta \in (0,\infty), \quad \delta \star \left[\begin{array}{cc} \underline{A}_j & \underline{B}_j \\ \underline{C}_j & \underline{D}_j \end{array}\right] = \left[\begin{array}{c} \delta \star \widetilde{T}_{\Omega_j^l} \star \left[\begin{array}{c|c} A_j^l & B_j^l \\ \hline C_j^l & D_j^l \end{array}\right] \\ \delta \star T_{\Omega_j^l} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{array}\right]$$

with $\Omega_j^l = (\underline{\omega}_j,\ \overline{\omega}_j)$

$$T_{\Omega_j^l} = \left[\begin{array}{c|c} -\underline{\omega}_j I & (1-\frac{\underline{\omega}_j}{\overline{\omega}_j})I \\ \hline I & (\frac{1}{\overline{\omega}_j})I \end{array}\right]$$

and

$$\begin{array}{lll} \text{if} \quad \Delta_j^l(\omega) = \omega I, & & \widetilde{T}_{\Omega_j^l} = \left[\begin{array}{c|c} -\overline{\omega}_j I & (\underline{\omega}_j - \overline{\omega}_j)I \\ \hline \overline{\omega}_j I & \overline{\omega}_j I \end{array}\right] \\ \text{if} \quad \Delta_j^l = \frac{1}{\omega} I, & & \widetilde{T}_{\Omega_j^l} = T_{\Omega_j^l} \end{array}$$

The two following propositions are then equivalent.

(i) *There exists a minimum phase proper stable filter* F *of order bounded by* n*, with poles* p_i, $i \in \{1, \cdots, n\}$ *such that:*

$$\left\{ \begin{array}{l} \text{for all } i \in \{1, \cdots, N_u\}, \ \text{for all } \omega \in \Omega_i^u, \quad |F(j\omega)| < U_i(\omega) \\ \text{for all } j \in \{1, \cdots, N_l\}, \ \text{for all } \omega \in \Omega_j^l, \quad L_j(\omega) < |F(j\omega)| \end{array} \right. \tag{26}$$

(ii) *There exist* X *a symmetric matrices in* $R^{(n+1)\times(n+1)}$ *such that there exist a* *symmetric matrix P, N_u complex matrices P_i^u and N_l complex matrices P_j^l such that:*

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P + P A_b & P B_b \\ B_b^T P & 0 \end{bmatrix} > 0 \tag{27}$$

$$\forall i, P_i^u + P_i^{u*} > 0, \quad \begin{bmatrix} \overline{C}_i^* \\ \overline{D}_i^* \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \overline{C}_i & \overline{D}_i \end{bmatrix} + \begin{bmatrix} \overline{A}_i^* P_i^{u*} + P_i^u \overline{A}_i & P_i^u \overline{B}_i \\ \overline{B}_i^* P_i^{u*} & 0 \end{bmatrix} < 0 \tag{28}$$

$$\forall j, P_j^l + P_j^{l*} > 0, \quad \begin{bmatrix} \underline{C}_j^* \\ \underline{D}_j^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -X \end{bmatrix} \begin{bmatrix} \underline{C}_j & \underline{D}_j \end{bmatrix} + \begin{bmatrix} \underline{A}_j^* P_j^{l*} + P_j^l \underline{A}_j & P_j^l \underline{B}_j \\ \underline{B}_j^* P_j^{l*} & 0 \end{bmatrix} < 0 \tag{29}$$

Moreover, if the previous conditions are satisfied, the solution F of the Particular problem formulation *is given by $F(j\omega) = B(j\omega)^* X B(j\omega)$.*

Proof This Theorem can be proved by a slight modification of Theorem 4.1.

In the subsection 5.2, this Theorem is adapted in order to design FIR filters.

5 Applications

5.1 H_∞ weighting function design

For this numerical application, a Matlab 6 based program handling arbitrary numbers of upper and lower constraints of the form (3) was developed. The optimization problems are solved using the Matlab toolbox, LMI Control Toolbox [GNLC95].

We want to find a weighting function W whose magnitude satisfies upper and lower constraints given in table 1 (see also figure 2). Consider a system $G(p)$ controlled by a one

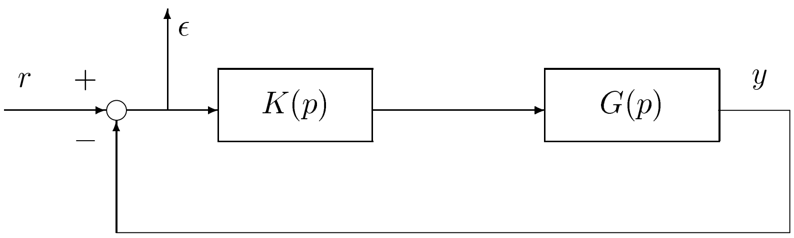

Figure 3: Plant $G(p)$ with the control law $K(p)$

degree of freedom control law $K(p)$ (see figure 3). We want to design a control law ensuring that the closed loop system output y is able to track with a small error, step and sinusoïdal reference signals r. Such a property is obtained if the function transfer from the input r to the tracking error $\epsilon = r - y$ has its magnitude bounded by a weighting function magnitude satisfying the considered set of constraints.

By applying our approach, we obtain a 3^{d} order filter W:

$$W(s) = \frac{1.1367(s + 0.002689)(s^2 + 0.1976s + 90.44)}{(s + 53.38)(s^2 + 1.494s + 1.391)}.$$

Our programme was unable to compute a 2^{d} order filter. As a consequence, there does not exist a 2^{d} order filter that meets the specification.

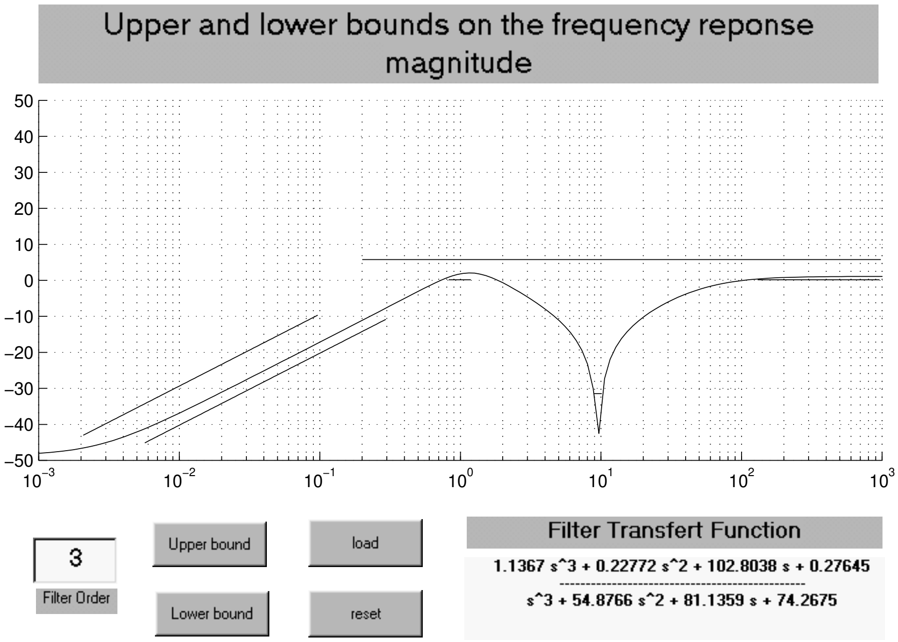

Figure 4: H_∞ weighting function

5.2 (FIR) discrete time filter design

Lowpass filter design

A finite impulse response (FIR) filter is defined as:

$$H(\omega) = \sum_{i=0}^{n-1} h(i)e^{-j\omega}.$$

We consider the classical lowpass filter design problem (see figure 5). For a given order $n-1$, lowpass filter $H(\omega)$ has to ensure:

1. a given passband ripple specification, that is, given a passband frequency ω_p and a given strictly positive α:

$$\alpha^{-1} \leq |H(\omega)| \leq \alpha, \quad \omega \in (0, \omega_p)$$

2. a minimum stopband attenuation gain specification, that is, given a stopband frequency ω_s ($\omega_s > \omega_p$) and for the smallest positive maximum gain attenuation β:

$$|H(\omega)| \leq \beta, \quad \omega \in (\omega_s, \pi).$$

To sum up, the corresponding optimization problem can be written as:

$$\begin{array}{lll} \text{minimize} & \beta & \\ \text{on} & \beta, \ \ h(i), & \\ \text{subject to} & \alpha^{-1} \leq |H(\omega)| \leq \alpha, & \omega \in (0, \omega_p) \\ & |H(\omega)| \leq \beta, & \omega \in (\omega_s, \pi) \end{array} \tag{30}$$

Remark Note that our approach allows to solve the Infinite Impulse Response (IIR) filter design. Nevertheless, due to the special interest of FIR filters in Signal Processing, we just focus on the FIR case.

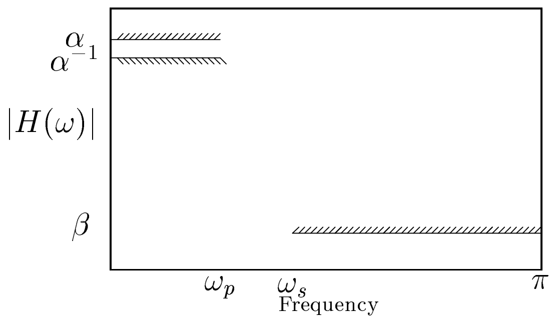

Figure 5: Lowpass filter design specification

How to solve this problem using our approach? As a matter of fact, the main problem is that $H(\omega)$ is not a rational function of ω. Let us introduce the new variable ν such that:

$$e^{-j\omega} = \frac{2 - j\nu}{2 + j\nu}$$

that is, $\nu = 2\tan\left(\frac{\omega}{2}\right)$, which is the classical bilinear (ou tustin) transformation (see *e.g.* [DV75, RG75b]). Using this change of variable, the optimization problem (30) is then equivalent to the minimization problem

$$\begin{array}{lll} \text{minimize} & \beta & \\ \text{on} & \beta,\ h(i), & \\ \text{subject to} & \alpha^{-1} \leq |H_c(\nu)| \leq \alpha, & \nu \in (0, \nu_p) \\ & |H_c(\nu)| \leq \beta, & \nu \in (\nu_s,\ +\infty) \end{array} \tag{31}$$

where $\nu_p = 2\tan\left(\frac{\omega_p}{2}\right)$, $\nu_s = 2\tan\left(\frac{\omega_s}{2}\right)$ and

$$H_c(\nu) = \sum_{n=0}^{n-1} h(i)\frac{2 - j\nu}{2 + j\nu}. \tag{32}$$

In this problem, we have two upper bound constraints ($N_u = 2$) with

1. $\Omega_1^u = (0,\ \nu_p)$ and $U_1(\nu) = \alpha$;
2. $\Omega_2^u = (\nu_s,\ \pi)$ and $U_1(\nu) = \beta$

and one lower bound constraint ($N_l = 1$) with $\Omega_1^l = (0,\ \nu_p)$ and $L_1(\nu) = \alpha^{-1}$.

Due to the special structure of H_c, a more interesting choice of $B(j\nu)$ can be proposed:

$$B(j\nu) = [\ \left(\frac{2-j\nu}{2+j\nu}\right)^n \quad \left(\frac{2-j\nu}{2+j\nu}\right)^{n-1} \quad \cdots \quad 1\ ]^T.$$

Using this choice, in order to obtain $H_c(\nu)$ with the structure (32), $|H_c(\nu)|^2 = B(j\nu)^* X B(j\nu)$. Note that the previous choice, via spectral factorization, the new optimization variable X allows to generate all the FIR filters of order n. Let us denote (A_b, B_b, C_b, D_b) such that

$$B(j\nu) = j\nu \star \left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right].$$

Theorem 4.1 is applied with:

$$T_{\Omega_1^u} = \left[\begin{array}{c|c} -\nu_p I & -\nu_p I \\ \hline \nu_p I & \nu_p I \end{array}\right] \quad \text{and} \quad T_{\Omega_2^u} = \left[\begin{array}{c|c} 0 & I \\ \hline I & \nu_s I \end{array}\right]$$

and $T_{\Omega_1^l} = T_{\Omega_1^u}$. Furthermore,

$$\delta \star \begin{bmatrix} \overline{A}_1 & \overline{B}_1 \\ \overline{C}_1 & \overline{D}_1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \delta \star T_{\Omega_1^u} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

$$\delta \star \begin{bmatrix} \overline{A}_2 & \overline{B}_2 \\ \overline{C}_2 & \overline{D}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \delta \star T_{\Omega_2^u} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

$$\delta \star \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \\ \underline{C}_1 & \underline{D}_1 \end{bmatrix} = \begin{bmatrix} \alpha^{-1} \\ \delta \star T_{\Omega_1^l} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

We thus obtain the following optimization problem:

minimize $\quad \beta^2$

on $\quad \beta^2,\ X_N,\ P,\ P_1^u,\ P_2^u,\ P_l^u$

subject to

$$P = P^T, \qquad \begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P + PA_b & PB_b \\ B_b^T P & 0 \end{bmatrix} > 0$$

$$P_1^u + P_1^{u*} > 0, \quad \begin{bmatrix} \overline{C}_1^* \\ \overline{D}_1^* \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \overline{C}_1 & \overline{D}_1 \end{bmatrix} + \begin{bmatrix} \overline{A}_1^* P_1^{u*} + P_1^u \overline{A}_1 & P_1^u \overline{B}_1 \\ \overline{B}_1^* P_1^{u*} & 0 \end{bmatrix} < 0$$

$$P_2^u + P_2^{u*} > 0, \quad \begin{bmatrix} \overline{C}_2^* \\ \overline{D}_2^* \end{bmatrix} \begin{bmatrix} -\beta^2 & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \overline{C}_2 & \overline{D}_2 \end{bmatrix} + \begin{bmatrix} \overline{A}_2^* P_2^{u*} + P_2^u \overline{A}_2 & P_2^u \overline{B}_2 \\ \overline{B}_2^* P_2^{u*} & 0 \end{bmatrix} < 0$$

$$P_1^l + P_1^{l*} > 0, \quad \begin{bmatrix} \underline{C}_1^* \\ \underline{D}_1^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -X \end{bmatrix} \begin{bmatrix} \underline{C}_1 & \underline{D}_1 \end{bmatrix} + \begin{bmatrix} \underline{A}_1^* P_1^{l*} + P_1^l \underline{A}_1 & P_1^l \underline{B}_1 \\ \underline{B}_1^* P_1^{l*} & 0 \end{bmatrix} < 0$$

This problem is a convex optimization involving LMI constraints. In fact, it reduces to the minimization of a linear cost under LMI constraints [BEFB94], that is,

$$\begin{array}{cc} \text{minimize} & c^T \xi \\ \text{on} & \xi \\ \text{subject to} & F(\xi) > 0 \end{array}$$

This problem can be solved using the Matlab toolbox, LMI Control Toolbox [GNLC95].

6 Conclusion

The proposed approach was presented in the case of the design of filters satisfying hard upper and lower magnitude bound. Nevertheless, note that this approach can be readily extended to the case of (all-pass) filter design satisfying phase constraints.

The salient features of the approach are that:

1. the problem is equivalently cast as a *finite dimensional convex* optimization problem involving Linear Matrix Inequality constraints, which can be efficiently solved using the available softwares;

2. the proposed formulation is really flexible, allowing to solve problems using a natural form;

3. no extra assumption of the obtained filter (as the filter poles are fixed or the phase filter is linear) are introduced;

4. the minimum order filter satisfying upper and lower magnitude bounds is obtained.

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A Proof of Lemma 3.1

$\boxed{(i) \Rightarrow (ii)}$

Let us rewrite the filter transfer function F as $F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ where $N(j\omega)$ and $D(j\omega)$ are two polynomials of order p and of order n with $n \geq p$, that is:

$$\begin{aligned} N(j\omega) &= b_p(j\omega)^p + b_{p-1}(j\omega)^{p-1} + \ldots + b_1 j\omega + b_0 &(33)\\ D(j\omega) &= a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1 j\omega + a_0 &(34)\end{aligned}$$

Consider the polynomial base $B(j\omega)$ of order n defined in (8). Equations (33) and (34) can be written as:

$$\begin{aligned} N(j\omega) &= \begin{bmatrix} 0 & \ldots & 0 & b_p & \ldots & b_0 \end{bmatrix} B(j\omega) &(35)\\ D(j\omega) &= \begin{bmatrix} a_n & \ldots & \ldots & \ldots & a_0 \end{bmatrix} B(j\omega) &(36)\end{aligned}$$

It is then straightforward that by introducing X_N and X_D symmetric matrices in $R^{(n+1)\times(n+1)}$ such that

$$\begin{aligned} X_N &= \begin{bmatrix} 0 & \ldots & 0 & b_p & \ldots & b_0 \end{bmatrix}^T \begin{bmatrix} 0 & \ldots & 0 & b_p & \ldots & b_0 \end{bmatrix} &(37)\\ X_D &= \begin{bmatrix} a_n & \ldots & \ldots & \ldots & a_0 \end{bmatrix}^T \begin{bmatrix} a_n & \ldots & \ldots & \ldots & a_0 \end{bmatrix} &(38)\end{aligned}$$

we have

$$\begin{aligned} N(j\omega)^* N(j\omega) &= B(j\omega)^* X_N B(j\omega) > 0 &(39)\\ D(j\omega)^* D(j\omega) &= B(j\omega)^* X_D B(j\omega) > 0 &(40)\end{aligned}$$

Thus, we obtain conditions (11).

Furthermore, the following condition

$$|F(j\omega)| < U(\omega) \tag{41}$$

leads to

$$\begin{aligned}
&\Longrightarrow \ |N(j\omega)| < |D(j\omega)|U(\omega) \\
&\Longrightarrow \ N(j\omega)^* N(j\omega) < U(\omega)^* D(j\omega)^* D(j\omega) U(\omega) \\
&\Longrightarrow \ B(j\omega)^* X_N B(j\omega) - U(\omega)^* B(j\omega)^* X_D B(j\omega) U(\omega) < 0
\end{aligned}$$

(from equations (39) and (40)). We thus obtain condition (10).

$\boxed{(ii) \Rightarrow (i)}$

Let us consider X_N and X_D two real symmetric matrices such that conditions (11) hold. Using spectral factorization, there exist two Hurwitz polynomials $N(j\omega)$ and $D(j\omega)$ such that

$$\begin{aligned}
N(j\omega)^* N(j\omega) &= B(j\omega)^* X_N B(j\omega) && (42) \\
D(j\omega)^* D(j\omega) &= B(j\omega)^* X_D B(j\omega) && (43)
\end{aligned}$$

Furthermore, we assume that, in addition, X_N and X_D satisfy equation (10).

$$\begin{bmatrix} B(j\omega)U(\omega) \\ U(\omega) \end{bmatrix}^* \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} B(j\omega)U(\omega) \\ U(\omega) \end{bmatrix} < 0 \qquad (44)$$

$$\begin{aligned}
&\Longrightarrow \ B(j\omega)^* X_N B(j\omega) - U(\omega)^* B(j\omega)^* X_D B(j\omega) U(\omega) < 0 \\
&\Longrightarrow \ N(j\omega)^* N(j\omega) < U(\omega)^* D(j\omega)^* D(j\omega) U(\omega) \\
&\Longrightarrow \ |N(j\omega)| < |D(j\omega)|U(\omega)
\end{aligned}$$

So there exist a filter

$$F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$$

proper, stable and minimum phase such that condition (9) holds.

QED

B Proof of Lemma 3.2

$\boxed{\mathbf{(i) \Rightarrow (ii)}}$

We now introduce the following LFT system:

$$\begin{cases} q = Ap + Bw \\ z = Cp + Dw \\ p = \delta q \end{cases}$$

Let us assume that:

$$\forall \delta \in (0,\ +\infty), \qquad \Phi^*(\delta) M \Phi(\delta) < 0.$$

that is the property (i) is satisfied. Then

$$\forall \delta \in (0, +\infty), \qquad \forall w \in \mathbb{C}^{n_w}, \quad w^* \Phi^*(\delta) M \Phi(\delta) w < 0.$$

This is equivalent to: $\forall \delta \in (0, +\infty)$, $\forall w \in \mathbb{C}^{n_w}$,

$$z^* M z < 0. \tag{45}$$

Furthermore, for any complex definite positive matrix P, the vectors p and q, such that $p = \delta q$, satisfy the following quadratic inequality:

$$q^* P p + p^* P^* q \geq 0 \tag{46}$$

since $q^* P p + p^* P^* p = \delta q^* (P + P^*) q > 0$ as $\delta > 0$ and P definite positive.

The vectors q and z can be replace by their explicit expression in vectors p and w. The problem can thus be rewritten in the following way.
All vectors x and w satisfying the constraint:

$$\begin{bmatrix} p \\ w \end{bmatrix}^* \begin{bmatrix} A^* P^* + PA & PB \\ B^* P^* & 0 \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} \geq 0,$$

must also satisfy the inequality :

$$\begin{bmatrix} p \\ w \end{bmatrix}^* \begin{bmatrix} C^* M C & C^* M D \\ D^* M C^* & D^* M D \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} < 0.$$

By applying $\mathcal{S}$-procedure theorem [Jaker], this property holds if a positive scalar τ such as for *all* vectors p and w such that

$$\begin{bmatrix} p \\ w \end{bmatrix}^* \begin{bmatrix} \tau(A^* P^* + PA) + C^* M C & \tau PB + CMD \\ \tau B^* P^* + D^* M C^* & D^* M D \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} < 0.$$

Thus by replacing τP by P, we can conclude that the property (i) holds if there is a complex defined positive matrix P such that:

$$\begin{bmatrix} C^* \\ D^* \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^* P^* + PA & PB \\ B^* P^* & 0 \end{bmatrix} < 0 \tag{47}$$

$\boxed{\textbf{(ii)} \Rightarrow \textbf{(i)}}$

Let us assume that property (ii) holds, that is, that there exists a complex defined positive matrix P such that

$$\begin{bmatrix} A^* P^* + PA + C^* M C & PB + CMD \\ B^* P^* + D^* M C^* & D^* M D \end{bmatrix} < 0$$

This matrix inequality still holds after postmultiplication [4] by $\begin{bmatrix} (\delta^{-1}I - A)^{-1}B \\ I \end{bmatrix}$ and premultiplication by $\begin{bmatrix} B^*(\delta^{-1}I - A^*)^{-1} & I \end{bmatrix}$. After some cumbersome calculations involving a square completion, we thus obtained the following equation:

$$\Phi(\delta)^* M \Phi(\delta) < -\delta^{-1} B^* (\delta^{-1}I - A^*)^{-1} (P + P^*) (\delta^{-1}I - A)^{-1} B. \tag{48}$$

As $\delta > 0$ and $P + P^* > 0$, we thus obtained:

$$\Phi(\delta)^* M \Phi(\delta) < 0. \tag{49}$$

QED

C Proof of Theorem 3.1

From Lemma 3.1, there exists a stable minimum phase filter $F(j\omega)$ of order bounded by n such that:

$$\forall \omega \in \Omega, |F(j\omega)| < U(\omega)$$

if and only if there exist two real symmetric matrices X_N and X_D such that condition (11) and condition (10) hold. From the Kalman Yakubovic Popov Lemma, conditions (11) are equivalent to condition (14) and condition (15).

Note that in equation (10), the parameter ω is restricted to the bounded interval $\Omega = (\overline{\omega},\ \underline{\omega})$. In order to apply Lemma 3.1, we introduce a change of variable, called in the Automatic Control context loop-shifting [DV75]. The interest of such change of variable is to be written as an LFT. The new variable δ is defined in order that $\omega \in \Omega \Longleftrightarrow \delta \in (0, +\infty)$. We straightforwardly obtain:

$$\left\{ \begin{array}{ccc} \omega & = & \overline{\omega}\frac{\delta + \overline{\omega}}{\delta + \underline{\omega}} \\ \frac{1}{\omega} & = & \frac{1}{\overline{\omega}}\frac{\delta + \underline{\omega}}{\delta + \overline{\omega}} \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{ccc} \omega & = & \delta \star T_\Omega \\ \frac{1}{\omega} & = & \delta \star \widetilde{T}_\Omega \end{array} \right. \tag{50}$$

with T_Ω and $\widetilde{T}_\Omega$ defined in Theorem 3.1. With this change of variable, equation (10) can be written as

$$\forall \delta \in (0,\ +\infty), \qquad \Phi(\delta)^* \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \Phi(\delta) < 0 \tag{51}$$

with

$$\Phi(\delta) = \delta \star \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with A, B, C and D complex matrices defined as in Theorem 3.1. From the Lemma 3.2, condition (16) is obtained from condition (51).

QED

[4] This requires that the matrix $(\delta^{-1}I - A)$ is of full row.