POPOV-ZAMES-FALB MULTIPLIERS AND CONTINUITY OF THE INPUT/OUTPUT MAP

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Abstract: It has been conjectured in Kulkarni and Safonov [2002] that there might exist feedback systems with monotone nonlinearities for which positivity-based multiplier results assure stability, but which are not incrementally stable. The conjecture is demonstrated to be true via a simple example.

Keywords: Nonlinear systems, incremental stability, Lur'e systems, Popov multipliers, Zames-Falb mutilpliers, Lipschitz continuous systems, jump resonance.

1. INTRODUCTION

Incremental stability was recently proposed as a powerful tool for analyzing qualitative properties (Fromion *et al.* [1996]) and quantitative properties (Fromion *et al.* [1995, 1999, 2001]) of nonlinear systems.

We focus our attention in this note to nonlinear systems of the Lur'e-type, which consist of a feedback interconnection of a Linear Time-Invariant (LTI) element and a memoryless single-input-single-output nonlinear element. The issue of precise tests for incremental stability is, in the general case, a difficult problem though there is a result showing that the problem is equivalent to solution of certain Hamilton-Jacobi-Isaac inequalities (Fromion *et al.* [1995], Romanchuk and James [1996]) or the stability of some Linear differential inclusion (Fromion *et al.* [2003]). On this basis, it is possible to prove that exact testing incremental stability of Lur'e-type systems is an NP hard problem. In this context, the sufficient conditions for incremental stability based on incremental conicity conditions proposed in Zames [1966*a*] (see also Sandberg [1964]) is of first interest. Nevertheless, incremental conicity conditions can be conservative (see Fromion *et al.* [2003]). It is then of the first interest to know if it is possible, as in the finite-gain stability context to use multipliers in order to decrease the possible conservatism of incremental stability conditions, see *e.g.* Ly *et al.* [1994], Safonov [1980], Desoer and Vidyasagar [1975], Willems [1971], Zames [1966*b*], Zames and Falb [1967, 1968]. This question has been addressed in the work presented in Kulkarni and Safonov [2002]. The authors of Kulkarni and Safonov [2002] proved that Popov-Zames-Falb-type and related dynamical multipliers fail to preserve the incremental positivity of monotone nonlinearities. That implies that the multipliers belonging to the Popov-Zames-Falb set of multipliers, are currently known only to decrease the conservatism of the incremental stability conditions only in the trivial case in which the multipliers are constant (Kulkarni and Safonov [2002]).

By the way, even if dynamical type multipliers can not be used for proving incremental stability, that does not imply that the class of systems which satisfy Popov-like conditions are not incrementally stable. In a first analysis, this question seems probably naive. But, if we note that the systems which satisfy Popov like condition are actually continuous at constant inputs, it is of a great interest to know if the continuity property remains true when non-constant inputs are considered. In Kulkarni and Safonov [2002], it has been conjectured that there might exist feedback systems with monotone nonlinearities for which positivity-based multiplier results assure stability, but which are not incrementally stable nor continuous. We will show that the conjecture made in Kulkarni and Safonov [2002] is true: there exist Lur'e systems with a monotone nonlinearity for which positivity-based multiplier results assure finite gain stability but which are not incrementally stable.

1.1 Notation and definitions

The $\mathcal{L}_1$-norm of $f : [t_0, \infty) \mapsto \mathbb{R}^n$ is $\|f\|_1 = \int_{t_0}^{\infty} \|f(t)\| dt$. The $\mathcal{L}_2$-norm of $f : [t_0, \infty) \mapsto \mathbb{R}^n$ is $\|f\|_2 = \sqrt{\int_{t_0}^{\infty} \|f(t)\|^2 dt}$. The inner product between two functions f and g belonging to $\mathcal{L}_2$ is $< f, g >= \int_{t_0}^{\infty} f(t)' g(t) dt$. The *causal truncation* at $T \in [t_0, \infty)$, denoted by $P_T f$ gives $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise. The *extended space*, $\mathcal{L}_2^e$ is composed with the functions whose causal truncations belong to $\mathcal{L}_2$. For convenience, $\|P_T f\|_2$ is denoted by $\|f\|_{2,T}$.

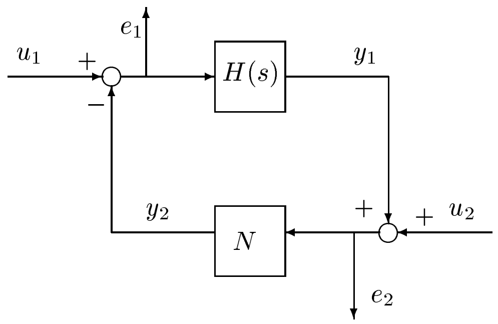

Fig. 1. Internal stability

Let us consider that the nonlinear system, namely Σ, is defined as the interconnection between a stable LTI system

$$H = \begin{cases} \dot{\xi}(t) = A\xi(t) + Be_1(t);\ \xi(t_0) = \xi_0 \\ y_1(t) = C\xi(t) \end{cases} \quad (1)$$

where $\xi(t), \xi_0 \in \mathbb{R}^n$, $y_1(t)$ and $y_2(t) \in \mathbb{R}$ and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$ and the following nonlinearity:

$$e_1(t) = u_1(t) - N(y_1(t) + u_2(t))$$

where $u_1(t)$ and $u_2(t) \in \mathbb{R}$ and where N is defined from $\mathbb{R}$ into $\mathbb{R}$ and it is assumed to be a Lipschitz continuous, nondecreasing and unbiased function of its argument. N generates an operator (this operator is classically called, in the Russian literature, the *Nemytsky operator*), defined from $\mathcal{L}_2$ into $\mathcal{L}_2$ which is incremental passive (positive) on $\mathcal{L}_2$, *i.e.*, $< N(u_1) - N(u_2) | u_1 - u_2 >\geq 0$ for any $u_1, u_2 \in \mathcal{L}_2$.

The system Σ is said to be finite gain stable if there exists $\gamma \geq 0$ such that $\|\Sigma(u)\|_2 \leq \gamma \|u\|_2$ for all $u \in \mathcal{L}_2$. Σ has a finite incremental gain if there exists $\eta \geq 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta \|u_1 - u_2\|_2$ for all $u_1, u_2 \in \mathcal{L}_2$. Σ is said to be incrementally stable (Lipschitz continuous) if it is stable, *i.e.*, it maps $\mathcal{L}_2$ to $\mathcal{L}_2$, and has a finite incremental gain. Σ is said to be continuous at $u_1 \in \mathcal{L}_2$ if it maps $\mathcal{L}_2$ into $\mathcal{L}_2$ and if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \epsilon$ for any $u_2 \in \mathcal{L}_2$ such that $\|u_1 - u_2\|_2 \leq \delta$.

2. BACKGROUND AND MAIN RESULT

A classical way to prove $\mathcal{L}_2$ gain stability of a nonlinear system of the Lur'e-type, is to use the circle or the small gain theorem or the passive theorem. It is moreover possible, when the nonlinearity satisfies specific properties, to consider positivity-based multiplier results (see *e.g.* Zames [1966*b*], Zames and Falb [1968], Willems [1971], Desoer and Vidyasagar [1975]). In our case, since the nonlinearity is assumed to be nondecreasing, we can use multiplier belonging to the Zames-Falb multipliers set, $\mathcal{M}$, which corresponds to the set of multipliers which posses a Fourier transform of this form:

$$M(j\omega) = m_0 - Z(j\omega)$$

on which, $z(t)$, the impulse response associated to $Z(jw)$, is nonnegative, *i.e.*, $z(t) \geq 0$ for any $t \in \mathbb{R}$, and where the $\mathcal{L}_1$ norm of the multiplier is nonnegative too, *i.e.*, $m_0 - \int_{-\infty}^{\infty} \|z(t)\| dt \geq 0$.

Let us note that the classical Popov multipliers are limiting case of Zames-Falb multipliers since

$$1 + qj\omega = \lim_{\epsilon \to 0} \frac{1 + qj\omega}{1 + \epsilon j\omega}, \quad \frac{1}{1 + qj\omega} = \lim_{\epsilon \to 0} \frac{1 + \epsilon j\omega}{1 + jq\omega}.$$

Following this remark, $\mathcal{M}$ is called the *Popov-Zames-Falb set of multipliers*. On this basis, one has this classical and powerful result:

Theorem 1. (Zames and Falb [1968]) Let H be a stable LTI system given by (1) and N be an unbiased nondecreasing nonlinearity. If there exists $M \in \mathcal{M}$ such that

$$Re(M(j\omega)H(j\omega)) \geq \delta > 0 \text{ for all } \omega \in \mathbb{R} \quad (2)$$

then Σ is finite gain stable.

In Kulkarni and Safonov [2002], the authors look for conditions allowing to extend this result to the incremental framework. They demonstrate that

Popov-Zames-Falb and related dynamical multipliers fail to preserve the incremental positivity of an incremental positive nonlinearity. They proved in fact that only the constant multipliers belonging to $\mathcal{M}$ allow to preserve the incremental positivity.

Since multiplier stability tests generally provided only sufficient conditions, the obtained result does not imply that the system which satisfies the assumptions of Theorem 1 is not incrementally stable. In fact, as it is pointed out in Kulkarni and Safonov [2002], systems satisfying Theorem 1 are continuous with respect to any constant inputs. Following that fact, it remains to study if the continuity property remains true when non constant inputs are considered. The authors of Kulkarni and Safonov [2002] conjectured that there exist systems which satisfy the assumptions of Theorem 1 but which are not incrementally stable. We prove in sequel that this conjecture is true, more precisely, one has:

Proposition 2. (Main Result). The conditions of Theorem 1 do not necessarily ensure the incremental stability of Σ.

Proof: This proposition is proved in the following section.

3. PROOF OF PROPOSITION 2

In this section we demonstrate by means of a counterexample that the conditions of Theorem 1 are not sufficient to ensure the incremental stability of Σ.

3.1 Preliminary remark

We firstly note that Theorem 1 is stated in positive form. Following "loop-shifting" and "multiplier transformations" it is possible to consider nonlinearities belonging to any incremental sectors (*e.g.* Desoer and Vidyasagar [1975], Willems [1971], Zames and Falb [1968], Zames [1966*a*,*b*], Safonov and Wyetzner [1987]). Indeed, if a nonlinearity belongs to the incremental sector $[a, b]$, *i.e.*, for all $x_1 \neq x_2 \in \mathbb{R}$

$$a \leq \frac{N(x_1) - N(x_2)}{x_1 - x_2} \leq b$$

then the use of this "loop-shifting" transformation:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \triangleq \begin{bmatrix} 1 & -1/\tilde{b} \\ -a & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

leads to a nonlinearity, $\tilde{y} = \tilde{N}(\tilde{x})$, belonging to the incremental sector $[0, \infty)$ if $\tilde{b} = b + \epsilon$ and $\epsilon > 0$, *i.e.*, for all $x_1 \neq x_2 \in \mathbb{R}$

$$0 \leq \frac{\tilde{N}(x_1) - \tilde{N}(x_2)}{x_1 - x_2} \leq \frac{(b-a)(b+\epsilon)}{\epsilon}.$$

Applying the inverse transformation of (3) to the linear part of the interconnected system leads to this linear system [1] :

$$\tilde{H}(s) = \frac{H(s) + 1/\tilde{b}}{1 + aH(s)}.$$

The main interest of this technique is mainly due to the fact that the loop-shifting transformations do not affect the stability property of the closed-loop system, *e.g.* the closed-loop system defined by H and N is incrementally stable if and only if the closed-loop system defined by $\tilde{H}$ and $\tilde{N}$ is incrementally stable.

Following this preliminary remark, if we are able to find an interconnected system with a nonlinearity belonging to an incremental sector $[a, b]$ and such that there exists $M \in \mathcal{M}$ such that

$$Re\left(M(j\omega)\frac{H(j\omega) + 1/\tilde{b}}{1 + aH(j\omega)}\right) \geq \delta > 0 \text{ for all } \omega \in \mathbb{R} \tag{4}$$

which is not incrementally stable then there exists an increasing nonlinearity, *i.e.*, $\tilde{N}$, and a LTI system, *i.e.*, $\tilde{H}$, such that conditions of Theorem 1 are satisfied and such the interconnected system is not incrementally stable.

3.2 A Saturation Nonlinearity System that Satisfies Zames-Falb Theorem

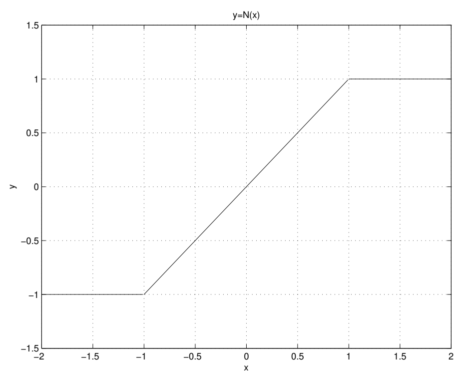

Fig. 2. The saturation nonlinearity $N(x)$

We now develop the desired counterexample. Let us define, Σ, as a specific interconnection between

$$H(s) = \frac{909}{(s^2 + 0.1s + 1)(s + 100)} \tag{5}$$

[1] The nonlinearity and the LTI system are connected through a negative sign.

and the following nonlinearity $N(x)$ defined by

$$\begin{cases} -1 & \text{for} \quad x \leq -1-\delta \\ ax^3+bx^2+cx+d & \text{for} \quad x \in [-1-\delta, -1+\delta] \\ x & \text{for} \quad x \in [-1+\delta, 1-\delta] \\ ax^3-bx^2+cx-d & \text{for} \quad x \in [1-\delta, 1+\delta] \\ 1 & \text{for} \quad x \geq 1+\delta \end{cases} \tag{6}$$

with $\delta = 0.01$, $a = -2.47\ 10^{-6}$, $b = 250$, $c = 500.5$ and $d = 249.5$. Since the gradient of N is such that $0 \leq \frac{\partial N(x)}{\partial x} \leq 1$ then N is incrementally inside sector $[0, 1]$. The linear system H has state-space realization $H(s) = C(Is - A)^{-1}B$ where

$$A = \begin{bmatrix} -100.1 & -0.0859 & -0.0061 \\ 128 & 0 & 0 \\ 0 & 128 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 8 & 0 & 0 \end{bmatrix}' ; c = \begin{bmatrix} 0 & 0 & 0.0069 \end{bmatrix}';$$

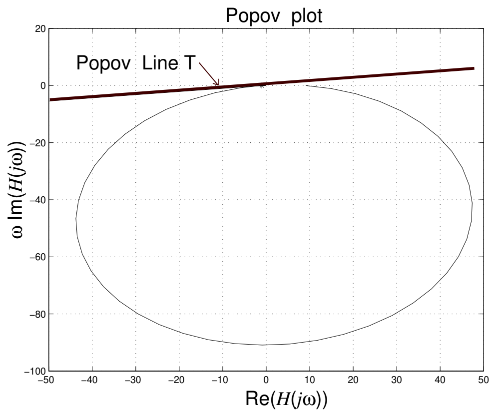

Fig. 3. Popov plot

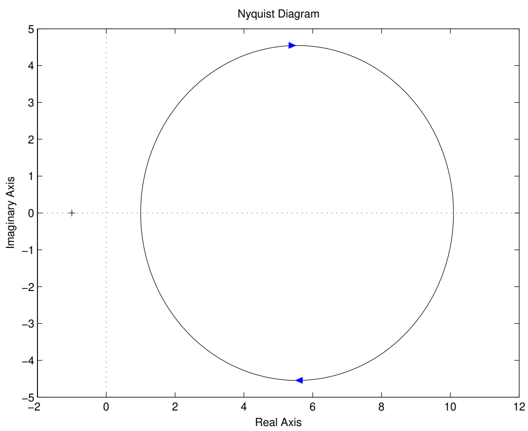

Fig. 4. Nyquist plot of $\frac{1+9s}{1+10^{-6}s}(H(s) + 1/\tilde{b})$

The Popov plot (see Figure 3) lies +0.09 to the right of the Popov line having slope 1/9 passing through the real axis point $-1/\tilde{b} = -0.99999$; equivalently,

$$\inf_{w\in\mathbb{R}} Re\left[(1 + jw9)(H(jw) + 1/\tilde{b})\right] \geq 0.09.$$

This inequality also holds with a slight perturbation to the Popov multiplier in the Zames-Falb class $\mathcal{M}$ (see Figure 4) , *i.e.*,

$$M(s) = \frac{1+9s}{1+10^{-6}s} \in \mathcal{M},$$
$$\inf_{w\in\mathbb{R}} Re\left[\frac{1+9j\omega}{1+10^{-6}j\omega}(H(j\omega) + 1/\tilde{b})\right] \geq 0.9$$

Thus, the Zames-Falb stability conditions are satisfied, establishing that the system (5)-(6) is finite-gain stable. It remains to show that the system is not incrementally stable.

3.3 Proof that Σ is not incrementally stable

To show that the system Σ of (5)-(6) is not incrementally stable, we make use of the the following proposition, which we deduce from a result of Fromion *et al.* [2003]:

Proposition 3. Suppose N belongs to the incremental sector $[a, b]$. The interconnection between H and N is not incrementally stable if there exists a time-varying and measurable matrix, $A(t)$, belonging to the polytope of matrices $\mathcal{A} = \{A(t)|A(t) = A + Bk(t)C|k(t) \in [a, b]\}$ such the following associated linear system, $\dot{z}(t) = A(t)z(t)$, is instable.

Proof: The proof is in the appendix.

Following Proposition 3, we select in the polytope of matrices $\mathcal{A}$, a $\pi/2$ -periodic matrix defined by:

$$A(t) = \begin{cases} A - BC \text{ for } t \in [0, \tau_1], \\ \quad\quad A \text{ for } t \in (\tau_1, \pi/2]. \end{cases}$$

with $\tau_1 = 0.34$ and where

$$A = \begin{bmatrix} -100.1 & -0.0859 & -0.0061 \\ 128 & 0 & 0 \\ 0 & 128 & 0 \end{bmatrix}$$

$$A - BC = \begin{bmatrix} -100.1 & -0.0859 & -0.0616 \\ 128 & 0 & 0 \\ 0 & 128 & 0 \end{bmatrix}$$

To prove that this periodic system is instable, we use the Floquet technique. Let us then compute the transition matrix associated to $\dot{z}(t) = A(t)z(t)$, *i.e.*

$$\Phi(T_0, 0) = e^{(A-BC)\tau_1} e^{A(\pi/2-\tau_1)} = \begin{bmatrix} -0.05 & -0.04 & 0 \\ -2.9 & -2.3 & -0.01 \\ 84.8 & 66.1 & -0.06 \end{bmatrix}$$

The periodic time varying system is instable since the module of eigenvalues of $\Phi(T_0, 0)$ are:

$$|\lambda_1| = 1.9847 \ |\lambda_2| = 0 \ \text{ and } |\lambda_3| = 0.4441$$

and one of them is strictly greater than one. Thus, by Proposition 3, the the system Σ given by (5)-(6) is not incrementally stable, which completes the proof.

4. DISCUSSION

In classical situation, Proposition 3 might lead one to firstly consider the stability of set of all the constant matrices belonging to the polytope $\mathcal{A}$. However, for nonlinear systems satisfying the incremental conicity conditions of Theorem 1, it is easy to prove that all the constant matrices belonging to the polytope $\mathcal{A}$ are necessarily stable (see Ly *et al.* [1994]). Thus, of necessity, our counterexample involves an instable *time-varying* $A(t)$.

Though we do not describe all the details here, the basic idea that we employed to discover the counterexample (5)-(6) involved using describing function method identify a system with a jump resonance that satisfies the Popov criterion. Describing function analysis allows to one to deduce the frequency and the amplitude of a periodic input around which the associated time-varying periodic linearization is possibly instable. Indeed, describing function analysis of the counterexample system (5)-(6) predicts a jump resonance for a sinusoidal input $u_2(t) = \sin(2t)$, *i.e* the system has two possible steady-states for the same periodic input and it can be jump from one steady-state to the other. Steady-state periodic responses for simulations with the input $u_2(t) = \sin(2t)$, but with slightly different initial conditions, are shown in Figure 5.

It is easy to prove that time-invariant incrementally stable systems have necessarily a unique steady-state periodic response for any periodic input signal belonging to $\mathcal{L}_2^e$; for otherwise, there would be an instable boundary in the state-space where initial states on either side of the boundary would lead to different steady-state periodic responses. This would mean that for any initial state on this boundary the system response near such periodic input signals would be discontinuous with respect to the topology induced by the $\mathcal{L}_2^e$-norm; *i.e.*, the system would not be incrementally stable.

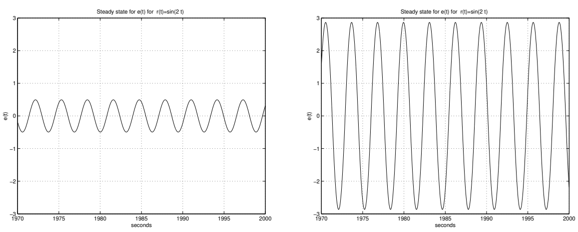

Fig. 5. Two different steady states for the same input $u_2(t) = \sin(2t)$.

5. CONCLUSION

The theory and the counterexample in this paper prove that when Popov and Zames-Falb multiplier stability results establish finite-gain stability of systems with incrementally conic nonlinearities, the system may nevertheless fail to be incrementally stable (*i.e.*, the closed-loop response may fail to be Lipschitz continuous with respect to the $\mathcal{L}_2^e$-norm). The input signals about which continuity fails for such systems are always time-varying signals.

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Appendix A. PROOF OF PROPOSITION 3

A.1 Some recalls

Let us recall that a causal operator Σ, defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ has a linearization (a Gâteaux derivative) at $u_0 \in \mathcal{L}_2^e$ if for any $T \in [t_0, \infty)$ and for any $h \in \mathcal{L}_2^e$, there exists a continuous linear operator $D\Sigma_G[u_0]$ from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ such that

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma(u_0 + \lambda h) - \Sigma(u_0)}{\lambda} - D\Sigma_G[u_0](h) \right\|_{2,T} = 0.$$

We now recall a powerful result in the context of incrementally bounded systems.

Proposition 4. (Fromion and Scorletti [2003]) Let us assume that Σ has a linearization for any input in $\mathcal{L}_2^e$ and let the state space representation of each linearization of Σ be minimal. Then Σ is incrementally stable only if all its linearizations are exponentially stable.

A.2 Proof of Proposition 3

The linearization of Σ along a specific input $u_r = (u_{1r}, u_{2r})'$ corresponds to the interconnection between this LTI system

$$\bar{H} = \begin{cases} \dot{\bar{\xi}}(t) = A\bar{\xi}(t) + B\bar{e}_{1r}(t); \ \bar{\xi}(t_0) = 0 \\ \bar{y}_{1r}(t) = C\bar{\xi}(t) \end{cases}$$

and the following time-varying gain

$$\bar{e}_{1r}(t) = \bar{u}_{1r}(t) - k(t)(\bar{y}_{1r}(t) + \bar{u}_{2r}(t)) \qquad \text{(A.1)}$$

with

$$k(t) \triangleq \frac{\partial N}{\partial x}(y_{1r}(t) + u_{2r}(t))$$

and where $y_{1r}(t)$ is associated to Σ for the input $u_r(t) = (u_{1r}(t), u_{2r}(t))'$ and for the initial condition ξ_0. Let us note that the realization of the linearization is bounded and minimal since the realization of H is minimal and $k(t)$ is bounded (see Lemma 3 in Silverman and Anderson [1968]). So, following Proposition 4, a necessary condition for the incremental stability of Σ is the exponential stability of linearizations defined by

$$\dot{z} = Az(t) - Bk(t)Cz(t) \qquad \text{(A.2)}$$

where, by definition, each $k(t)$ belongs to $[a, b]$ for any $t \in [t_0, \infty)$.

In the other hand, $\mathcal{A}$ defined a linear differential inclusion given by

$$\dot{z} = Az(t) - Bw(t)Cz(t) \qquad \text{(A.3)}$$

where input $w(t)$ is a measurable signal such that $a \le w(t) \le b$ for any $t \in [t_0, \infty)$.

Proposition 3 is proved if the instability associated to the LDI system (A.3) implies the instability of a possible linearization of Σ. Thus, the necessity is proved if for the same initial condition, the solutions of system (A.3) are the solutions of system (A.2), *i.e.*, for any measurable input $w(t)$ such that $a \le w(t) \le b$, there at least exists an input u_{2r} belonging to $\mathcal{L}_2^e$, such that

$$w(t) = \frac{\partial N}{\partial x}(y_{1r}(t) + u_{2r}(t)) \text{ a.e.} \qquad \text{(A.4)}$$

This fact can be proved in several steps. The first step is to prove that it is always possible to choose the input of N. To this purpose, let us assume that the input of N is $\nu(t)$. Let us consider the output of the open-loop system associated to the connection between H and N, *i.e.*, $y_{1\nu}(t) = H(N(\nu(t))$. Now, if we consider the closed-loop system and if we define u_{2r} as

$$u_{2r}(t) = \nu(t) - y_{1\nu}(t)$$

then by definition the input of N is $\nu(t)$.

Let us now prove that for any measurable $w(t)$, there exists $u_{2r}(t) \in \mathcal{L}_2^e$ such that (A.4) is satisfied. To this purpose, let us recall that a measurable function is a step function limit (see *e.g.* Chae [1994]) then for any $w(t)$, there exist step functions, ϕ_n such that

$$\lim_{n \to \infty} \phi_n(t) = u(t) \text{ a.e.}$$

Moreover, since $\frac{\partial N}{\partial x}$ is a continuous function, there exists for any ϕ_n, at least a step function ψ_n, such that

$$\frac{\partial N}{\partial \sigma}(\psi_n) = \phi_n.$$

We then deduce that there exists an input belonging to $\mathcal{L}_2^e$ defined by

$$u_{2r}(t) = \lim_{n \to \infty} \psi_n(t) - y_{1r}(t)$$

and such that (A.4) is satisfied. Indeed, $u_{2r}(t)$ is the sum of two functions belonging to $\mathcal{L}_2^e$ since the closed-loop is assumed well-defined that ensures that $e_{ir}(t) \in \mathcal{L}_2^e$ and $|\psi_n(t)|$ is bounded by $K \triangleq \max\{|a|, |b|\}$. We thus deduce that $f(t)$ defined by $f(t) = \lim_{n \to \infty} \psi_n(t)$, is square integrable on any finite support since $\|f(t)\|^2$ is a bounded and measurable function on any finite interval of time. This last claim allows to conclude the proof. $\square$