Technical report GREYC Auto 2005

Parameterized H_∞ controller design for adaptive trade-off by finite dimensional LMI optimization

M. Dinh[△, ‡], G. Scorletti[△], V. Fromion[‡], E. Magarotto[△]

△ LAP, ISMRA, 6 bd du Maréchal Juin, F14050 Caen cedex, France.

{scorletti@greyc.ismra.fr}

tel: 00 33 2 31 45 27 12 fax: 00 33 2 31 45 27 60

‡ LASB, INRA–Montpellier, 2 place Viala, 34060 Montpellier, France.

{fromion@ensam.inra.fr}

Abstract

In this paper, we consider the design of an H_∞ trade-off dependent controller, that is, a controller such that, for a given Linear Time-Invariant plant, a set of performance trade-offs parameterized by a scalar θ is satisfied. The controller state space matrices are explicit functions of θ. This new problem is a special case of the design of a parameter dependent controller for a parameter dependent plant, which has many application in Automatic Control. This last design problem can be naturally formulated as a convex but infinite dimensional optimization problem involving parameter dependent Linear Matrix Inequality (LMI) constraints. In this paper, we propose finite dimensional (parameter independent) LMI constraints which are equivalent to the parameter dependent LMI constraints. The parameter dependent controller design is then formulated as a convex finite dimensional LMI optimization problem. The obtained result is then applied to the trade-off dependent controller design. Numerical examples emphasize the strong interest of our finite dimensional optimization problem with respect to alternative approaches and with respect to the trade-off dependent control application.

Keywords parameter dependent LMI, parameter dependent H_∞ control, trade-off dependent control, gain scheduling control, parameter dependent Lyapunov function.

1 Introduction

Trade-off dependent controller design During the last twenty years, dramatic advances were accomplished in the design of Linear Time-Invariant (LTI) controllers for LTI plants using the frequency domain approach. The so-called H_∞ control approach [Zam81, DGKF89, SP96a] is at now a mature design method. The existing methods focus on designing one particular LTI controller for one particular set of design specifications corresponding to one particular performance trade-off.

Nevertheless, in some control problems, for a given plant, an important issue is to retune in situ the controller in order to ensure different performance trade-offs. Such retunings can be performed during the controller exploitation [Ala01]. In these conditions, a new controller design by an Automatic Control engineer has to be avoided. A promising solution to this practical problem is the design of a trade-off dependent controller, that is, a controller whose gains are explicit functions on a continuous set of trade-offs. Retuning the controller just amounts to select a different trade-off which is easy and affordable without expertise in Automatic Control. Another possible application can be the on-line performance retuning for *e.g.* ship control with the rejection of wave disturbances which depend on the sea conditions [KYM$^+$01], active suspension in order to adapt it to road conditions [FB02], etc.. In these cases, the trade-off parameter is in a continuous interval. This is the key fact for using a trade off dependent controller instead of, *e.g.*, a finite number of controllers.

Using classical design methods, when specifications can be ensured using a low complexity controller, *e.g.* Proportional Integral, engineers use to investigate the link between controller gains and design specifications (such as time response, control input energy..) in order to obtain (re)tuning rules. This link (generally qualitative) can be established based on *e.g.* know-how, classical rules of automatic control.. Nevertheless, for ensuring more stringent specifications, more complex (multivariable) controllers are usually designed using modern methods (such as Linear Quadratic Gaussian, Model Predictive Control, H_∞ control..). The obtained controllers are defined by numerous parameters whose links with the design specifications are not crystal clear. This prevents the controller (re)tuning in order to ensure different trade-offs. Here again, if a trade-off dependent controller has been designed, the controller (re)tuning just amounts to choose the trade-off.

To our best knowledge, a complete solution to the trade-off dependent controller design was not previously proposed. Let us introduce a parameter $\theta \in [0, 1]$ which parameterizes the performance trade-offs. The problem is then to design a controller whose gains explicitly depend on this parameter θ. In the H_∞ control approach, the design of a controller is formulated as an optimization problem on weighted closed loop transfer functions. The considered closed loop transfer functions and the weighting functions are defined by the generalized plant [DGKF89]. The desired performance specifications are introduced through the choice of the weighting functions. As a consequence, the performance trade-off can be defined by choosing the weighting functions depending on θ. Even if, in our case, the plant does not depend on θ, the generalized plant depends on.

Parameter dependent controller design Our problem is thus a subcase of the design of a parameter dependent controller for a (generalized) parameter dependent plant. In the first part of this paper, we focus on this problem. Performance is considered through the use of the H_∞ norm. Nevertheless, other performance criteria (such as H_2, multiobjective...) can be considered in a similar way.

The design of a parameter dependent controller for a (generalized) parameter dependent plant has a strong interest since it encompasses numerous control design problems such as: gain scheduled control [SR99, FS03], saturated system control [Meg96], spatial system control [dCP02], adaptive control [FFM95], low cost identification [BSG$^+$04] to cite a few.

In this paper, we propose a solution to the parameter dependent control with an application to the trade-off dependent control. Some other applications of this solution, listed above, are probably more important or challenging. Nevertheless, in addition to its own interest, the performance of the obtained trade-off dependent controllers can be analyzed in the numerical examples using basic automatic control knowledge. Furthermore, the "best performance" can be readily evaluated. Thus, we prefer to focus on the trade-off dependent control since it allows to clearly evaluate the benefit of our solution.

Infinite dimensional optimization The parameter dependent controller design can be naturally formulated as a convex but infinite dimensional optimization problem as it involves parameter dependent Linear Matrix Inequalities. Its decision variables are (unknown) functions of the parameter. This infinite dimensional nature forbids a practical computation of a solution. The same difficulty arises in robustness analysis [FAG96, GAC96] or Linear Parameter Varying (LPV) control [Pac94, AGB95]. For these particular problems, the basic idea is to derive finite dimensional Linear Matrix Inequality (LMI) optimization problems [BEFB94]. Different approaches were proposed with possibly introducing conservatism, see section 2.5 for a discussion. One of the main features of these approaches is the choice of function sets for the decision variables. The most general choice was, up to now, rational with a priori chosen degree and denominator.

Proposed approach In this paper, we consider the more general problem of replacing a parameter dependent LMI by a finite number of parameter independent LMIs. Our approach is based on an extended version of the Kalman-Yakubovich-Popov Lemma and an elementary property of the real valued polynomials. Due to its potential important applications, some extension of Kalman-Yakubovich-Popov Lemma was recently largely investigated [RSF03], with an emphasis on the necessity [IMF00, IH03, Sch03]. It is strongly related to the μ computation for special uncertainty sets where the μ upper bound [FTD91, MSF97] gives the actual value of μ. Moreover, recent papers focus on numerical algorithms dedicated to the LMI based condition of the Kalman-Yakubovich-Popov Lemma (see [VBWH03] and the references therein). This solution is dramatically more efficient than a direct use of general purpose LMI solvers such as [GNLC95].

In this paper, we investigate applications of the extended Kalman-Yakubovich-Popov Lemma to problems of interest. We first reveal that a rational dependent parameter LMI with rational decision variables can be equivalently recast as a finite number of parameter independent LMIs in the case of one parameter[1]. In contrast with previous results, the denominator of the rational decision variables is *e.g.* no longer a priori chosen (Lemma 2.1, section 2.4). Such a contribution allows to dramatically improve the obtained result. This solution is part of our continuing investigation into the transformation of an infinite dimensional optimization problem into a finite one [SE98, BGSA01, RSF03, SRF04].

The obtained result is applied to propose a solution to the parameter dependent controller design in the form of a finite dimensional optimization problem involving LMIs. In this application, the interest of rational decision variables is crystal clear: the state space matrices of the parameter dependent controller are then rational functions in θ, which is a desirable feature for real-time implementation. This solution is the second contribution of this paper, perhaps one of the most interesting. The third contribution of the paper is then to derive a solution to the trade-off dependent controller design problem. This paper is based on the conference paper [DSFM03, DSFM04].

Paper outline Section 2 focus on the parameter dependent controller design problem: the problem is formulated in section 2.1. In section 2.2, its solution as an infinite dimensional optimization problem is presented. An equivalent finite dimensional optimization problem is then proposed in section 2.3. The proof, developed in section 2.4, is based on a solution to the general problem of replacing a parameter dependent LMI by a finite number of parameter independent LMIs. This solution is discussed with respect to existing results in section 2.5. Section 3 is an application to the trade-off dependent controller design with two numerical examples. Both numerical examples first emphasize the interest of the proposed approach to parameter dependent LMIs. For the sake of comparison, an alternative approach is developed for the considered problem in Appendix, section A. Second, both numerical examples emphasize on the practical use of the trade-off dependent controller design.

Notations and definitions I_n and $0_{m\times p}$ denotes respectively the $n \times n$ identity matrix and the zero matrix of size $m \times p$. The subscript is omitted when it is evident from the context. $P > 0$ denotes that the matrix P is positive definite. $\dim(T)$ is the dimension of the matrix T. The Redheffer star product [ZDG95] is denoted by $\star$. A Linear Fractional Transformation (LFT) is a particular Redheffer star product defined, with $(I - A\Delta)$

[1] In the case of several parameters, the proposed conditions are only sufficient, see the Appendix.

invertible, by:

$$\Delta \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = D + C\Delta(I - A\Delta)^{-1}B.$$

Elementary operations on LFT (addition, product, etc..) are defined, see *e.g.* [ZDG95].

2 Parameter dependent design for a parameter dependent plant

2.1 Problem formulation

Let us consider the LTI system $P(s,\theta)$ defined by a parameter dependent state space representation:

$$\left\{\begin{array}{rcrcrcr} \dot{x}(t) & = & A(\theta)x(t) & + & B_w(\theta)w(t) & + & B_u(\theta)u(t) \\ z(t) & = & C_z(\theta)x(t) & + & D_{zw}(\theta)w(t) & + & D_{zu}(\theta)u(t) \\ y(t) & = & C_y(\theta)x(t) & + & D_{yw}(\theta)w(t) & & \end{array}\right. \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ the command input, $y(t) \in \mathbb{R}^{n_y}$ the measured output, $z(t) \in \mathbb{R}^{n_z}$ the controlled output, $w(t) \in \mathbb{R}^{n_w}$ the disturbance input and θ a time-invariant scalar parameter (conventionally $\theta \in [0,1]$). The state space matrices of $P(s,\theta)$ are assumed to be rational functions of θ, well-posed on $[0,1]$. We then consider the following problem.

Extended H_∞ control problem Given $P(s,\theta)$ as defined in (1) and $\gamma > 0$ find, if there exist, a parameter dependent controller

$$K(s,\theta) = \frac{1}{s}I_n \star \left[\begin{array}{c|c} A_K(\theta) & B_K(\theta) \\ \hline C_K(\theta) & D_K(\theta) \end{array}\right] \quad (2)$$

where $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ are rational functions of θ, of limited degree and well-posed on $[0,1]$, such that, for any $\theta \in [0,1]$:

1. the closed loop system $P(s,\theta) \star K(s,\theta)$ is asymptotically stable;
2. $\|P(s,\theta) \star K(s,\theta)\|_\infty < \gamma$.

The state space matrices $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ of the controller (2) are required to be rational in θ of limited degree in order to obtain a controller implementation of reasonable complexity. A more complex dependence on θ is useless since it would be necessary to approximate these functions by, for example, rational or polynomial ones of limited degree for a practical implementation. For the sake of briefness, the control objective is defined using the H_∞ norm. Nevertheless, the proposed approach can be readily applied *e.g.* to the H_2 control problem [DGKF89] or to the multiobjective control problem [SGC97].

2.2 Proposed approach

In the following theorem, the design of a parameter dependent H_∞ controller is formulated as an optimization problem. It is obtained by a direct extension of the standard H_∞ control solution proposed in [SGC97].

Theorem 2.1 *Given $\gamma > 0$, there exists a parameter dependent controller*

$$K(s,\theta) = \frac{1}{s} I_n \star \left[\begin{array}{c|c} A_K(\theta) & B_K(\theta) \\ \hline C_K(\theta) & D_K(\theta) \end{array} \right]$$

such that, for any $\theta \in [0,1]$:

1. *the closed loop system $P(s,\theta) \star K(s,\theta)$ is asymptotically stable;*
2. $\|P(s,\theta) \star K(s,\theta)\|_\infty < \gamma$

if and only if there exist

- *symmetric parameter dependent matrices $\mathcal{X}(\theta) \in \mathbb{R}^{n\times n}$ and $\mathcal{Y}(\theta) \in \mathbb{R}^{n\times n}$ well-posed on $[0,1]$;*
- *parameter dependent matrices $\mathcal{A}(\theta) \in \mathbb{R}^{n\times n}$, $\mathcal{B}(\theta) \in \mathbb{R}^{n\times n_y}$, $\mathcal{C}(\theta) \in \mathbb{R}^{n_u\times n}$ and $\mathcal{D}(\theta) \in \mathbb{R}^{n_u\times n_y}$ well-posed on $[0,1]$*

satisfying (3) and (4) for any $\theta \in [0,1]$:

$$\begin{bmatrix} \mathcal{X}(\theta) & I \\ I & \mathcal{Y}(\theta) \end{bmatrix} > 0 \tag{3}$$

$$\left[\begin{array}{c|c|c|c} \begin{array}{c} A(\theta)\mathcal{X}(\theta) + \mathcal{X}(\theta)A(\theta)^T + \dots \\ B_u(\theta)\mathcal{C}(\theta) + (B_u(\theta)\mathcal{C}(\theta))^T \end{array} & (.)^T & (.)^T & (.)^T \\ \hline \begin{array}{c} \mathcal{A}(\theta) + \dots \\ (A(\theta) + B_u(\theta)\mathcal{D}(\theta)C_y(\theta))^T \end{array} & \begin{array}{c} A(\theta)^T\mathcal{Y}(\theta) + \mathcal{Y}(\theta)A(\theta) + \dots \\ \mathcal{B}(\theta)C_y(\theta) + (\mathcal{B}(\theta)C_y(\theta))^T \end{array} & (.)^T & (.)^T \\ \hline (B_w(\theta) + B_u(\theta)\mathcal{D}(\theta)D_{yw}(\theta))^T & (\mathcal{Y}(\theta)B_w(\theta) + \mathcal{B}(\theta)D_{yw}(\theta))^T & -\gamma I & (.)^T \\ \hline C_z(\theta)\mathcal{X}(\theta) + D_{zu}(\theta)\mathcal{C}(\theta) & C_z(\theta) + D_{zu}(\theta)\mathcal{D}(\theta)C_y(\theta) & D_{zw}(\theta) + D_{zu}(\theta)\mathcal{D}(\theta)D_{yw}(\theta) & -\gamma I \end{array} \right] < 0 \tag{4}$$

where $(.)^T$ denotes the transpose of the symmetric block.

A state space representation of a parameter dependent controller is then obtained with

$$\begin{aligned} \begin{bmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{bmatrix} &= \begin{bmatrix} L(\theta) & -J(\theta) & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix} \times \dots \\ & \left(\begin{bmatrix} I_n & 0 \\ 0 & B_u(\theta) \\ 0 & I_{n_u} \end{bmatrix} \mathcal{V}(\theta) \begin{bmatrix} \mathcal{X}(\theta)^{-1} & 0 \\ -C_y(\theta) & I_{n_y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A(\theta) & 0 \\ 0 & 0 \end{bmatrix} \right) \end{aligned} \tag{5}$$

where

$$\left[\begin{array}{cc} L(\theta) & -J(\theta) \end{array}\right] = \left(\left[\begin{array}{c} I_n \\ I_n \end{array}\right] \mathcal{X}(\theta) \left[\begin{array}{cc} I_n & \mathcal{Y}(\theta) \end{array}\right] \left[\begin{array}{ccc} I_n & 0 & 0 \\ 0 & -I_n & I_n \end{array}\right] \right) \star I_n$$

and where

$$\mathcal{V}(\theta) = \left[\begin{array}{cc} \mathcal{A}(\theta) & \mathcal{B}(\theta) \\ \mathcal{C}(\theta) & \mathcal{D}(\theta) \end{array}\right].$$

The optimization problem involving constraint (3) and constraint (4) is convex in the decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$, which is a desirable feature. Unfortunately, it is also infinite dimensional. As parameterized by θ, there is an infinite number of constraints. As functions of θ, the decision variables are in an infinite dimensional space. In this form, this prevents an efficient computation of the solution.

However, the problem considered in Theorem 2.1 is a little bit more general than the one considered in EXTENDED H_∞ CONTROL PROBLEM. Remind that the controller state space matrices $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ of the controller (2) are required to be rational functions of θ of limited degree. From equation (5), the decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ are then enforced to be rational functions in θ of limited degree. To this purpose, the optimization problem introduced in Theorem 2.1 is modified with:

$$\mathcal{X}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{X}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \quad \mathcal{Y}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{Y}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \quad \mathcal{V}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{V}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \tag{6}$$

where for $i = 0, \ldots, N$, $\mathcal{X}_i = \mathcal{X}_i^T \in \mathbb{R}^{n \times n}$, $\mathcal{Y}_i = \mathcal{Y}_i^T \in \mathbb{R}^{n \times n}$, $\mathcal{V}_i \in \mathbb{R}^{(n+n_u)\times(n+n_y)}$, and for $i = 1, \ldots, N$, $d_i \in \mathbb{R}$. In (6), only N is *a priori* chosen[2]. The integer N is a trade-off parameter. A small N allows to obtain a low complexity controller, that is, a controller whose state space matrices are rational functions of small degree, with the possible drawback of a poor performance. Performance can be improved by increasing N, with the possible drawback of a large complexity controller. The example presented in section 3.3 illustrate that good performance can be obtained with a small N.

With respect to EXTENDED H_∞ CONTROL PROBLEM, an interesting optimization problem is thus :

Given N, find the decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ defined by (6) such that for any $\theta \in [0,\ 1]$, constraint (3) and constraint (4) are satisfied.

In the next section, this infinite dimensional optimization problem is equivalently recast as a finite dimensional convex optimization problem involving Linear Matrix Inequalities which can be efficiently computed.

[2] N is the degree of the denominator when the rational function is written as the fraction of two polynomials in $\frac{1}{\theta}$. Thus, it is not necessary that $d_N \neq 0$ since the higher term coefficient is 1.

2.3 Finite dimensional solution

Before presenting the result, let us first associate to (6):

$$\mathcal{R}_{\mathcal{X}} = \left[\begin{array}{cccc} \mathcal{X}_N & \cdots & \mathcal{X}_1 & \mathcal{X}_0 \end{array}\right] \quad \mathcal{R}_{\mathcal{Y}} = \left[\begin{array}{cccc} \mathcal{Y}_N & \cdots & \mathcal{Y}_1 & \mathcal{Y}_0 \end{array}\right] \quad \mathcal{R}_{\mathcal{V}} = \left[\begin{array}{cccc} \mathcal{V}_N & \cdots & \mathcal{V}_1 & \mathcal{V}_0 \end{array}\right] \quad \mathcal{R}_{d,p} = \left[\begin{array}{cccc} d_N I_p & \cdots & d_1 I_p & I_p \end{array}\right]$$

$$J_p(c_i) = \left[\begin{array}{ccccc|c} 0 & I_p & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & I_p & 0 \\ -c_N I_p & \cdots & \cdots & \cdots & -c_1 I_p & I_p \\ \hline I_p & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & I_p & 0 \\ -c_N I_p & \cdots & \cdots & \cdots & -c_1 I_p & I_p \end{array}\right] \tag{7}$$

where c_i, $i = 1, \ldots, N$ are real scalars such that, for any $\theta \in [0,1]$, $1 + \sum_{i=1}^{N} \theta^i c_i \neq 0$. Let us introduce $\mathcal{L}(A_\Phi, B_\Phi, C_\Phi, D_\Phi, M, \mathcal{S}, \mathcal{G})$, with $\dim(\mathcal{S}) = \dim(\mathcal{G}) = \dim(A_\Phi)$, defined by:

$$\left[\begin{array}{c} C_\Phi^T \\ D_\Phi^T \end{array}\right] M \left[\begin{array}{cc} C_\Phi & D_\Phi \end{array}\right] + \left[\begin{array}{cc} A_\Phi^T(\mathcal{S}-\mathcal{G}) + (\mathcal{S}+\mathcal{G})A_\Phi - 2\mathcal{S} & (\mathcal{S}+\mathcal{G})B_\Phi \\ B_\Phi^T(\mathcal{S}-\mathcal{G}) & 0 \end{array}\right].$$

Theorem 2.2 *Given N, there exist decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ defined by (6) and well-posed on $[0,1]$ such that, for any $\theta \in [0,1]$, constraint (3) and constraint (4) are satisfied if and only if there exist*

- *symmetric matrices $\mathcal{X}_i \in \mathbb{R}^{n\times n}$ and $\mathcal{Y}_i \in \mathbb{R}^{n\times n}$, and matrices $\mathcal{V}_i \in \mathbb{R}^{(n+n_u)\times(n+n_y)}$, $i = 0, \ldots, N$;*
- *scalars d_i, $i = 1, \ldots, N$*

such that

(i) there exist a symmetric positive definite matrix $\mathcal{S}_0$ and a skew-symmetric matrix $\mathcal{G}_0$ such that

$$\mathcal{L}\left(A_{\Omega_0}, B_{\Omega_0}, C_{\Omega_0}, D_{\Omega_0}, \left[\begin{array}{cc} 0 & -\mathcal{W} \\ -\mathcal{W}^T & 0 \end{array}\right], \mathcal{S}_0, \mathcal{G}_0\right) < 0 \tag{8}$$

with

$$\mathcal{W} \triangleq \left[\begin{array}{cc} \mathcal{R}_{\mathcal{X}} & 2\mathcal{R}_{d,n} \\ 0 & \mathcal{R}_{\mathcal{Y}} \end{array}\right] \text{ and } \theta I \star \left[\begin{array}{c|c} A_{\Omega_0} & B_{\Omega_0} \\ \hline C_{\Omega_0} & D_{\Omega_0} \end{array}\right] \triangleq \left[\begin{array}{cc} \multicolumn{2}{c}{I_{2n}} \\ \hline \theta I \star J_n(c_i) & 0 \\ 0 & \theta I \star J_n(c_i) \end{array}\right].$$

(ii) there exist a symmetric positive definite matrix $\mathcal{S}$ and a skew-symmetric matrix $\mathcal{G}$ such that

$$\mathcal{L}\left(A_\Omega, B_\Omega, C_\Omega, D_\Omega, \begin{bmatrix} 0 & \mathcal{Z}(\gamma) \\ \mathcal{Z}(\gamma)^T & 0 \end{bmatrix}, \mathcal{S}, \mathcal{G}\right) < 0 \tag{9}$$

with

$$\mathcal{Z}(\gamma) = \left[\begin{array}{ccccc} \mathcal{R}_\mathcal{V} & 0 & 0 & 0 & 0 \\ \hline 0 & \mathcal{R}_\mathcal{X} & \mathcal{R}_{d,n} & 0 & 0 \\ 0 & 0 & \mathcal{R}_\mathcal{Y} & 0 & 0 \\ \hline 0 & \mathcal{R}_{d,n} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{d,n_w} & 0 \\ 0 & 0 & 0 & \gamma\mathcal{R}_{d,n_w} & 0 \\ 0 & 0 & 0 & 0 & \gamma\mathcal{R}_{d,n_z} \end{array}\right]$$

and with

$$\theta I \star \left[\begin{array}{c|c} A_\Omega & B_\Omega \\ \hline C_\Omega & D_\Omega \end{array}\right] \triangleq \begin{bmatrix} F_1(\theta)^T \\ F_2(\theta)F_3(\theta) \end{bmatrix}$$

where

$$F_1(\theta) = \left[\begin{array}{cc|cc|cccc} 0 & B_u(\theta) & A(\theta) & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & A(\theta)^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_w(\theta)^T & B_w(\theta)^T & 0 & -\frac{1}{2}I_{n_w} & 0 \\ 0 & D_{zu}(\theta) & C_z(\theta) & 0 & 0 & D_{zw}(\theta) & 0 & -\frac{1}{2}I_{n_z} \end{array}\right]$$

$$F_2(\theta) = \left[\begin{array}{c|cccc} \theta I \star J_{n+n_y}(c_i) & 0 & 0 & 0 & 0 \\ 0 & \theta I \star J_n(c_i) & 0 & 0 & 0 \\ 0 & 0 & \theta I \star J_n(c_i) & 0 & 0 \\ 0 & 0 & 0 & \theta I \star J_{n_w}(c_i) & 0 \\ 0 & 0 & 0 & 0 & \theta I \star J_{n_z}(c_i) \end{array}\right]$$

$$F_3(\theta) = \left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & C_y(\theta) & D_{yw}(\theta) & 0 \\ \hline \multicolumn{4}{c}{I_{n+n+n_w+n_z}} \end{array}\right]$$

The state space representation of a parameter dependent controller is then obtained using (5) with

$$\mathcal{X}(\theta) = \frac{\sum_{i=0}^N \theta^i \mathcal{X}_i}{1 + \sum_{i=1}^N \theta^i d_i}, \quad \mathcal{Y}(\theta) = \frac{\sum_{i=0}^N \theta^i \mathcal{Y}_i}{1 + \sum_{i=1}^N \theta^i d_i}, \quad \mathcal{V}(\theta) = \frac{\sum_{i=0}^N \theta^i \mathcal{V}_i}{1 + \sum_{i=1}^N \theta^i d_i}.$$

Computation: for a given value of γ, the optimization problem defined by (8) and (9) is an LMI feasibility problem since $\mathcal{W}$ and $\mathcal{Z}(\gamma)$ are affine in the decision variables $\mathcal{R}_\mathcal{X}$, $\mathcal{R}_\mathcal{Y}$,

$\mathcal{R}_{\mathcal{V}}$ and d_i, $i = 1, \ldots, N$. Another interesting problem is to minimize γ over LMI constraints (8) and (9). This minimization is a quasi convex optimization problem[3], the minimum value of γ can be found by performing a dichotomy on γ.

Remark The results presented in Theorem 2.2 do not depend on the choice of the scalars c_i. They can be chosen in order to improve the numerical resolution of the optimization problem defined by (8) and (9). For instance, it is chosen in order to limit the order of an LFT realization of $\Omega(\theta)$, thus reducing the computational burden of (9). As the obtained result is insensitive to this choice as long as $1 + \sum_{i=1}^{N} c_i \theta^i$ does not vanish on $[0, 1]$, the introduction of the scalars c_i comes from purely computational considerations.

2.4 Proof of Theorem 2.2

The problem considered in Theorem 2.1 is a particular case of an *infinite* dimensional convex optimization problem involving *parameter dependent* LMI constraints. Following [RSF03], the finite dimensional optimization problem of Theorem 2.2 is derived from Theorem 2.1 along two steps. The first step is the introduction of a finite parameterization of the decision variables. From equation (6), the decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ are naturally parameterized by a finite number of coefficients: the matrices $\mathcal{X}_i$, $\mathcal{Y}_i$ $\mathcal{V}_i$, $i = 0, \cdots, N$ and the scalars d_i, $i = 1, \cdots, N$. In order to obtain a finite number of optimization constraints, the second step is the application of the following lemma.

Lemma 2.1 *Let $H_1(\theta)$ and $H_2(\theta)$ be two matrices of rational functions of θ, well-posed on $[0, 1]$. Let C be a matrix and N an integer.*

There exists $\Upsilon(\theta)$ a (possibly structured) matrix of rational functions of θ of degree N, well-posed on $[0, 1]$:

$$\Upsilon(\theta) = \frac{\sum_{i=0}^{N} \theta^i \Upsilon_i}{1 + \sum_{i=1}^{N} \theta^i d_i}$$

such that

$$\forall \theta \in [0, 1], \quad H_1(\theta)(C + \Upsilon(\theta))H_2(\theta) + (H_1(\theta)(C + \Upsilon(\theta))H_2(\theta))^T < 0 \tag{10}$$

if and only if there exist $N + 1$ matrices Υ_i, $i = 0, \ldots, N$, and N scalar d_i, $i = 1, \ldots, N$, such that the two following conditions are satisfied:

[3]Quasi convexity can be proved by a simple adaptation of the proof of the (LMI) Generalized Eignevalue Problems, see [BEFB94].

(i) there exist a symmetric positive definite matrix $\mathcal{S}_d$ *and a skew symmetric matrix* $\mathcal{G}_d$ *such that:*

$$\mathcal{L}\left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -c_N & \cdots & \cdots & \cdots & -c_1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} & & 0 & & \\ \hline 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -c_N & \cdots & \cdots & \cdots & -c_1 \end{bmatrix}, \begin{bmatrix} 1 \\ \hline 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \cdots \right.$$
$$\left. \cdots \begin{bmatrix} 0 & -\mathcal{R}_{d,1} \\ -\mathcal{R}_{d,1}^T & 0 \end{bmatrix}, \mathcal{S}_d, \mathcal{G}_d \right) < 0 \tag{11}$$

(ii) there exist a symmetric positive definite matrix $\mathcal{S}$ *and a skew symmetric matrix* $\mathcal{G}$ *such that*

$$\mathcal{L}\left(A_H, B_H, C_H, D_H, \begin{bmatrix} 0 & \mathcal{U}(\Upsilon_i, d_i) \\ \mathcal{U}(\Upsilon_i, d_i)^T & 0 \end{bmatrix}, \mathcal{S}, \mathcal{G}\right) < 0 \tag{12}$$

where

$$\theta I \star \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] \triangleq \begin{bmatrix} H_1(\theta)^T \\ \bar{H}(\theta) H_2(\theta) \end{bmatrix}$$

and where $\mathcal{U}(\Upsilon_i, d_i)$ *is an affine function of* Υ_i *and of* d_i *such that*

$$\mathcal{U}(\Upsilon_i, d_i)\bar{H}(\theta) = \frac{(\Upsilon_0 + C) + \sum_{i=1}^N \theta^i (\Upsilon_i + d_i C)}{1 + \sum_{i=1}^N \theta^i c_i} \tag{13}$$

Remark The factorization (13) is always possible. Such a factorization is not unique. For instance, a factorization is given by $\bar{H}(\theta) = \theta I \star J_p(c_i)$, with p the number of columns of $\Upsilon(\theta)$, $J_p(c_i)$ defined by (7) and $\mathcal{U}(\Upsilon_i, d_i) = \begin{bmatrix} \Upsilon_N + d_N C & \cdots & \Upsilon_1 + d_1 C & \Upsilon_0 + C \end{bmatrix}$.

Interpretation Lemma 2.1 conditions are obtained through an extension of the Kalman-Yakubovich-Popov Lemma. The first point is that any matrix $\Phi(\theta)$ of rational functions of θ, well-posed for $\theta = 0$, has an LFT representation, that is, there exists four matrices A_Φ, B_Φ, C_Φ and D_Φ such that[ZDG95]:

$$\Phi(\theta) = \theta I \star \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & D_\Phi \end{array}\right]. \tag{14}$$

Lemma 2.2 (Extended Kalman-Yakubovich-Popov Lemma) *Let $\Phi(\theta)$ be a rational matrix function of θ, well-posed on $[0,1]$, defined by its LFT realization as in (14). Let M be a matrix.*
Then the condition

$$\forall \theta \in [0,1], \qquad \Phi(\theta)^T M \Phi(\theta) < 0$$

holds if and only if there exist a symmetric positive definite matrix $\mathcal{S}$ and a skew-symmetric matrix $\mathcal{G}$ such that

$$\mathcal{L}(A_\Phi, B_\Phi, C_\Phi, D_\Phi, M, \mathcal{S}, \mathcal{G}) < 0.$$

Let us discuss the application of Lemma 2.2 for proving Lemma 2.1 :

- Condition (11) is equivalent to the strict positivity of the rational function $\dfrac{1+\sum_{i=1}^N \theta^i d_i}{1+\sum_{i=1}^N \theta^i c_i}$ for any $\theta \in [0,\ 1]$. The polynomial $1+\sum_{i=1}^N \theta^i c_i$ has a constant sign on $[0,\ 1]$ as it does not vanish on $[0,\ 1]$. The polynomial $1+\sum_{i=1}^N \theta^i d_i$ is enforced to get a constant sign on $[0,\ 1]$: well posed of $\Upsilon(\theta)$ is thus ensured.

- Condition (12) is equivalent to:

$$\begin{aligned}\forall \theta \in [0,1],\ H_1(\theta)\left(\frac{(\Upsilon_0+C)+\sum_{i=1}^N \theta^i(\Upsilon_i+d_iC)}{1+\sum_{i=1}^N \theta^i c_i}\right)H_2(\theta)+\cdots \\ \cdots+\left(H_1(\theta)\left(\frac{(\Upsilon_0+C)+\sum_{i=1}^N \theta^i(\Upsilon_i+d_iC)}{1+\sum_{i=1}^N \theta^i c_i}\right)H_2(\theta)\right)^T<0.\end{aligned} \tag{15}$$

Thus, combining both items, condition (10) is obtained. Note that the contribution of Lemma 2.1 is allowed by the following fact: the sign of a real valued polynomial with no roots on an interval is constant on this interval. This elementary property was intensively used for formulating the approximation of real valued functions by real valued rational functions as a convex optimization problem (see *e.g.* [Che82]). It explains why the extension to the rational approximation of complex valued functions (and the model reduction problem) is a much more difficult problem.

Proof of Lemma 2.1 From the previous discussion, condition (i) is a necessary and sufficient condition for the well-posedness of $\Upsilon(\theta)$ on $[0,1]$. Condition (ii) ensures that condition (10) is satisfied.

Let us focus on condition (i). $\Upsilon(\theta)$ is well-posed on $[0,1]$ if and only if the polynomial $1+\sum_{i=1}^N d_i\theta^i$ does not vanish on $[0,1]$. As the polynomial is real valued, with real coefficients, its sign is then constant for any $\theta \in [0,1]$. The sign is positive since for $\theta=0$, the value of the polynomial is 1. Let us introduce the polynomial $1 + \sum_{i=1}^{N} c_i\theta^i$ that does not vanish on $[0, 1]$. Then, the polynomial $1 + \sum_{i=1}^{N} d_i\theta^i$ does not vanish on $[0, 1]$ if and only if

$$\forall\theta \in [0, 1], \ \frac{1 + \sum_{i=1}^{N} d_i\theta^i}{1 + \sum_{i=1}^{N} c_i\theta^i} > 0. \tag{16}$$

Since $\dfrac{1 + \sum_{i=1}^{N} d_i\theta^i}{1 + \sum_{i=1}^{N} c_i\theta^i} = \mathcal{R}_{d,1} \times \theta \star J_1(c_i)$, condition (16) is then equivalent to

$$\forall\theta \in [0, 1], \quad \begin{bmatrix} 1 \\ \theta \star J_1(c_i) \end{bmatrix}^T \begin{bmatrix} 0 & -\mathcal{R}_{d,1} \\ -\mathcal{R}_{d,1}^T & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \theta \star J_1(c_i) \end{bmatrix} < 0 \tag{17}$$

Lemma 2.2 is now applied with

$$M = \begin{bmatrix} 0 & -\mathcal{R}_{d,1} \\ -\mathcal{R}_{d,1}^T & 0 \end{bmatrix} \text{ and } \quad \Phi(\theta) = \begin{bmatrix} 1 \\ \theta \star J_1(c_i) \end{bmatrix}.$$

Condition (17) is satisfied if and only if there exist a symmetric positive definite matrix $\mathcal{S}_d$ and a skew symmetric matrices $\mathcal{G}_d$ such that condition (11) is satisfied.

Let us now consider condition (*ii*). Using (16), condition (10) is equivalent to condition (15). Since

$$\frac{(\Upsilon_0 + C) + \sum_{i=1}^{N} \theta^i(\Upsilon_i + d_i C)}{1 + \sum_{i=1}^{N} \theta^i c_i} = \mathcal{U}(\Upsilon_i, d_i)\bar{H}(\theta)$$

condition (15) is equivalent to

$$\forall\theta \in [0, 1], \ \begin{bmatrix} H_1(\theta)^T \\ \overline{H}(\theta)H_2(\theta) \end{bmatrix}^T \begin{bmatrix} 0 & \mathcal{U}(\Upsilon_i, d_i) \\ \mathcal{U}(\Upsilon_i, d_i)^T & 0 \end{bmatrix} \begin{bmatrix} H_1(\theta)^T \\ \overline{H}(\theta)H_2(\theta) \end{bmatrix} < 0. \tag{18}$$

Lemma 2.2 is now applied with

$$M = \begin{bmatrix} 0 & \mathcal{U}(\Upsilon_i, d_i) \\ \mathcal{U}(\Upsilon_i, d_i)^T & 0 \end{bmatrix} \text{ and } \quad \Phi(\theta) = \begin{bmatrix} H_1(\theta)^T \\ \overline{H}(\theta)H_2(\theta) \end{bmatrix}$$

Condition (18) is satisfied if and only if there exist a symmetric positive definite matrix $\mathcal{S}$ and a skew symmetric matrices $\mathcal{G}$ such that condition (12) is satisfied. □

Remark Lemma 2.1 can be extended to the case when H_1, H_2 and Υ are rational functions of *several* parameters $\theta_1, \cdots, \theta_m$, see Appendix. In this case, the conditions corresponding to (11) and (12) are no longer necessary.

The interest of Lemma 2.1 with respect to previously published results will be discussed in section 2.5. Theorem 2.2 is now proved by applying Lemma 2.1.

Proof of Theorem 2.2 Lemma 2.1 is first applied to condition (4). Condition (4) can be factorized in the form of (10) with $H_1(\theta) = F_1(\theta)$, $H_2(\theta) = F_3(\theta)$ and

$$C + \Upsilon(\theta) = \left[\begin{array}{c|cccc} \mathcal{V}(\theta) & & & 0 & \\ \hline \multirow{2}{*}{0} & \mathcal{X}(\theta) & I_n & 0 & 0 \\ & 0 & \mathcal{Y}(\theta) & 0 & 0 \\ \hline & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_w} & 0 \\ & 0 & 0 & \gamma I_{n_w} & 0 \\ & 0 & 0 & 0 & \gamma I_{n_z} \end{array}\right].$$

Lemma 2.1 is then applied with $\bar{H}(\theta) = F_2(\theta)$ and $\mathcal{U}(\Upsilon_i, d_i) = \mathcal{Z}(\gamma)$. Note that, in the special case of (4), condition (16) is implied by condition (15) and thus can be dropped. It only remains condition (12) in Lemma 2.1, which leads to condition (9).

Condition (3) can be factorized in the form of (10) with $H_1(\theta) = I$, $H_2(\theta) = I$ and

$$C + \Upsilon(\theta) = \begin{bmatrix} -\mathcal{X}(\theta) & -2I_n \\ 0 & -\mathcal{Y}(\theta) \end{bmatrix}.$$

Lemma 2.1 is then applied with $\bar{H}(\theta) = \begin{bmatrix} \theta I \star J_n(c_i) & 0 \\ 0 & \theta I \star J_n(c_i) \end{bmatrix}$ and $\mathcal{U}(\Upsilon_i, d_i) = -\mathcal{W}$. Note that we use the same scalars c_i for both conditions (3) and (4). Here again, it only remains (12) in Lemma 2.1, which is (8). □

2.5 Discussion of Lemma 2.1

Optimization problems involving parameter dependent LMIs with parameter dependent decision variables usually arise when considering robustness analysis of systems with parametric uncertainties or when considering LPV analysis and design, see Table 1. In this section, we propose a non exhaustive review of important results related to our problem.

The LMIs and/or the decision variables can depend on several (real) parameters. Many approaches were proposed in order to obtain a convex finite dimensional optimization for particular problems. By particular problems, we mean that the considered inequalities consist in special choices of $H_1(\theta)$, C, $\Upsilon(\theta)$ and $H_2(\theta)$ in (10) (with θ a vector). For instance, many approaches consider a parameter dependent LMI where $C = 0$ and $H_2(\theta) = I$, see *e.g.* the robustness analysis [DS98, FAG96, Bli02] with: $A(\theta)^T P(\theta) + P(\theta)A(\theta) < 0$. This LMI

robustness analysis	[FAG96, dOBG99, DS00]
LPV analysis	[GAC96, TdS01, DS98, IS01, LH97, Sch98]
LPV control	[AGB95, YS97, BPPB93, BP94, Pac94, Hel95, AG95] [SE98, Sch01, TdS02, Bec95, WYPB96] [Lim99, Bli03, MKS98]
infinite dimensional optimization	[TA00]

Table 1: Parameter dependent LMIs in control

paper	$H_1(\theta)$	$\Upsilon(\theta)$
[DS98]	rational	rational with a priori chosen denominator
[FAG96]	affine	affine
[Bli02]	affine	polynomial

Table 2: Considered $H_1(\theta)$ and $\Upsilon(\theta)$ for robustness analysis

can be reformulated as (10) with $H_1(\theta) = A(\theta)^T$ and $\Upsilon(\theta) = P(\theta)$. The class of functions considered for $H_1(\theta)$ and $\Upsilon(\theta)$ are presented Table 2.

Another important special case involves the decision variable $\Upsilon(\theta)$ which is independent of θ such as the ($\mathcal{L}_2$-gain) analysis (control) by a parameter independent Lyapunov function [AGB95, BP94, Sch01]:

$$\mathcal{N}_P^T \begin{bmatrix} A(\theta)^T P + PA(\theta) & PB(\theta) & C(\theta)^T \\ B(\theta)^T P & -\gamma I & D(\theta)^T \\ C(\theta) & D(\theta) & -\gamma I \end{bmatrix} \mathcal{N}_P < 0.$$

This LMI can be reformulated as (10) with

$$H_1(\theta) = \mathcal{N}_P^T \begin{bmatrix} A(\theta)^T & 0 & 0 & C(\theta)^T \\ B(\theta)^T & -\frac{1}{2}I & 0 & D(\theta)^T \\ 0 & 0 & -\frac{1}{2}I & 0 \end{bmatrix} \quad \Upsilon(\theta) = \begin{bmatrix} P & 0 & 0 \\ 0 & \gamma I & 0 \\ 0 & 0 & \gamma I \\ 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

and $H_2(\theta) = \mathcal{N}_P$. The considered functions $H_1(\theta)$ and $H_2(\theta)$ are presented table 3.

These proposed approaches can be interpreted by a two steps solution:

1. The decision variables are restricted to lie in convex finite dimensional subsets of the set of functions.

2. Different methods to turn the infinite number of constraints into a finite one can be chosen. Three classes of methods can be exhibited.

Note that there exists a strong interplay between both steps.

paper	$H_1(\theta)$	$H_2(\theta)$
[AGB95]	affine	constant
[BP94]	multi affine	constant
[Sch01]	rational	constant

Table 3: Considered $H_1(\theta)$ and $H_2(\theta)$ for LPV analysis

The infinite number of decision variables

Let us focus on the first step. For sake of illustration, a classification of the considered subsets in the different published results is presented in Table 4. When using a subset instead of

Papers	subset of functions
[Pac94, Hel95, AG95, SE98, Sch01]	constant
[AGB95, GAC96, FAG96, TA00, YS97, dOBG99]	affine
[BPPB93, BP94, DS00]	multi affine
[TdS01]	quadratic
[Bli03]	polynomial
[DS98, TdS02, IS01]	rational with a priori chosen denominator
[Bec95, WYPB96]	generated by a finite basis
[Lim99, LH97]	continuous piecewise affine
[MKS98, Sch98]	spline

Table 4: Considered subsets of decision variables

the set of functions, obtained conditions are conservative. However, in some cases, obtained conditions are not conservative if the considered subset is "large enough" [CTB99]: for instance the subset of polynomials of sufficiently high order [Bli03]. Our problem fits one of these cases [Bli04]. Unfortunately, the order is not known *a priori* except for specific problems [ZTI03]. Along the same idea, splines are used in conjunction with a gridding in the parameter(s) space. It is shown in [MKS98, Sch98] that if the gridding is fine enough, there is no conservatism in choosing splines instead of the space of functions. Here again, it is not known *a priori* how thin should be the gridding.

In fact, in order to compare the conservatism of obtained conditions when using two different subsets, one should includes the other. The including one yields less (but not necessarily strictly less) conservative conditions than the included one. Thus, using polynomial functions (of degree higher than 2) yields less conservative conditions than using constant, affine, multi affine and quadratic functions. However, it is generally not possible to compare with the use of rational functions with *a priori* chosen denominator or with the use of functions generated by a finite basis.

Connection with our approach In Lemma 2.1, the decision variables are considered to be rational with free denominator. Note that no previously considered subsets of decision variables fit our problem. For a given degree, the set of rational functions with free denominator strictly includes the set of polynomial functions and the set of rational functions with *a priori* chosen denominator. Thus, our obtained conditions are less conservative.

Perhaps, the most related set is the set of "rational functions with *a priori* chosen denominators". However, the difference is strong: there is no *a priori* indication for choosing the denominator "correctly" (this fact will be illustrated in section 3.3). Thus, an important contribution of Lemma 2.1 is to allow a more general set of decision variables, which is suitable to our Automatic Control problem.

The infinite number of constraints

For the second step, the different approaches can be classified into three main classes:

Gridding class the basic idea is to grid the parameter(s) space, and then to ensure the parameter dependent LMI at each point defined by the gridding [Bec95, WYPB96, MKS98, Sch98]. The advantage is that the derivations are really simple. The drawback is that this does not ensure the parameter dependent LMI for all the possible values of the parameter(s). The drawback disappears if the gridding is sufficiently thin. A size for the gridding has been given in [WYPB96]. The disadvantage is then the extremely high cost of the computation due to the number of points defined by the gridding.

Polytopic-like class in some cases, ensuring a parameter dependent LMI for particular values of the parameter(s) allows to ensure this LMI for every considered value of the parameter(s). An important case is when the parameter(s) lies (lie) in a polytope and when the LMI depends in an affine way on the parameter(s) (property of polytopic problems used in [BPPB93, PBPB93, BP94, AGB95]) or in a multi affine way [BP94]. In this case, the particular values correspond to the vertices of the polytope. The drawback of the polytopic property is that this finite number grows exponentially with the number of parameter(s). Nevertheless, many approaches to turn an infinite number of LMIs into a finite one are based on it, even if the LMI is not a (multi) affine function of the parameter(s).

In this case, a first approach consists in extending the idea for more general dependence, even if extra conditions must be introduced at the price of conservatism. In [TA00], a monotonicity argument is used in order to check some conditions only at two particular vertices when considering a quadratic dependent LMI. In [GAC96], a multi convexity argument is used in order to check the LMI only at the vertices of the polytope, also for a quadratic dependent LMI. The result has been generalized in [AT98] for polynomial dependent LMIs. Lim applied the latter technique for a piecewise dependence [Lim99].

A second approach consists in transforming a parameter dependent LMI into an affine one. A first transformation is to embed the LMI into an affine one, see [YS97, TS94] for quadratic LMIs. This embedding introduces conservatism. A second transformation is obtained by introducing slack variables using the reciprocical projection lemma [dOBG99, TdS01, DS00, TAN01]. In [TdS02], rational LMIs are considered. Conservatism is introduced because the slack variables do not depend on the parameter(s).

The advantage of this class of methods is that they allows for quite simple derivations and that they use a minimum number of decision variables (except for the use of slack variables). The disadvantage is the exponential growth in the number of LMIs with the number of parameter(s) (except for the monotonicity approach with a linear growth). Moreover for a direct application of a polytopic technique, one must consider a specific dependence of the LMI on the parameter(s).

LFT class the third class deals with rational dependent LMIs. Such an LMI can be interpreted (recast) as a positivity condition on a (static) LFT system [SP96b, SE98, Sch01, IS01, GOL98]. The parameter(s) is (are) characterized by a set of quadratic constraints parameterized by symmetric scalings [Pac94, AG95, LZD96], symmetric and skew-symmetric scalings [SE98, Hel95, FAG96], full block multipliers [DS98, Sch01, IS01]. Using S-procedure, a finite number of LMIs is obtained. Generally, conservatism is introduced, which can be reduced by a repetitive application of the idea [Bli01]. With symmetric and skew-symmetric multipliers, there is no conservatism for LMIs which depend on one real parameter [MSF97]. This technique has been applied in a piecewise context in [LH97, Sch98]. In the case of frequency dependent LMIs, this approach leads to the famous Kalman-Yakubovich-Popov lemma. The disadvantage of this class (with respect to the polytopic-like one) is that the derivations are more complicated and that there are more decision variables. The advantages of this class (with respect to the polytopic-like one) is that there is no exponential growth in the number of LMIs with the number of parameters and that more general dependence of the LMI on the parameter(s) can be considered.

Some approaches combine both classes of approaches (polytopic-like and LFT) [Sch01, Iwa97, WB02], with the disadvantage of the exponential growth in the number of LMIs with the number of parameter(s).

Connection with our approach The method developed in Lemma 2.1 belongs to the LFT class in the spirit of [SE98, RSF03].

3 Application to the design of a parameter dependent controller for a set of parameterized trade-offs

In this section, the solution to the parameter dependent controller design presented in section 2.3 is applied to the design of a trade-off dependent controller.

3.1 Problem formulation

In the H_∞ control approach, the design of a controller K is recast as an optimization problem on weighted closed loop transfer functions. The considered closed loop functions are defined by P^w (which depends on the plant):

$$\left\{\begin{array}{rcrcrcr} \dot{x}^w(t) & = & A^w x^w(t) & + & B_p^w p(t) & + & B_u^w u(t) \\ q(t) & = & C_q^w x^w(t) & + & D_{qp}^w p(t) & + & D_{zu}^w u(t) \\ y(t) & = & C_y^w x^w(t) & + & D_{yp}^w p(t) & & \end{array}\right. .$$

The desired performance specifications are introduced through the choice of the weighting functions W_i and W_o.

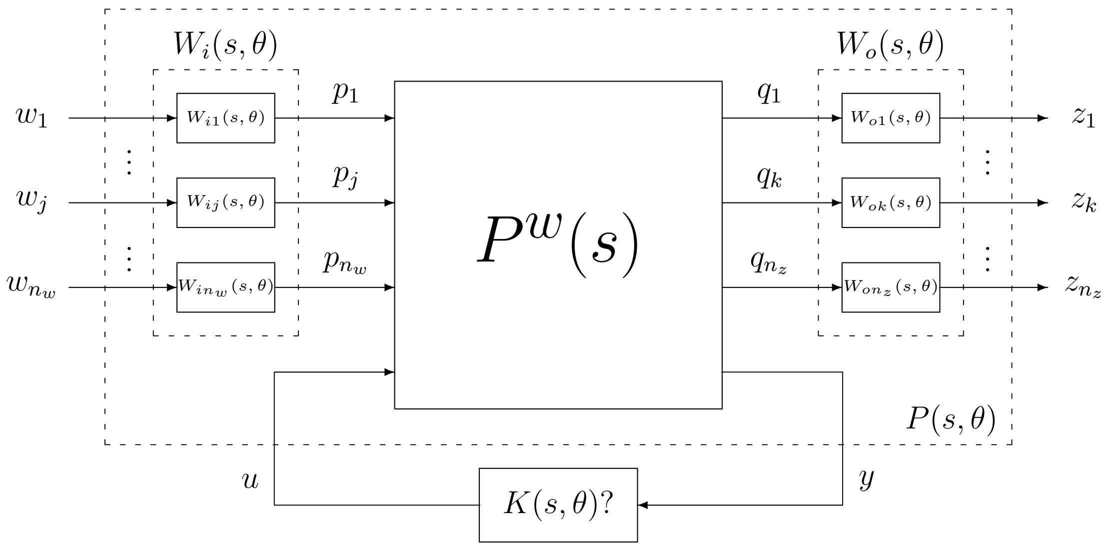

Figure 1: Trade-off dependent controller design problem

A set of performance trade-offs parameterized by a scalar $\theta \in [0,1]$ is then defined by weighting functions that are depending on θ:

$$W_i(s,\theta) = \tfrac{1}{s} I \star \left[\begin{array}{c|c} A_{W_i}(\theta) & B_{W_i}(\theta) \\ \hline C_{W_i}(\theta) & D_{W_i}(\theta) \end{array}\right], \quad \text{and} \quad W_o(s,\theta) = \tfrac{1}{s} I \star \left[\begin{array}{c|c} A_{W_o}(\theta) & B_{W_o}(\theta) \\ \hline C_{W_o}(\theta) & D_{W_o}(\theta) \end{array}\right]$$

whose state space representation are assumed rational functions of θ, well-posed on $[0, 1]$. The generalized plant is then defined as (see Figure 1):

$$P(s,\theta) = \begin{bmatrix} W_o(s,\theta) & 0 \\ 0 & I \end{bmatrix} P^w(s) \begin{bmatrix} W_i(s,\theta) & 0 \\ 0 & I \end{bmatrix}. \tag{19}$$

The problem is, given $\gamma > 0$, to compute a trade-off dependent controller, that is, a controller $K(s,\theta)$ whose state space representation are (explicit) rational functions of θ such that

$$\forall \theta \in [0,\ 1], \quad \|P(s,\theta) \star K(s,\theta)\|_\infty < \gamma \quad \text{with} \quad P(s,\theta) \text{ given by (19).} \tag{20}$$

As the state space matrices of $P(s,\theta)$ are rational in θ, the trade-off dependent control problem is a subcase of the parameter dependent control problem, problem considered in section 2. The solution presented in Theorem 2.2 can then be applied.

Remind that in order to get a controller whose state space matrices are reasonably complex functions of θ, the decision variables in Theorem 2.2 are enforced to be rational functions of θ of limited degree. The question of the performance loss introduced by this constraint arises. A possible evaluation can be obtained by (i) finding the smallest γ, denoted γ_r, such that there exists $K(s,\theta)$ of the considered structure satisfying (20), (ii) comparing γ_r with the obtained γ, denoted γ_{best} ("best achievable performance"), by considering a controller without any constraint on its state space matrices (except well-posedness). Of course, the effective computation of γ_{best} with its corresponding controller is an open problem. Nevertheless, an estimation of γ_{best} (a lower bound) can be straightforwardly obtained by computing γ_{θ_i} for a "lot" of values θ_i where γ_{θ_i} is the smallest γ such that there exists $K_{\theta_i}(s)$ with $\|P(s,\theta_i) \star K_{\theta_i}(s)\|_\infty < \gamma$. An estimation is then $\gamma_{best} \approx \max_{\theta_i} \gamma_{\theta_i}$. In the sequel, the obtained controller for a given θ_i is denoted $K_{\theta_i}(s)$ and it is referred to as a *"pointwise"* controller. For purpose of comparison, a criterion is given in percent: $100\frac{\gamma_r - \gamma_{best}}{\gamma_{best}}$.

In the next subsections, our approach is evaluated through two examples. The first objective of the first example is to discuss the benefits of our approach with respect to less involved alternative approaches. In the case when strong assumptions are made on the state space matrices of the generalized plant[4] $P(s,\theta)$, alternative conditions, based on the polytopic approach can be proposed by considering specific classes of decision variables. The details and the derivations are presented in Appendix, section A: it can be interpreted as an extension of the approach presented in [GAC96]. In order to apply these alternative conditions, the considered example has to be quite academic. The comparison to an approach based on quadratic (parameter independent) Lyapunov functions is also presented. It is a reminiscence of several approaches to LPV control [Pac94, BP94, AG95, AGB95, SE98, Sch01].

[4]such as the matrix $A(\theta)$ does not depend on θ.

The second objective of the first example is to evaluate the obtained performance with our approach with respect to the best achievable performance.

As a second example, we focus on the DC motor control with more realistic specifications. Here again, we evaluate the obtained performance with our approach with respect to the best achievable performance. In addition, we evaluate the obtained performance with our approach with respect to the performance obtained by the same approach but with an *a priori* fixed denominator for the rational decision variables. Remind that a contribution of this paper is to allow to optimize on the denominator of the rational decision variables.

In these examples, the choice of the weighting functions W_i and W_o as rational functions of θ is discussed. Optimization problem are solved using `Matlab` 6.5 with the LMI control toolbox [GNLC95].

3.2 First example

Let us consider a first order plant $G(s)$:

$$G(s) = \frac{1}{s+1} = \frac{1}{s} \star \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right]$$

controlled by a one degree of freedom controller. The purpose is to design a trade-off dependent controller ensuring that the closed loop system output is able to track step and low frequency sinusoidal reference signals with different transient times (5.4 s for $\theta = 0$ and 1.1 s for $\theta = 1$) and the most limited possible control energy. The trade-off is between transient time and control input energy. Such a problem is addressed by mixed sensitivity H_∞ design [SP96a] (see Figure 2). The usual H_∞ problem is for a given trade-off, that is,

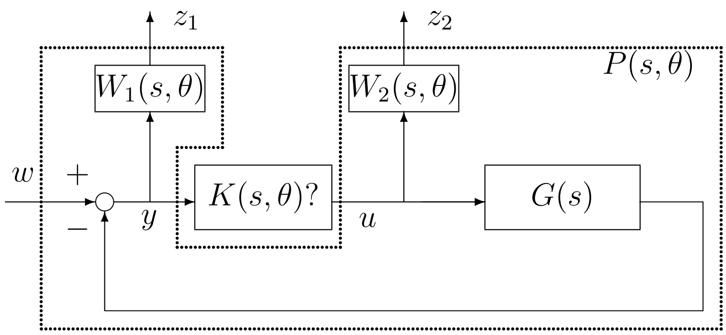

Figure 2: Mixed sensitivity problem

for a given $\theta_i \in [0, 1]$: find $K_{\theta_i}(s)$ such that

$$\left\| \begin{array}{c} W_1(s,\theta_i)\dfrac{1}{1+G(s)K_{\theta_i}(s)} \\ W_2(s,\theta_i)\dfrac{K_{\theta_i}(s)}{1+G(s)K_{\theta_i}(s)} \end{array} \right\|_\infty < \gamma. \tag{21}$$

In this example, for the sake of comparison, we first apply and compare the less involved alternative approach presented in section A and our proposed approach. In the approach of section A, the decision variables are degree one rational functions of θ with fixed denominator. Affine decision variables are an interesting subcase as it was largely considered, see *e.g.* [GAC96]. We first compare our approach to the alternative approach with affine decision variables and then with the degree one rational decision variables.

First comparison

Choice of the weighting functions In order to apply the approach presented in Appendix, section A, only the numerators of the weighted functions W_1 and W_2 are allowed to depend on θ. For a given value $\theta_i \in [0, 1]$:

1. W_1 is chosen for ensuring tracking performance:

$$W_1(s, \theta_i) = k\frac{s + \beta_{\theta_i}}{s + \epsilon} \tag{22}$$

   where

   (a) ϵ is set to a small value (0.0017) for ensuring small error tracking;

   (b) β_{θ_i} is set with respect to transient time response: the bigger β_{θ_i}, the smaller the transient time;

   (c) k is a lower bound of the modulus margin: $k = 0.5$ for a guaranteed -6 dB modulus margin.

2. W_2 is chosen for ensuring control input energy limitation:

$$W_2(s, \theta_i) = \frac{b_{\theta_i} s + c_{\theta_i}}{s + a} \tag{23}$$

   W_2 is such that the inverse of W_2 is a low pass filter. The smaller the bandwidth, the smaller the control energy.

When γ is approximately less than 1 [SP96a], the specifications imposed by the choice of W_1 and W_2 are met. In our problem, a trade-off is defined by β_{θ_i} and the bandwidth of W_2 (which depends on b_{θ_i} and c_{θ_i}). We consider the two extreme trade-offs:

1. for $\theta_i = 1$, fast response with high control energy: $\beta_1 = 3.45$ is chosen for ensuring the time response of 1.1 s and $a = 1580$, $b_1 = 100$ and $c_1 = 500$ for ensuring $\gamma_0 = 1.035 \approx 1$;

2. for $\theta_i = 0$, slow response with low control energy: $\beta_0 = 0.86$ ensure the time response of 5.4 s and $a = 1580$, $b_1 = 1800$ and $c_1 = 504$ for ensuring $\gamma_1 = 1.04 \approx 1$.

Note that between both extreme trade-offs, β is divided by 4. The pair of weighting functions $W_1(s,\theta)$ and $W_2(s,\theta)$, for intermediate trade-offs, are obtained by interpolating numerator coefficients of the weighting functions $W_1(s,0)$ and $W_1(s,1)$ and by interpolating numerator coefficients of the weighting functions $W_2(s,0)$ and $W_2(s,1)$:

$$W_1(s,\theta)=k\frac{s+((1-\theta)\beta_0+\theta\beta_1)}{s+\epsilon} \quad \text{and} \quad W_2(s,\theta)=\frac{((1-\theta)b_0+\theta b_1)s+((1-\theta)c_0+\theta c_1)}{s+a}$$

Note that, with this interpolation technique, we do not a priori ensure that for all $\gamma_i \in (0,\ 1)$, $\gamma_i \approx 1$. Actually, $\gamma_{best} \approx 1.38$ (computed with 101 equally spaced values of θ_i).

Computation of the trade-off dependent controllers We apply two approaches: the approach presented in Appendix, section A (Theorem A.1) with polynomial decision variables of degree one (affine, $d=0$) and our proposed approach (Theorem 2.2) with rational decision variables of degree 1 and[5] 4. The obtained results are presented in Table 5.

Table 5: First comparison results

	Theorem A.1 $d=0$	Theorem 2.2 $N=1$	Theorem 2.2 $N=4$
γ_r	1.92	1.7	1.4
$100\frac{\gamma_r-\gamma_{best}}{\gamma_{best}}$ (upper bound)	$\approx 39\%$	$\approx 16\%$	$\approx 1.5\%$

Note the poor result obtained with Theorem A.1: a dramatic improvement can be obtained with our approach involving degree one rational decision variable. First, Theorem A.1 is based on a sufficient condition whereas Theorem 2.2 is base on a *necessary* and sufficient condition. Second, the class of decision variables considered in Theorem 2.2 contains the class of decision variables considered in Theorem A.1.

Note the excellent result obtained with Theorem 2.2 with fourth order rational decision variables. The obtained performance is very closed to the best achievable performance, that is the performance obtained without constraining the choice of the decision variables. This example emphasizes the strong interest of rational functions and the strong interest of optimizing on their denominator. This second point will be illustrated in section 3.3.

Performance analysis We first compare the trade-off dependent controller obtained with Theorem A.1 to the trade-off dependent controller obtained with Theorem 2.2 ($N=4$). We focus, for the sake of example, on the tracking specification by inspecting the Bode magnitude of $S(s,\theta_i)=\frac{1}{1+G(s)K(s,\theta_i)}$, $\theta_i \in \{0, 0.5, 1\}$ represented in Figure 3 for both trade-

[5]with $1+c_1\theta=1+0.5\theta$. and $1+c_1\theta+c_2\theta^2+c_3\theta^3+c_4\theta^4=(1+2\theta)(1+3\theta)(1+5\theta)$. It is not necessary that $c_4 \neq 0$.

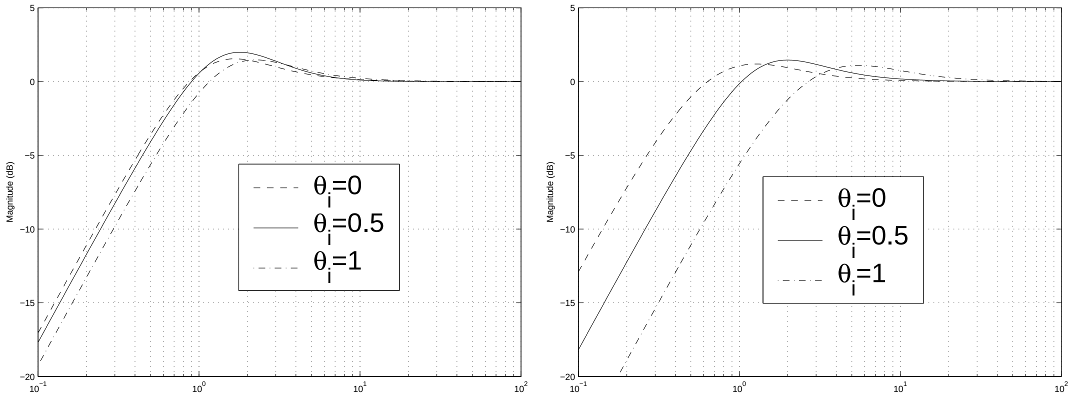

Figure 3: $|S(s, \theta_i)|$, $\theta_i \in \{0, 0.5, 1\}$, Theorem A.1 (left) and Theorem 2.2 ($N = 4$) (right)

off dependent controllers. The bandwidth of $|S|$ is multiplied by 1.3 only between the extreme values of θ whereas it is multiplied by 4 (as needed) in the second case.

Let us now compare the trade-off dependent controller obtained with Theorem 2.2 to the pointwise controllers by inspecting the Bode magnitude of $S(s, \theta_i)$ and $K(s, \theta_i)S(s, \theta_i)$ ($\theta_i \in \{0, 0.5, 1\}$) Figure 4 (thick line for the trade-off dependent controller and thin line

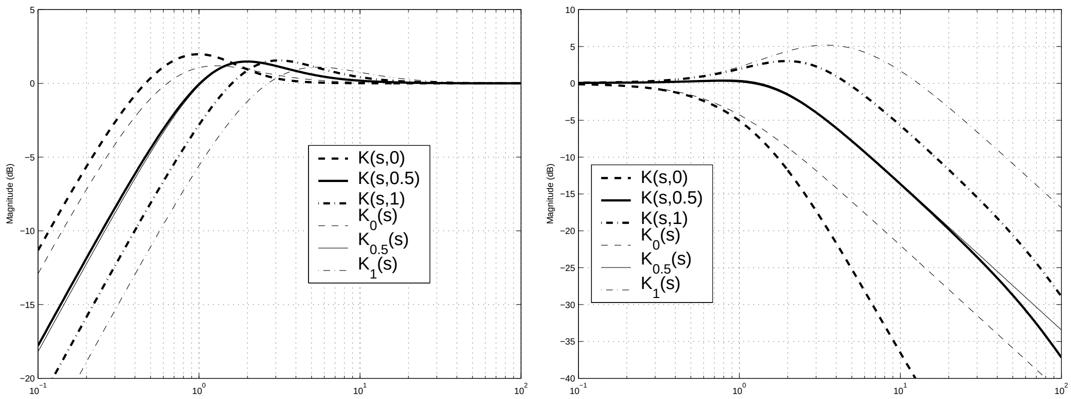

Figure 4: $|S|$ (left) and $|KS|$ (right), $\theta_i \in \{0, 0.5, 1\}$

for the pointwise controllers). For $\theta = 0.5$, the performance of the pointwise controller is recovered by the trade-off parameter dependent controller. Note that for a given θ_i, the bandwidth of $S(s, \theta)$ for the trade-off dependent controller is smaller to the the bandwidth of $S(s, \theta)$ for the pointwise controller. The transient time responses for the extreme values of θ with the trade-off dependent controller is then larger than with the pointwise controllers. As our trade-off parameter dependent controller (closely) achieves the best performance, what is the origin of the discrepancy between it and the pointwise controllers?

This problem is due to the choice of the weighting functions $W_1(s, \theta)$ and $W_2(s, \theta)$ for $\theta \in (0, 1)$. For θ_i, the specifications are defined by $W_1(s, \theta_i)$ and $W_2(s, \theta_i)$ if there exists

$K_{\theta_i}(s)$ such that (21) is satisfied with $\gamma \approx 1$. If γ_{θ_i} is largely greater than 1, such a controller does not exist. Remind that in this example, $\gamma_{best} \approx 1.38$. Thus, with the considered weighting functions, the best trade-off controller ensures $\gamma_{best} > 1$ for the extreme values and thus a slower time response. From Figure 5 where γ_{θ_i} is plotted (dash line with 101 linearly spaced values), we see that the maximum value of γ_{θ_i} is obtained for θ_i close to 0.5,

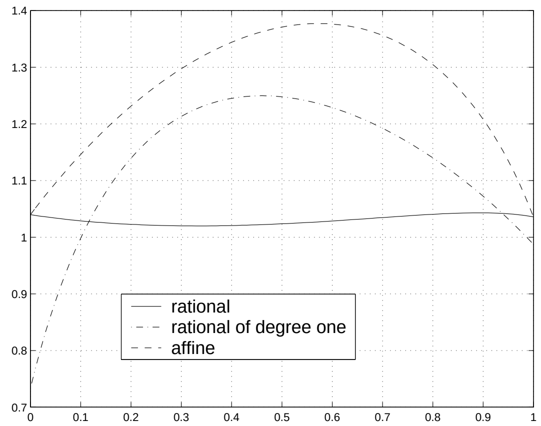

Figure 5: Best achievable performance curve for sections 3.2 and 3.2: $\gamma_{\theta_i}(\theta_i)$

which explains that both controllers have the same performance, for this value of θ.

It is thus necessary to obtain weighting functions $W_1(s,\theta)$ and $W_2(s,\theta)$ such that, for all θ_i, γ_{θ_i} is close to 1. To this purpose, an interesting solution is to introduce pairs of weighting functions for additional values θ_i such that $\gamma_{\theta_i} \approx 1$. In order to obtain $W_1(s,\theta)$ and $W_2(s,\theta)$ as functions of θ, a rational approximation of their state space representation matrices can be straightforwardly performed using a least square technique[6]. For a given degree, a rational function allows a better approximation than a polynomial one. Approximation is preferred to interpolation for a compromise between the limitation of the rational function degree and the quality of the result. To this purpose, we choose a third pair of weighting functions for[7] $\theta = 0.6$:

$$W_1(s,0.6) = 0.5\frac{s+1.73}{s+0.0017} \qquad W_2(s,0.6) = 500\frac{s+1}{s+1580}.$$

such that $\gamma_{0.6} = 1.03$. In order to be able to apply Theorem A.1 again, we consider approximation by degree one rational functions. Note that the denominators of different rational functions are enforced to be the same. For this purpose, an approach based on least square minimization is performed on $\begin{bmatrix} C_{W_1}(\theta) & D_{W_1}(\theta) \\ C_{W_2}(\theta) & D_{W_2}(\theta) \end{bmatrix}$ with $\begin{bmatrix} A_{W_1} & B_{W_1} \end{bmatrix} = \begin{bmatrix} -0.0017 & 1.32 \end{bmatrix}$

[6]A more interesting weighting function choice will be investigated in the second example. Unfortunately, in this case, the assumption of Theorem A.1 are no longer satisfied.

[7]This value of correspond to the worst case trade-off with the affine interpolation.

and $\begin{bmatrix} A_{W_2} & B_{W_2} \end{bmatrix} = \begin{bmatrix} -1580 & 397 \end{bmatrix}$. We then obtain

$$\begin{bmatrix} C_{W_1}(\theta) & D_{W_1} \\ C_{W_2}(\theta) & D_{W_2}(\theta) \end{bmatrix} = \begin{bmatrix} 0.14 & 0.5 \\ -7170 & 1800 \end{bmatrix} + \frac{\theta}{1+1.17\theta} \begin{bmatrix} 2.27 & 0 \\ 7330 & -3700 \end{bmatrix}.$$

In Figure 5, with these weighting functions, γ_{θ_i} is plotted versus θ_i (dash dot line with 101 points). Clearly, we do not have $\gamma_{\theta_i} \approx 1$. It is thus necessary to consider rational functions of larger degree for the approximation, that is $\begin{bmatrix} C_z(\theta) & D_{zw}(\theta) & D_{zu}(\theta) \end{bmatrix}$ will be a rational function of degree strictly more than one. Assumption of Theorem A.1 is then no longer satisfied. Note that this alternative approach, less involved than our proposed approach, breaks down on this academic example[8].

Second comparison

In this section, we propose a choice of the weighting functions $W_1(s, \theta)$ and $W_2(s, \theta)$ adapted to the performance specifications for any $\theta \in [0, 1]$. A trade-off dependent controller is design using Theorem 2.2. It is compared to LPV synthesis. The design of rational parameter dependent controllers was intensively considered in the LPV control, but with (parameter independent) quadratic Lyapunov functions [Pac94, BP94, AG95, AGB95, SE98, Sch01]. Remind that obtained conditions are computationally more attractive than ours. For the sake of comparison, we focus on the interest of a parameter dependent Lyapunov functions versus parameter independent Lyapunov functions.

Choice of the weighting functions In order to obtain convenient weighting functions, we separately approximate C_{W_1} and $\begin{bmatrix} C_{W_2} & D_{W_2} \end{bmatrix}$. We obtain:

$$C_{W_1}(\theta) = 0.33 + \frac{0.33\theta}{1-0.67\theta} \ ;$$

$$\begin{bmatrix} C_{W_2}(\theta) & D_{W_2}(\theta) \end{bmatrix} = \begin{bmatrix} -7170 & 1800 \end{bmatrix} + \frac{\theta}{1+1.17\theta} \begin{bmatrix} 14670 & -3680 \end{bmatrix} .$$

In Figure 5, with these weighting functions, γ_{θ_i} is plotted versus θ_i (full line with 101 points). Clearly, we have[9] $\gamma_{\theta_i} \approx 1$.

Computation of the trade-off dependent controllers In order to compute a trade-off dependent controller with a parameter independent Lyapunov function, we slightly modify Theorem 2.2 by considering :

$$\mathcal{X}(\theta) = \mathcal{X}_0, \quad \mathcal{Y}(\theta) = \mathcal{Y}_0, \quad \mathcal{V}(\theta) = \frac{\mathcal{V}_0 + \theta\mathcal{V}_1 + \theta^2\mathcal{V}_2}{1 + d_1\theta + d_2\theta^2}$$

[8] If we adapt the approaches of [AGB95, FAG96, TA00, YS97, dOBG99, BPPB93, BP94, TdS01] to our problem, affine functions would also be considered for the interpolation of the weighting functions: the same problem would arise.

[9] between 1.02 for $\theta = 0.3$ and 1.045 for $\theta = 0.9$, with a variation of 2.5%

with *a priori* fixed d_i: $1 + d_1\theta + d_2\theta^2 = (1 - 0.67\theta)(1 + 1.17\theta)$.

On the other hand, our approach (Theorem 2.2) is applied with $N = 2$ and $1 + c_1\theta + c_2\theta^2$ the polynomial given by the interpolation of the weighting functions: $(1-0.67\theta)(1+1.17\theta)$. This choice allows to limit the size of the matrices A_Ω, B_Ω, C_Ω, D_Ω.

The obtained results are presented in Table 6. Note that Theorem 2.2 with $N = 2$, that is low degree rational functions, allows to obtain a performance very close to the best achievable performance. The obtained result with a parameter independent Lyapunov function is very poor.

Table 6: Second comparison results

	Parameter independent Lyapunov function	Theorem 2.2 $N = 2$
γ_r	2.26	1.07
$100\frac{\gamma_r - \gamma_{best}}{\gamma_{best}}$ (upper bound)	$\approx 115\%$	$\approx 2.5\%$

Performance analysis We first analyze the performance of the trade-off dependent controller obtained with a parameter independent Lyapunov function by inspecting the Bode magnitude of $S(s, \theta_i)$ and $K(s, \theta_i)$ ($\theta_i \in \{0, 0.1, \ldots, 0.9, 1\}$), Figure 6. The eleven closed loop transfer functions have the same diagram of Bode: the controller is in fact not dependent on the trade-off parameter θ. The obtained result is really very poor.

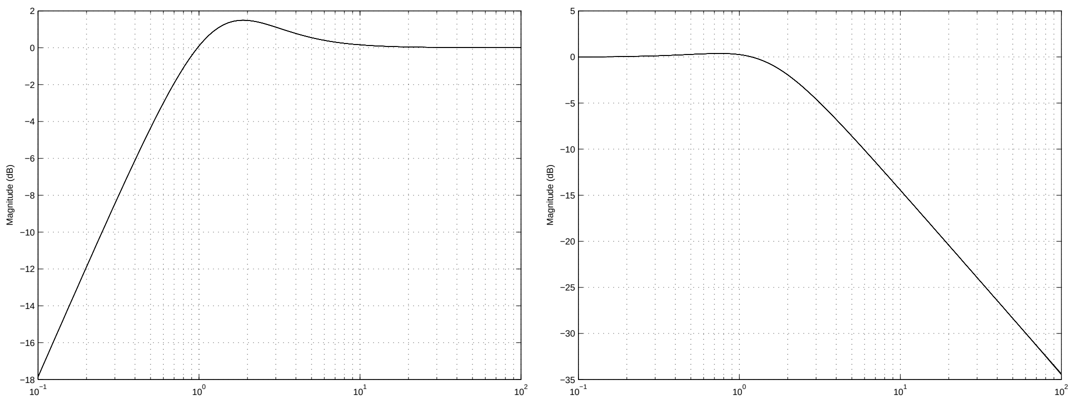

Figure 6: $|S(s, \theta)|$ and $|K(s, \theta)S(s, \theta)|$ by step of 0.1 in θ

Let us now compare the trade-off dependent controller obtained with Theorem A.1 and $N = 2$ to the pointwise controllers by inspecting the Bode magnitude of $S(s, \theta_i)$ and $K(s, \theta_i)S(s, \theta_i)$ (see Figure 7), the tracking of a step reference (see Figure 8) and the Bode diagram of $K(s, \theta_i)$ (see Figure 9) (for each figure $\theta_i \in \{0, 0.5, 1\}$, thick line for the trade-off

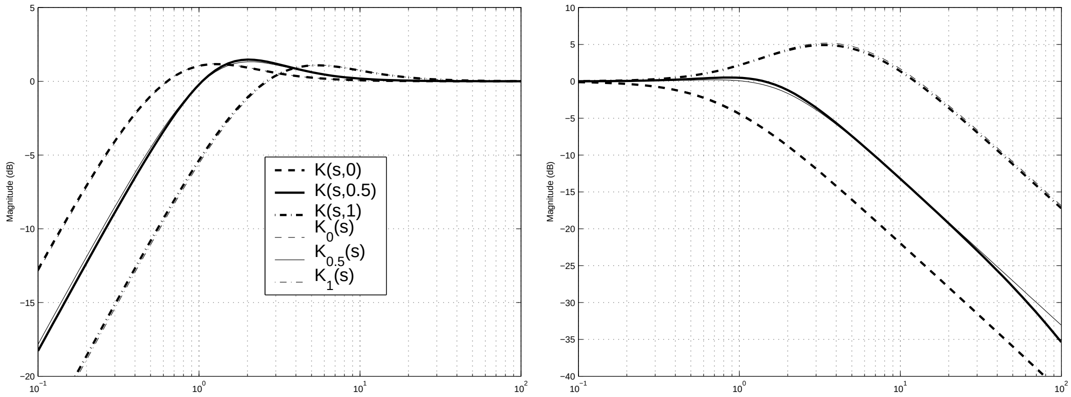

Figure 7: $|S|$ (left) and $|KS|$ (right), $\theta_i \in \{0, 0.5, 1\}$

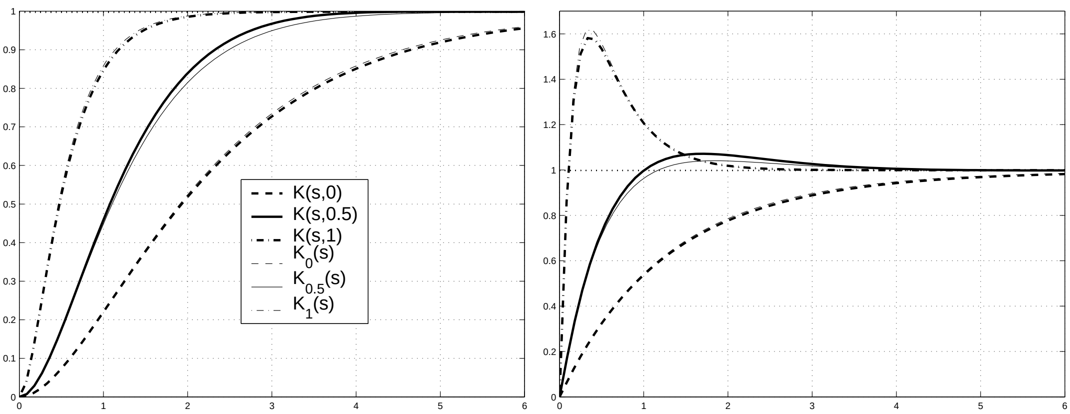

Figure 8: Plant output (left) and control signal (right) with $K(s, \theta_i)$ (left) and $K_{\theta_i}(s)$ (right), $\theta_i \in \{0, 0.5, 1\}$

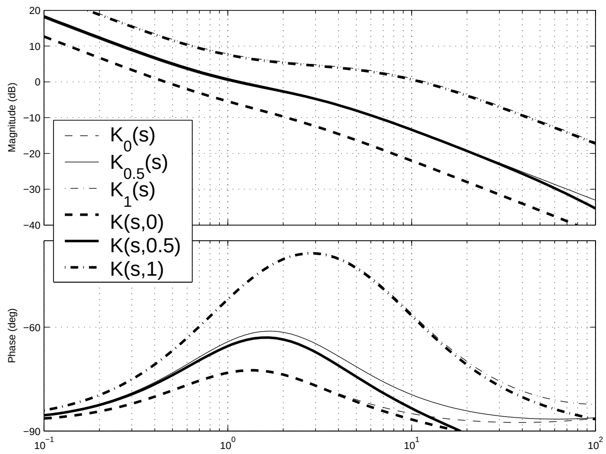

Figure 9: Bode diagrams of controllers, $\theta_i \in \{0, 0.5, 1\}$

dependent controller and thin line for the pointwise controllers). The obtained trade-off dependent controller satisfies the design (rise time) specification. It actually recovers the performance obtained with the pointwise controllers using low degree rational functions, which is really good.

This point emphasizes the importance of the weighting function choice as functions of θ. In the next example, we illustrate that a more direct choice is possible.

3.3 DC motor control

In the previous example, the interest of rational decision variables was illustrated. In this example, we focus on the benefit on optimizing on the decision variable denominator.

The considered plant is a DC motor which can be modeled by

$$G(s) = \frac{235}{s(\frac{s}{66}+1)} = \frac{1}{s}I \star \left[\begin{array}{cc|c} -66 & 0 & 32 \\ 32 & 0 & 0 \\ \hline 0 & 15 & 0 \end{array}\right].$$

It is controlled by a one degree of freedom controller. The purpose is to design a trade-off dependent controller ensuring that the closed loop system output is able to track, with a small error, step and low frequency sinusoidal reference signals with different transient times (from 0.02 s for $\theta = 1$ up to 0.06 s for $\theta = 0$) and with the most limited possible control input energy. The closed loop system has to reject step and low frequency sinusoidal input disturbance signals. For different trade-offs between transient time and control input energy, such a problem is addressed by the weighted H_∞ problem presented in Figure 10 [SP96a]. The usual H_∞ problem is for a given trade-off, that is for a given $\theta_i \in [0, 1]$: find $K_{\theta_i}(s)$

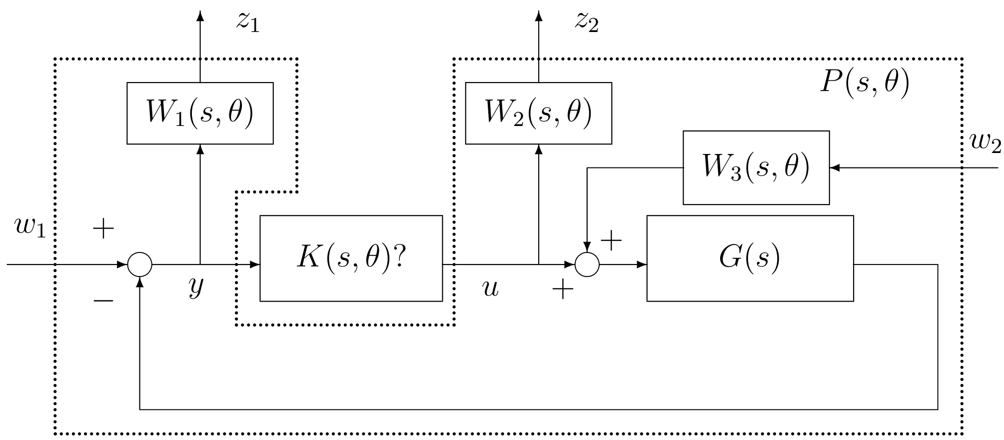

Figure 10: Weighted H_∞ problem

such that

$$\left\| \begin{array}{cc} W_1(s,\theta_i)S_{\theta_i}(s) & W_1(s,\theta_i)G(s)S_{\theta_i}(s)W_3(s,\theta_i) \\ W_2(s,\theta_i)K_{\theta_i}(s)S_{\theta_i}(s) & W_2(s,\theta_i)T_{\theta_i}(s)W_3(s,\theta_i) \end{array} \right\|_\infty < \gamma \tag{24}$$

with $S_{\theta_i}(s) = \dfrac{1}{1+G(s)K_{\theta_i}(s)}$ and $T_{\theta_i}(s) = \dfrac{G(s)K_{\theta_i}(s)}{1+G(s)K_{\theta_i}(s)}$.

Choice of the weighting functions The weighting functions $W_i(s,\theta)$, $i \in \{1,2\}$ have the following form [Fon95]:

$$\frac{1}{s} \star \left[\begin{array}{c|c} -\omega_{ci}(\theta)\sqrt{\dfrac{|G_{\infty i}^2-1|}{|G_{0i}^2-1|}} & (G_{0i}-G_{\infty i})\sqrt{\dfrac{|G_{\infty i}^2-1|}{|G_{0i}^2-1|}} \\ \hline \omega_{ci}(\theta) & G_{\infty i} \end{array} \right]$$

where $G_{0i} = |W_i(0,\theta)|$, $G_{\infty i} = \lim_{\omega\to\infty}|W_i(j\omega,\theta)|$ (with $(G_{0i}-1)(G_{\infty i}-1) < 0$) and $\omega_{ci}(\theta) > 0$, the crossover frequency, such that $|W_i(j\omega_{ci}(\theta),\theta)| = 1$.

1. W_1 is chosen for ensuring tracking performance.

   (a) The considered trade-off can be defined by ω_{c1}, as it is directly related to the transient time response (20 rad/s for 0.06 s up to 80 rad/s for 0.02 s). Thus, ω_{c1} is directly related to θ by $\omega_{c1}(\theta) = 20 + 60\theta$. The parameter θ can be then interpreted as the crossover frequency of $W_1(s,\theta)$, up to an affine transformation. Note that, with this choice of θ, an assumption of Theorem A.1 (A does not depend on θ) is no longer satisfied.

   (b) G_{01} is an upper bound on the static error: we set $-40dB$.

   (c) $G_{\infty 1}$ is a lower bound on the modulus margin: we set $-6dB$.

2. W_2 is chosen for ensuring control energy limitation: the smallest ω_{c2} the smallest the control energy. We first obtain the smallest possible value of ω_{c2} for three values of θ (23.33 rad/s and $\gamma_0 = 0.991$ for $\theta = 0$, 180 rad/s and $\gamma_{0.5} = 0.986$ for $\theta = 0.5$, 700 rad/s and $\gamma_1 = 0.992$ for $\theta = 1$). Using a least square method, we then obtain $\omega_{c2}(\theta) = 23.33 + \dfrac{204\theta}{1-0.7\theta}$. In addition, we choose $20\log(G_{02}) = 10dB$ and $20\log(G_{\infty 2}) = -60dB$.

3. $W_3(s,\theta)$ is chosen in order to specify the input disturbance rejection. For simplicity, W_3 is chosen as a constant gain: $W_3(s,\theta) = 0.05$.

$P(s,\theta)$ is then obtained with the parameter dependent matrices $A(\theta)$ and $C_z(\theta)$ rational functions with the denominator $1 - 0.7\theta + 0\theta^2$.

With these weighting functions, γ_{θ_i} is computed for several values of $\theta_i \in [0,1]$ with a step of 0.01: we have$\gamma_{\theta_i} \approx 1$. An estimation of γ_{best} is 0.998.

Computation of the trade-off dependent controllers Trade-off dependent controllers are obtained by applying Theorem 2.2 along three way:

1. with $N = 2$ and the denominator of the decision variables a priori chosen. A natural choice for the denominator is $1 - 0.7\theta + 0\theta^2$ as it is the denominator of the matrices $A(\theta)$ and $C_z(\theta)$;

2. with $N = 2$ to evaluate the effect of optimizing on the decision variable denominator;

3. with $N = 3$ for improving the previous result.

In order to improve the numerical resolution, we choose $1 + c_1\theta = 1 - 0.7\theta$, $c_2 = 0$ for $N = 2$ and $1 + c_1\theta + c_2\theta^2 = (1 - 0.7\theta)(1 + 3\theta)$, $c_3 = 0$ for $N = 3$ with $(1 - 0.7\theta)$ for limiting the size of the matrices A_Ω, B_Ω, C_Ω, D_Ω and $(1 + 3\theta)$ arbitrary.

Table 7: Obtained results

	Theorem 2.2 $N = 2$ a priori chosen denominator	Theorem 2.2 $N = 2$	Theorem 2.2 $N = 3$
γ_r	1.105	1.06	1
$100\frac{\gamma_r - \gamma_{best}}{\gamma_{best}}$ (upper bound)	$\approx 11\%$	$\approx 6\%$	$< 1\%$

The obtained results are presented in Table 7. Note that with the a priori chosen denominator, the obtained result is quite correct with a lower bound on the criterion of 11%. Nevertheless, with the same degree, this result is strongly improved with a smaller value: 6%. Note that, in this case, the denominator of the decision variables is $1 - 1.12\theta + 3.37\theta^2$, that is, a polynomial with complex roots, really different of the denominator of $A(\theta)$ and $C_z(\theta)$. Its *a priori* selection would be difficult. Theorem 2.2 with $N = 3$ allows to obtain a trade-off dependent controller whose performance is dramatically close to the best achievable performance.

Performance analysis Let us now compare the trade-off dependent controller obtained with Theorem 2.2 and $N = 3$ to the pointwise controllers by inspecting the Bode magnitude of $S(s, \theta_i)$, $K(s, \theta_i)S(s, \theta_i)$, $G(s, \theta_i)S(s, \theta_i)$ and $T(s, \theta_i)$ (see Figure 11), the tracking of a step reference and the rejection of a step disturbance (see Figure 12) and the Bode diagram of $K(s, \theta_i)$ (see Figure 13) (for each figure $\theta_i \in \{0, 0.5, 1\}$, thick line for the trade-off dependent controller and thin line for the pointwise controllers). The obtained trade-off dependent controller satisfies the design (rise time) specification. It actually recovers the performance obtained with the pointwise controllers using low degree rational functions, which is really good.

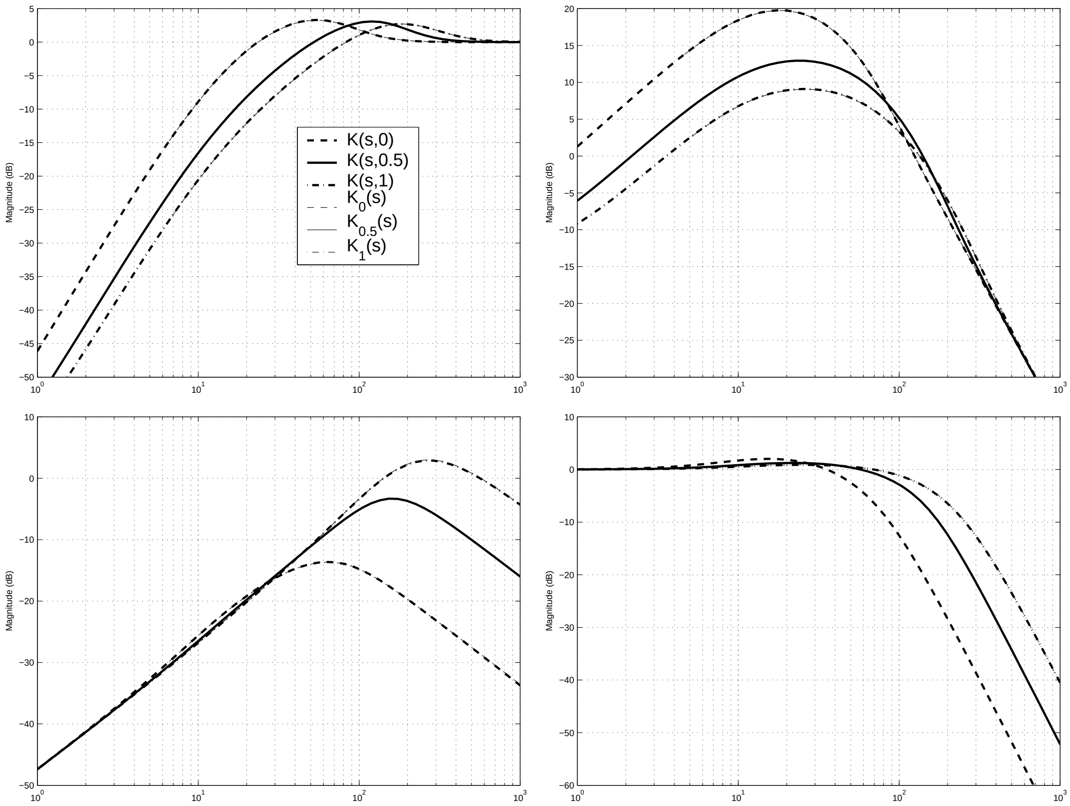

Figure 11: Bode magnitude of S (top left), GS (top right), KS (bottom left) and T (bottom right) for $\theta_i \in \{0, 0.5, 1\}$

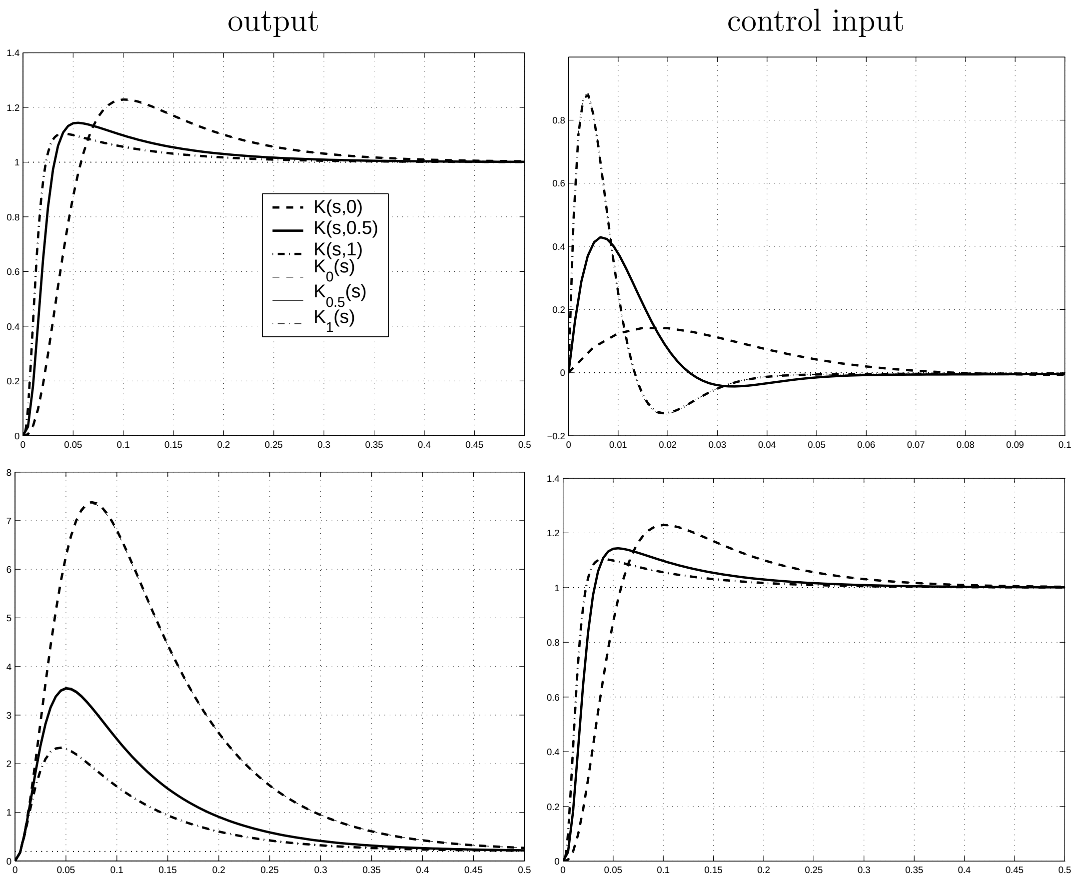

Figure 12: Transient responses to a unit step reference signal (top) and to a unit step disturbance signal (bottom) for $\theta_i \in \{0, 0.5, 1\}$ (top right plot has a different scale)

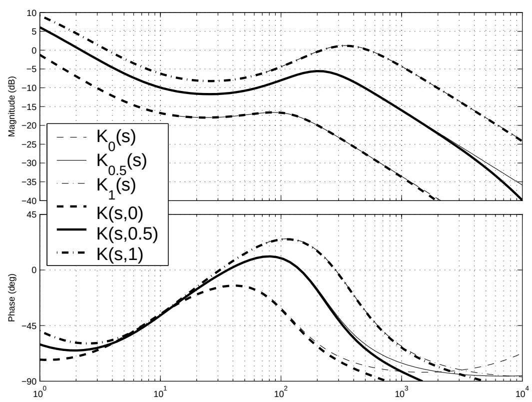

Figure 13: Bode plots of controllers (optimization with $N = 3$) for $\theta_i \in \{0, 0.5, 1\}$

An interesting feature has to be noticed from Figure 13. The structure of the controllers is a Proportional Integral (PI) with a lead effect (and a low pass filter). However, for $\theta = 0$, the lead effect is small and can be neglected; the trade-off dependent controller can be reduced to a PI controller (with a low pass filter). Whereas, for $\theta = 1$, the lead effect is important and cannot be neglected; the trade-off dependent controller is a PI plus a lead transfer function (and a low pass filter). It is a well-known fact that a DC motor can be controlled by a PI if the desired transient response is slow enough. Faster transient response involves a PI plus lead effect controller. Using classical rules of automatic control, know-how... a qualitative link between the performance specifications and the controller gains can be established. Our approach explicitly express the controller structure and the controller gains as analytic functions of the performance specifications, that is, a quantitative link.

4 Conclusion

In this paper, the design of a parameter dependent controller for a parameter dependent plant was recast as a (convex) finite dimensional optimization problem involving LMI constraints. We proved that a parameter (rational) dependent LMI constraint can be equivalently transformed in a parameter independent LMI constraint. An interesting contribution is that the denominator of the decision variables is optimized, which dramatically improves the existing results. This fact was emphasized through two examples. The obtained result can be extended to several parameters (see Appendix, section B). But, in this case, conservative is introduced.

The obtained result was applied to the design of a trade-off dependent controller. Two examples emphasize the interest of our approach based on rational functions. Using low order rational functions, we recover the performance obtained using pointwise controllers, that is, the best performance possible.

The solution of the parameter dependent controller design has a broader application. One of the most interesting application is probably the (classical) gain scheduled control [SR99, FS03]. Traditional solutions are based on gridding and interpolation, with well-known important difficulties. Our proposed solution is an alternative approach where the parameter dependent controller is readily obtained avoiding gridding and interpolation.

From a more general point of view, the proposed approach can be applied to control problems involving parameter dependent LMI conditions. Such formulations were proposed, for instance, for the control of nonlinear systems [SA78, LD95, HL96, HK96], control of saturated systems [Meg96] and control of spatial systems [dCP02, BPD02]. Under some technicalities, the proposed approach can be also adapted to control problems involving parameter dependent Riccati equations. Parameter dependent Riccati equations have been widely studied. Results on the existence and the analyticity of a solution have been given (see [Del84, RR88] and the references therein). But to the authors best knowledge, no efficient method to compute such a solution has been proposed. We hope that our proposed approach paves the way to an efficient solution to these problems.

References

[AG95] P. Apkarian and P. Gahinet. A convex characterization of gain-scheduled H_∞ controllers. *IEEE Transactions Automatic Control*, 40(5):853–864, May 1995.

[AGB95] P. Apkarian, P. Gahinet, and G. Becker. Self-scheduled H_∞ control of linear parameter-varying systems: a design example. *Automatica*, 31(9):1251–1261, September 1995.

[Ala01] D. Alazard. Extracting physical tuning potentiometers from a complex control law: application to flexible aircraft flight control. *AIAA Journal of Guidance, Control and Dynamics*, August 2001.

[AT98] P. Apkarian and H.D. Tuan. Parametrized LMIs in control theory. In *Conference on Decision and Control*, pages 152–157, Tampa, Florida, USA, December 1998.

[Bec95] G. Becker. Parameter-dependent control of an under-actuated mechanical system. In *Conference on Decision and Control*, pages 543–548, New Orleans, LA, USA, December 1995.

[BEFB94] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*, volume 15 of *Studies in Appllied Mathematics*. SIAM, Philadelphia, USA, June 1994.

[BGSA01] X. Bombois, M. Gevers, G. Scorletti, and B.D.O. Anderson. Robustness analysis tools for an uncertainty set obtained by prediction error identification. *Automatica*, 37(10):1629–1636, 2001.

[Bli01] Pierre-Alexandre Bliman. LMI characterization of the strong delay-independent stability of linear delay systems via quadratic lyapunov-krasovskii functionals. *Syst. Control Letters*, 43(2):263–274, 2001.

[Bli02] P-A. Bliman. Nonconservative LMI approach to robust stability for systems with uncertain scalar parameters. In *Conference on Decision and Control*, pages 305–310, Las Vegas, Nevada, USA, December 2002.

[Bli03] P-A. Bliman. Stabilization of LPV systems. In *Conference on Decision and Control*, pages 6103–6108, Maui, Hawaii, USA, December 2003.

[Bli04] P-A. Bliman. An existence result for polynomial solutions of parameter-dependent LMIs. *Systems and Control Letters*, 51(3-4):165–169, March 2004.

[BP94] G. Becker and A. Packard. Robust performance of linear parametrically varying systems using parametrically-dependent linear feedback. *Systems and Control Letters*, 23(3):205–215, September 1994.

[BPD02] B. Bamieh, F. Paganini, and M.A. Dahleh. Distributed control of spatially-invariant systems. *IEEE Transactions Automatic Control*, 47(7):1091–1107, July 2002.

[BPPB93] G. Becker, A. Packard, D. Philbrick, and G. Balas. Control of parametrically-dependent linear systems: a single quadratic Lyapunov approach. In *American Control Conference*, pages 2795–2799, San Fransisco, California, USA, June 1993.

[BSG+04] X. Bombois, G. Scorletti, M. Gevers, R. Hildebrand, and P. Van den Hof. Data-based controller design at the cheapest experimental cost. In *Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, December 2004.

[Che82] W. Cheney. *Approximation Theory.* AMS Chelsea Publishing, 1982.

[CTB99] Y.-S. Chou, A.L. Tits, and V. Balakrishnan. Stability multipliers and μ upper bounds: connections and implications for numerical verification of frequency domain conditions. *IEEE Trans. Aut. Control*, 44(5):906–913, 1999.

[dCP02] G. Ayres de Castro and F. Paganini. Convex synthesis of localized controllers for spatially invariant systems. *Automatica*, 38(3):445–456, March 2002.

[Del84] D.F. Delchamps. Analytic feedback control and the algebraic Riccati equation. *IEEE Transactions Automatic Control*, 29(11):1031–1033, November 1984.

[DGKF89] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis. State-space solutions to standard H_2 and H_∞ control problems. *IEEE Transactions Automatic Control*, 34(8):831–847, August 1989.

[dOBG99] M.C. de Oliveira, J. Bernussou, and J.C. Geromel. A new discrete-time robust stability condition. *Systems and Control Letters*, 37(4):261–265, July 1999.

[DS98] M. Dettori and C.W. Scherer. Robust stability analysis for parameter dependent systems using full block S-procedure. In *Proc. IEEE Conf. on Decision and Control*, pages 2798–2799, December 1998.

[DS00] M. Dettori and C.W. Scherer. New robust stability and performance conditions based on parameter-depedent multipliers. In *Conference on Decision and Control*, pages 4187–4192, Sydney, Australia, December 2000.

[DSFM03] M. Dinh, G. Scorletti, V. Fromion, and E. Magarotto. Parameterized H_∞ controller design for adaptative trade-off by finite dimensional lmi optimization. In *European Control Conference*, Cambridge, UK, September 2003.

[DSFM04] M. Dinh, G. Scorletti, V. Fromion, and E. Magarotto. Parameter dependent H_∞ controller design by finite dimensional LMI optimization: application to trade-off dependent control. In *IFAC World Congress*, submitted, December 2004.

[FAG96] E. Feron, P. Apkarian, and P. Gahinet. Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions. *IEEE Transactions Automatic Control*, 41(7):1041–1046, July 1996.

[FB02] I. Fialho and G. Balas. Road adaptive active suspension design using linear parameter-varying gain-scheduling. *IEEE Transactions on Control Systems Technology*, 10(1):43–54, January 2002.

[FFM95] G. Ferreres, V. Fromion, and M. M'Saad. Adaptive H_∞ control using coprime factors and set-membership identification with application to a missile autopilot. In *American Control Conference*, pages 698–702, Orlando, LA, USA, June 1995.

[Fon95] S. Font. *Méthodologie pour Prendre en Compte la* Robustesse *des Systèmes Asservis : Optimisation H_∞ et Approche Symbolique de la Forme Standard*. PhD thesis, Université de Paris-Sud et Ecole Supérieure d'Electricité, France, April 1995.

[FS03] V. Fromion and G. Scorletti. A theoritical framework for gain scheduling. *International Journal of Robust and Nonlinear Control*, 13(10):951–982, February 2003.

[FTD91] M. K. H. Fan, A. L. Tits, and J. C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Control*, 36(1):25–38, January 1991.

[GAC96] P. Gahinet, P. Apkarian, and M. Chilali. Affine parameter-dependent Lyapunov functions and real parametric uncertainty. *IEEE Transactions Automatic Control*, 41(3):436–442, March 1996.

[GCG93] F. Garofalo, G. Celentano, and L. Glielmo. Stability robustness of interval matrices via Lyapunov quadratic forms. *IEEE Transactions Automatic Control*, 38(2):281–284, February 1993.

[GNLC95] P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali. *LMI Control Toolbox*. The Mathworks Partner Series. The Mathworks, Inc., 1995.

[GOL98] L. El Ghaoui, F. Oustry, and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM J. Optimization*, 9(1):33–52, January 1998.

[Hel95] A. Helmersson. μ synthesis and LFT gain scheduling with mixed uncertainties. In *European Control Conference*, pages 153–158, Roma, Italy, July 1995.

[HK96] T. Hiroe and P.P. Khargonekar. Stability and H_∞ performance analysis of gain scheduled nonlinear control systems with $\mathcal{D}$-implementation under slowly varying command inputs. In *Conference on Decision and Control*, pages 1656–1662, Kobe, Japan, December 1996.

[HL96] Y. Huang and W-M. Lu. Nonlinear optimal control: Alternatives to Hamilton-Jacobi equation. In *Conference on Decision and Control*, pages 3942–3947, Kobe, Japan, December 1996.

[IH03] T. Iwasaki and S. Hara. Generalization of Kalman-Yakubovich-Popov lemma for restricted frequency inequalities. In *American Control Conference*, pages 3828–3833, Denver, Colorado, USA, June 2003.

[IMF00] T. Iwasaki, G. Meinsma, and M. Fu. Generalized $\mathcal{S}$-procedure and finite frequency KYP lemma. *Mathematical Problems in Engineering*, 6:305–320, December 2000.

[IS01] T. Iwasaki and G. Shibata. LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Transactions Automatic Control*, 46(8):1195–1208, August 2001.

[Iwa97] T. Iwasaki. Robust stability analysis with quadratic separator: Parametric time-varying uncertainty cases. In *Conference on Decision and Control*, pages 4880–4885, San Diego, California, US, December 1997.

[KYM+01] M.R. Katebi, I. Yamamoto, M. Matsuura, M.J. Grimble, H. Hirayama, and N. Okamoto. Robust dynamic ship positioning control system design and applications. 11(13):1257–1284, 2001.

[LD95] W.M. Lu and J.C. Doyle. H_∞ control of nonlinear systems: a convex characterization. *IEEE Trans. Aut. Control*, pages 1668–1675, September 1995.

[LH97] S. Lim and J.P. How. Analysis of LPV systems using a piecewise affine parameter-dependent Lyapunov function. In *Conference on Decision and Control*, pages 978–983, San Diego, California, USA, December 1997.

[Lim99] S. Lim. *Analysis and control of linear parameter-varying systems.* PhD thesis, Standford University, California, USA, 1999.

[LZD96] W.M. Lu, K. Zhou, and J.C. Doyle. Stabilization of uncertain linear systems: an LFT approach. *IEEE Transactions Automatic Control*, 41(1):50–65, January 1996.

[Meg96] A. Megretski. L_2 BIBO output feedback stabilization with saturated control. In *1996 IFAC Triennal World Congress*, pages 435–440, June 1996.

[MKS98] I. Masubuchi, A. Kume, and E. Shimemura. Spline-type solution to parameter-dependent LMIs. In *Conference on Decision and Control*, pages 1753–1758, Tampa, Florida, USA, December 98.

[MSF97] G. Meinsma, Y. Shrivastava, and M. Fu. A dual formulation of mixed μ and on the losslessness of (d, g) scaling. *IEEE Transactions Automatic Control*, 42(7):1032–1036, July 1997.

[Pac94] A. Packard. Gain scheduling via linear fractional transformations. *Systems and Control Letters*, 22:79–92, 1994.

[PBPB93] A. Packard, G. Becker, D. Philbrick, and G. Balas. Control of parameter-dependent systems: applications to H_∞ gain-scheduling. In *American Control Conference*, pages 329–333, San Fransisco, California, USA, May 1993.

[RR88] A.C.M. Ran and L. Rodman. On parameter dependence of solutions of algebraic Riccati equations. *Mathematics of Control, Signals, and Systems*, 1:269–284, 1988.

[RSF03] L. Rossignol, G. Scorletti, and V. Fromion. Filter design: a finite dimensional convex optimization approach. *International Journal of Robust and Nonlinear Control*, 13:1317–1335, October 2003.

[SA78] M. G. Safonov and M. Athans. Robustness and computation aspects of nonlinear stochastic estimators and regulators. *IEEE Trans. Aut. Control*, AC-23(4):717–725, August 1978.

[Sch98] C.W. Scherer. Robust performance analysis for parameter dependent systems using tensor product splines. In *Conference on Decision and Control*, pages 2216–2221, Tampa, Florida, USA, December 1998.

[Sch01] C.W. Scherer. LPV control and full block multipliers. *Automatica*, 73(3):361–375, March 2001.

[Sch03] C.W. Scherer. When are multiplier relaxations exact? In *IFAC Symposium on Robust Control Design*, 2003.

[SE98] G. Scorletti and L. El Ghaoui. Improved LMI conditions for gain scheduling and related problems. *International Journal of Robust and Nonlinear Control*, 8(10):845–877, August 1998.

[SGC97] C. Scherer, P. Gahinet, and M. Chilali. Multiobjective Output-Feedback control via LMI optimization. *IEEE Transactions Automatic Control*, 42(7):896–911, July 1997.

[SP96a] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control, Analysis and Design*. John Wiley & Sons, Chichester, England, 1996.

[SP96b] R. Smith and A. Packard. Optimal control of pertubed linear static systems. *IEEE Transactions Automatic Control*, 41(4):579–584, April 1996.

[SR99] D.J. Stilwell and W.J. Rugh. Interpolation of observer state feedback controllers for gain scheduling. *IEEE Transactions Automatic Control*, 44(6):1225–1229, June 1999.

[SRF04] G. Scorletti, L. Rossignol, and V. Fromion. Frequency dependent model phase reduction using convex optimization. In *Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, December 2004.

[TA00] H.D. Tuan and P. Apkarian. Monotonic relaxation for robust control: new characterization. In *American Control Conference*, pages 1914–1918, Chicago, Illinois, USA, June 2000.

[TAN01] H. D. Tuan, P. Apkarian, and T. Q. Nguyen. Robust and reduced-order filtering: new lmi-based characterizations and methods. *IEEE Trans. on Signal Processing*, 40(12):2975–2984, 2001.

[TdS01] A. Trofino and C.E. de Souza. Biquadratic stability of uncertain linear systems. *IEEE Transactions Automatic Control*, 46(8):1303–1307, August 2001.

[TdS02] A. Trofino and C.E. de Souza. Guaranteed cost control for rational parameter-dependent discrete-time systems. In *Conference on Decision and Control*, pages 1870–1875, Las Vegas, Nevada, USA, December 2002.

[TS94] K. Tanaka and M. Sano. A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer. *IEEE Transactions on Fuzzy Systems*, 2(2):119–134, may 1994.

[VBWH03] L. Vandenberghe, V. Balakrishnan, R. Wallin, and A. Hansson. On the implementation of primal-dual interior-point methods for semidefinite programming problems derived from the KYP lemma. In *Conference on Decision and Control*, pages 4658–4663, Hawaii, December 2003.

[WB02] F. Wang and V. Balakrishnan. Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems. *IEEE Transactions Automatic Control*, 47(5):720–734, May 2002.

[WYPB96] F. Wu, X.H. Yang, A. Packard, and G. Becker. Induced l_2-norm control for LPV systems with bounded parameter variations rates. *International Journal of Robust and Nonlinear Control*, 6(9-10):983–998, November 1996.

[YS97] J. Yu and A. Sideris. H_∞ control with parametric Lyapunov functions. *Automatica*, 30(1):57–69, January 1997.

[Zam81] G. Zames. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions Automatic Control*, AC-26(2):301–320, April 1981.

[ZDG95] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, New Jersey, 1995.

[ZTI03] X. Zhang, P. Tsiotras, and T. Iwasaki. Parameter-dependent Lyapunov functions for stability analysis of LTI parameter-dependent systems. In *Conference on Decision and Control*, volume 5, pages 5168–5173, Maui, Hawaii USA, December 2003.

A Alternative formulation using a polytopic approach

In this appendix, we develop the simpler approach applied in section 3.2 for the case of affine and rational of degree one decision variables. We consider an augmented plant $P(s,\theta)$ whose particular state space matrices depend on θ:

$$\left\{\begin{array}{rcrcrcr} \dot{x}(t) & = & Ax(t) & + & B_w w(t) & + & B_u u(t) \\ z(t) & = & C_z(\theta)x(t) & + & D_{zw}(\theta)w(t) & + & D_{zu}(\theta)u(t) \\ y(t) & = & C_y x(t) & + & D_{yw}w(t) & & \end{array}\right. . \tag{25}$$

Such a particular case is obtained when only output weighting functions depend on θ as follows:

$$W(s,\theta) = \frac{1}{s} I \star \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W(\theta) & D_W(\theta) \end{array} \right]. \tag{26}$$

Note that this is the case of the example presented in section 3.2. In addition, the matrix $\left[\begin{array}{ccc} C_z(\theta) & D_{zw}(\theta) & D_{zu}(\theta) \end{array}\right]$ is assumed to be a matrix of rational functions of degree one: that is, it can be written as:

$$\left[\begin{array}{ccc} C_z(\theta) & D_{zw}(\theta) & D_{zu}(\theta) \end{array}\right] = \left[\begin{array}{ccc} C_{z0} & D_{zw0} & D_{zu0} \end{array}\right] + \frac{\theta}{1+d\theta} \left[\begin{array}{ccc} C_{z1} & D_{zw1} & D_{zu1} \end{array}\right]$$

with for any $\theta \in [0,\ 1]$, $1 + d\theta \neq 0$. We then have the following Theorem.

Theorem A.1 *There exist decision variables* $\mathcal{X}(\theta) = \mathcal{X}_0 + \frac{\theta}{1+d\theta}\mathcal{X}_1$, $\mathcal{Y}(\theta) = \mathcal{Y}_0 + \frac{\theta}{1+d\theta}\mathcal{Y}_1$ *and*

$$\mathcal{V}(\theta) = \left[\begin{array}{cc} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{C}_0 & \mathcal{D}_0 \end{array}\right] + \frac{\theta}{1+d\theta} \left[\begin{array}{cc} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{C}_1 & \mathcal{D}_1 \end{array}\right]$$

such that for any $\theta \in [0,1]$, *constraint (3) and constraint (4) are satisfied if*

1.
$$C_{z1}\mathcal{X}_1 + D_{zu1}\mathcal{C}_1 = 0 \qquad D_{zu1}\mathcal{D}_1 = 0\ ; \tag{27}$$

2. *for* $\alpha \in \{0,1\}$*:*
$$\left[\begin{array}{cc} \mathcal{X}_0 + \frac{\alpha}{1+d}\mathcal{X}_1 & I \\ I & \mathcal{Y}_0 + \frac{\alpha}{1+d}\mathcal{Y}_1 \end{array}\right] > 0 \tag{28}$$
$$T_0 + \frac{\alpha}{1+d} T_1 < 0 \tag{29}$$

where

$$T_0 = \left[\begin{array}{c|c|c|c} \begin{array}{c} A\mathcal{X}_0 + \mathcal{X}_0 A^T + \dots \\ B_u\mathcal{C}_0 + (B_u\mathcal{C}_0)^T \end{array} & (\mathcal{A}_0)^T + A + B_u\mathcal{D}_0 C_y & B_w + B_u\mathcal{D}_0 D_{yw} & (C_{z0}\mathcal{X}_0 + D_{zu0}\mathcal{C}_0)^T \\ \hline \mathcal{A}_0 + (A + B_u\mathcal{D}_0 C_y)^T & \begin{array}{c} A^T\mathcal{Y}_0 + \mathcal{Y}_0 A + \dots \\ \mathcal{B}_0 C_y + (\mathcal{B}_0 C_y)^T \end{array} & \mathcal{Y}_0 B_w + \mathcal{B}_0 D_{yw} & (C_{z0} + D_{zu0}\mathcal{D}_0 C_y)^T \\ \hline (B_w + B_u\mathcal{D}_0 D_{yw})^T & (\mathcal{Y}_0 B_w + \mathcal{B}_0 D_{yw})^T & -\gamma I & (D_{zw0} + D_{zu0}\mathcal{D}0 D_{yw})^T \\ \hline C_{z0}\mathcal{X}_0 + D_{zu0}\mathcal{C}_0 & C_{z0} + D_{zu0}\mathcal{D}_0 C_y & D_{zw0} + D_{zu0}\mathcal{D}_0 D_{yw} & -\gamma I \end{array}\right], \tag{30}$$

$$T_1 = \left[\begin{array}{c|c|c|c} \begin{array}{c} A\mathcal{X}_1 + \mathcal{X}_1 A^T + \\ B_u\mathcal{C}_1 + (B_u\mathcal{C}_1)^T \end{array} & (\mathcal{A}_1)^T + B_u\mathcal{D}_1 C_y & B_u\mathcal{D}_1 D_{yw} & \begin{array}{c} (C_{z1}\mathcal{X}_0 + C_{z0}\mathcal{X}_1)^T + \dots \\ (D_{zu1}\mathcal{C}_0 + D_{zu0}\mathcal{C}_1)^T \end{array} \\ \hline \mathcal{A}_1 + (B_u\mathcal{D}_1 C_y)^T & \begin{array}{c} A^T\mathcal{Y}_1 + \mathcal{Y}_1 A + \dots \\ \mathcal{B}_1 C_y + (\mathcal{B}_1 C_y)^T \end{array} & \mathcal{Y}_1 B_w + \mathcal{B}_1 D_{yw} & \begin{array}{c} (C_{z1})^T + \dots \\ ((D_{zu1}\mathcal{D}_0 + D_{zu0}\mathcal{D}_1) C_y)^T \end{array} \\ \hline (B_u\mathcal{D}_1 D_{yw})^T & (\mathcal{Y}_1 B_w + \mathcal{B}_1 D_{yw})^T & 0 & \begin{array}{c} (D_{zw1})^T + \dots \\ ((D_{zu1}\mathcal{D}_0 + D_{zu0}\mathcal{D}_1) D_{yw})^T \end{array} \\ \hline \begin{array}{c} C_{z1}\mathcal{X}_0 + C_{z0}\mathcal{X}_1 + \dots \\ D_{zu1}\mathcal{C}_0 + D_{zu0}\mathcal{C}_1 \end{array} & \begin{array}{c} C_{z1} + \dots \\ (D_{zu1}\mathcal{D}_0 + D_{zu0}\mathcal{D}_1) C_y \end{array} & \begin{array}{c} D_{zw1} + \dots \\ (D_{zu1}\mathcal{D}_0 + D_{zu0}\mathcal{D}_1) D_{yw} \end{array} & 0 \end{array}\right]. \tag{31}$$

Remark Even when the decision variables are affine functions ($d = 0$), the obtained state space representation of the controller is not an affine function of θ but a rational one.

Computation Finding $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{Y}_0$, $\mathcal{Y}_1$, $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{D}_0$, $\mathcal{D}_1$ such that condition (27), condition (28) and condition (29) are satisfied introduces two kinds of constraints:

- *Linear Matrix Equality ones:* the set of decision variables which satisfy the conditions (27) can be linearly parameterized using the well-known Gauss-Seidel iteration, which leads to a new set of decision variables;
- *LMI ones:* minimizing γ such that there exist decision variables in this last set satisfying the conditions (28) and (29) is a linear cost minimization problem over LMI constraints [BEFB94].

Proof Let us focus on condition (4). In this case, it can be readily rewritten as:

$$\forall \theta \in [0,1],\ T_0 + \frac{\theta}{1+c\theta} T_1 + \left(\frac{\theta}{1+c\theta}\right)^2 T_2 < 0. \tag{32}$$

where T_0 and T_1 are defined in equations (30) and (31) respectively and where

$$T_2 = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & (C_{z1}\mathcal{X}_1 + D_{zu1}\mathcal{C}_1)^T \\ \hline 0 & 0 & 0 & (D_{zu1}\mathcal{D}_1 C_y)^T \\ \hline 0 & 0 & 0 & (D_{zu1}\mathcal{D}_1 D_{yw})^T \\ \hline C_{z1}\mathcal{X}_1 + D_{zu1}\mathcal{C}_1 & D_{zu1}\mathcal{D}_1 C_y & D_{zu1}\mathcal{D}_1 D_{yw} & 0 \end{array}\right]. \tag{33}$$

Equality conditions (27) impose $T_2 = 0$. Condition (32) is then equivalent to condition (29) by applying the following lemma [GCG93]:

Lemma A.1 *Consider a symmetric matrix $T(\theta)$ rational function of degree one. $T(\theta)$ is positive definite for each $\theta \in [0,1]$ if and only if $T(\alpha)$ is positive definite for $\alpha \in \{0,1\}$.*

Condition (3) is directly equivalent to condition (28) by applying Lemma A.1. □

B Extension of Lemma 2.1

Lemma 2.1 is here extended to parameter dependent LMI with several parameters:

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix}$$

with $\theta_i \in [0,1]$, $i = 1,\dots,m$. Let us first introduced the following set of structured block diagonal symmetric matrices:

$$\mathcal{S}(r_i) = \left\{ \mathcal{S} = \begin{bmatrix} \mathcal{S}_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \mathcal{S}_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mathcal{S}_m \end{bmatrix} \middle| \mathcal{S}_i = \mathcal{S}_i^T > 0,\, \dim(\mathcal{S}_i) = r_i,\ i = 1,\dots,m \right\},$$

and the following set of structured block diagonal skew-symmetric matrices:

$$\mathcal{G}(r_i) = \left\{ \mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \mathcal{G}_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mathcal{G}_m \end{bmatrix} \middle| \mathcal{G}_i = -\mathcal{G}_i^T,\, \dim(\mathcal{G}_i) = r_i,\, i = 1,\dots,m \right\}.$$

The extension of Lemma 2.1 is now presented.

Lemma B.1 *Let $H_1(\theta)$ and $H_2(\theta)$ be matrices of rational functions of θ, well-posed on $[0,1] \times \cdots \times [0,1]$. Let C be a constant matrix and N_j be m positive integers. Let $c_{i_1,\dots,i_m}$, $i_j = 0,\dots,N_j$, $j = 0,\dots,m$, $c_{0,\dots,0} = 1$ be scalars such that for any $\theta \in [0,1] \times \cdots \times [0,1]$, $\sum_{j=1}^{m} \sum_{i_j=0}^{N_j} c_{i_1,\dots,i_m} \theta_1^{i_1} \dots \theta_m^{i_m} \neq 0$.*

Then there exists $\Upsilon(\theta)$ a (possibly structured) matrix of rational functions of θ, well-posed on $[0,1] \times \cdots \times [0,1]$:

$$\Upsilon(\theta) = \frac{\sum_{j=1}^{m} \sum_{i_j=0}^{N_j} \Upsilon_{i_1,\dots,i_m} \theta_1^{i_1} \dots \theta_m^{i_m}}{\sum_{j=1}^{m} \sum_{i_j=0}^{N_j} d_{i_1,\dots,i_m} \theta_1^{i_1} \dots \theta_m^{i_m}}$$

with $d_{0,\dots,0} = 1$ such that

$$\forall \theta \in [0,1] \times \cdots \times [0,1], \quad H_1(\theta)(C + \Upsilon(\theta))H_2(\theta) + (H_1(\theta)(C + \Upsilon(\theta))H_2(\theta))^T < 0$$

if there exist matrices $\Upsilon_{i_1,\dots,i_m}$ and scalars $d_{i_1,\dots,i_m}$, $i_j = 0,\dots,N_j$, $j = 0,\dots,m$ with $d_{0,\dots,0} = 1$ such that the two following conditions are satisfied:
(i) there exist $\mathcal{S}_d \in \mathcal{S}(k_i)$ and $\mathcal{G}_d \in \mathcal{G}(k_i)$ such that:

$$\mathcal{L}\left(A_P, B_P, \left[\frac{0}{C_P}\right], \left[\frac{1}{D_P}\right], \begin{bmatrix} 0 & -\mathcal{T}(d_{i_1,\dots,i_m}) \\ -\mathcal{T}(d_{i_1,\dots,i_m})^T & 0 \end{bmatrix}, \mathcal{S}_d, \mathcal{G}_d \right) < 0$$

where $\mathcal{T}(d_{i_1,\ldots,i_m})$ *is an affine function of* $d_{i_1,\ldots,i_m}$ *such that for some positive* k_i, $i = 1,\ldots,m$

$$\mathcal{T}(d_{i_1,\ldots,i_m}) \times \begin{bmatrix} \theta_1 I_{k_1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \theta_i I_{k_i} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \theta_m I_{k_m} \end{bmatrix} \star \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & D_P \end{array}\right] \triangleq \frac{\sum_{j=1}^{m}\sum_{i_j=0}^{N_j} d_{i_1,\ldots,i_m}\theta_1^{i_1}\ldots\theta_m^{i_m}}{\sum_{j=1}^{m}\sum_{i_j=0}^{N_j} c_{i_1,\ldots,i_m}\theta_1^{i_1}\ldots\theta_m^{i_m}} \tag{34}$$

(ii) there exist $\mathcal{S} \in \mathcal{S}(l_i)$ *and* $\mathcal{G} \in \mathcal{G}(l_i)$ *such that*

$$\mathcal{L}\left(A_H, B_H, C_H, D_H, \begin{bmatrix} 0 & \mathcal{U}(\Upsilon_{i_1,\ldots,i_m}, d_{i_1,\ldots,i_m}) \\ \mathcal{U}(\Upsilon_{i_1,\ldots,i_m}, d_{i_1,\ldots,i_m})^T & 0 \end{bmatrix}, \mathcal{S}, \mathcal{G}\right) < 0$$

where

$$\begin{bmatrix} \theta_1 I_{l_1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \theta_i I_{l_i} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \theta_m I_{l_m} \end{bmatrix} \star \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] \triangleq \begin{bmatrix} H_1(\theta)^T \\ \bar{H}(\theta)H_2(\theta) \end{bmatrix}$$

and where $\mathcal{U}(\Upsilon_{i_1,\ldots,i_m}, d_{i_1,\ldots,i_m})$ *is an affine function of* $\Upsilon_{i_1,\ldots,i_m}$ *and* $d_{i_1,\ldots,i_m}$, $i_j = 0,\ldots,N_j$, $j = 0,\ldots,m$ *such that for some positive* l_i, $i = 1,\ldots,m$

$$\mathcal{U}(\Upsilon_{i_1,\ldots,i_m}, d_{i_1,\ldots,i_m})\bar{H}(\theta) = \frac{\sum_{j=1}^{m}\sum_{i_j=0}^{N_j}(\Upsilon_{i_1,\ldots,i_m} + d_{i_1,\ldots,i_m}C)\theta_1^{i_1}\ldots\theta_m^{i_m}}{\sum_{j=1}^{m}\sum_{i_j=0}^{N_j} c_{i_1,\ldots,i_m}\theta_1^{i_1}\ldots\theta_m^{i_m}}. \tag{35}$$

Remark The factorizations (34) and (35) are always possible, although not unique.

Remark In Lemma B.1, we obtain only sufficient conditions since Lemma 2.2 is no longer necessary and sufficient in the case of several parameters.