Parameter dependent H_∞ control by finite dimensional LMI optimization: application to trade-off dependent control

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Abstract

In this paper, we consider the design of an H_∞ trade-off dependent controller, that is, a controller such that, for a given Linear Time-Invariant plant, a set of performance trade-offs parameterized by a scalar θ is satisfied. The controller state space matrices are explicit functions of θ. This new problem is a special case of the design of a parameter dependent controller for a parameter dependent plant, which has many application in Automatic Control. This last design problem can be naturally formulated as a convex but infinite dimensional optimization problem involving parameter dependent Linear Matrix Inequality (LMI) constraints. In this paper, we propose finite dimensional (parameter independent) LMI constraints which are equivalent to the parameter dependent LMI constraints. The parameter dependent controller design is then formulated as a convex finite dimensional LMI optimization problem. The obtained result is then applied to the trade-off dependent controller design. A numerical example emphasizes the strong interest of our finite dimensional optimization problem with respect to the trade-off dependent control application.

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1 Introduction

Trade-off dependent controller design During the last twenty years, dramatic advances were accomplished in the design of Linear Time-Invariant (LTI) controllers for LTI plants using the frequency domain approach. The so-called H_∞ control approach [48, 16, 42] is at now a mature design method. The existing methods focus on designing one particular LTI controller for one particular set of design specifications corresponding to one particular performance trade-off.

Nevertheless, in some control problems, for a given plant, an important issue is to retune in situ the controller in order to ensure different performance trade-offs. Such retunings can be performed during the controller exploitation [1]. In these conditions, a new controller design by an Automatic Control engineer has to be avoided. A promising solution to this practical problem is the design of a trade-off dependent controller, that is, a controller whose gains are explicit functions on a continuous set of trade-offs. Retuning the controller just amounts to select a different trade-off which is easy and affordable without expertise in Automatic Control. Another possible application can be the on-line performance retuning for *e.g.* ship control with the rejection of wave disturbances which depend on the sea conditions [30], active suspension in order to adapt it to road conditions [20], etc.. In these cases, the trade-off parameter is in a continuous interval. This is the key fact for using a trade off dependent controller instead of, *e.g.*, a finite number of controllers.

Using classical design methods, when specifications can be ensured using a low complexity controller, *e.g.* Proportional Integral, engineers use to investigate the link between controller gains and design specifications (such as time response, control input energy..) in order to obtain (re)tuning rules. This link (generally qualitative) can be established based on *e.g.* know-how, classical rules of automatic control.. Nevertheless, for ensuring more stringent specifications, more complex (multivariable) controllers are usually designed using modern methods (such as Linear Quadratic Gaussian, Model Predictive Control, H_∞ control..). The obtained controllers are defined by numerous parameters whose links with the design specifications are not crystal clear. This prevents the controller (re)tuning in order to ensure different trade-offs. Here again, if a trade-off dependent controller has been designed,

the controller (re)tuning just amounts to choose the trade-off.

To our best knowledge, a complete solution to the trade-off dependent controller design was not previously proposed. Let us introduce a parameter $\theta \in [0, 1]$ which parameterizes the performance trade-offs. The problem is then to design a controller whose gains explicitly depend on this parameter θ. In the H_∞ control approach, the design of a controller is formulated as an optimization problem on weighted closed loop transfer functions. The considered closed loop transfer functions and the weighting functions are defined by the generalized plant [16]. The desired performance specifications are introduced through the choice of the weighting functions. As a consequence, the performance trade-off can be defined by choosing the weighting functions depending on θ. Even if, in our case, the plant does not depend on θ, the generalized plant depends on.

Parameter dependent controller design   Our problem is thus a subcase of the design of a parameter dependent controller for a (generalized) parameter dependent plant. In the first part of this paper, we focus on this problem. Performance is considered through the use of the H_∞ norm. Nevertheless, other performance criteria (such as H_2, multiobjective...) can be considered in a similar way.

The design of a parameter dependent controller for a (generalized) parameter dependent plant has a strong interest since it encompasses numerous control design problems such as: gain scheduled control [43, 22], saturated system control [32], spatial system control [11], adaptive control [19], low cost identification [8] to cite a few.

In this paper, we propose a solution to the parameter dependent control with an application to the trade-off dependent control. Some other applications of this solution, listed above, are probably more important or challenging. Nevertheless, in addition to its own interest, the performance of the obtained trade-off dependent controllers can be analyzed in the numerical examples using basic automatic control knowledge. Furthermore, the "best performance" can be readily evaluated. Thus, we prefer to focus on the trade-off dependent control since it allows to clearly evaluate the benefit of our solution.

Infinite dimensional optimization   The parameter dependent controller design can be naturally formulated as a convex but infinite dimensional optimization problem as it involves parameter dependent Linear Matrix Inequalities. Its decision variables are (unknown) functions of the parameter. This infinite dimensional nature forbids a practical computation of a solution.

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The same difficulty arises in robustness analysis [18, 23] or Linear Parameter Varying (LPV) control [34, 3]. For these particular problems, the basic idea is to derive finite dimensional Linear Matrix Inequality (LMI) optimization problems [9]. Different approaches were proposed with possibly introducing conservatism, see section 2.5 for a discussion. One of the main features of these approaches is the choice of function sets for the decision variables. The most general choice was, up to now, rational with a priori chosen degree and denominator.

Proposed approach In this paper, we consider the more general problem of replacing a parameter dependent LMI by a finite number of parameter independent LMIs. Our approach is based on an extended version of the Kalman-Yakubovich-Popov Lemma and an elementary property of the real valued polynomials. Due to its potential important applications, some extension of Kalman-Yakubovich-Popov Lemma was recently largely investigated [36], with an emphasis on the necessity [28, 27, 39]. It is strongly related to the μ computation for special uncertainty sets where the μ upper bound [17, 33] gives the actual value of μ. Moreover, recent papers focus on numerical algorithms dedicated to the LMI based condition of the Kalman-Yakubovich-Popov Lemma (see [45] and the references therein). This solution is dramatically more efficient than a direct use of general purpose LMI solvers such as [24].

In this paper, we investigate applications of the extended Kalman-Yakubovich-Popov Lemma to problems of interest. We first reveal that a rational dependent parameter LMI with rational decision variables can be equivalently recast as a finite number of parameter independent LMIs in the case of one parameter[1]. In contrast with previous results, the denominator of the rational decision variables is $e.g.$ no longer a priori chosen (Lemma 2.1, section 2.4). Such a contribution allows to dramatically improve the obtained result. This solution is part of our continuing investigation into the transformation of an infinite dimensional optimization problem into a finite one [40, 7, 36, 41].

The obtained result is applied to propose a solution to the parameter dependent controller design in the form of a finite dimensional optimization problem involving LMIs. In this application, the interest of rational decision variables is crystal clear: the state space matrices of the parameter dependent controller are then rational functions in θ, which is a desirable feature for real-time implementation. This solution is the second contribution of this paper, perhaps one of the most interesting. The third contribution of the paper is

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[1]In the case of several parameters, the proposed conditions are only sufficient, see the Appendix.

then to derive a solution to the trade-off dependent controller design problem. This paper is based on the conference paper [15]. An extended version of this paper is available [14].

Paper outline   Section 2 focus on the parameter dependent controller design problem: the problem is formulated in section 2.1. In section 2.2, its solution as an infinite dimensional optimization problem is presented. An equivalent finite dimensional optimization problem is then proposed in section 2.3. The proof, developed in section 2.4, is based on a solution to the general problem of replacing a parameter dependent LMI by a finite number of parameter independent LMIs. Section 3 is an application to the trade-off dependent controller design with a numerical example.

Notations and definitions I_n and $0_{m \times p}$ denotes respectively the $n \times n$ identity matrix and the zero matrix of size $m \times p$. The subscript is omitted when it is evident from the context. $P > 0$ denotes that the matrix P is positive definite. $\dim(T)$ is the dimension of the matrix T. The Redheffer star product [49] is denoted by $\star$. A Linear Fractional Transformation (LFT) is a particular Redheffer star product defined, with $(I - A\Delta)$ invertible, by:

$$\Delta \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = D + C\Delta(I - A\Delta)^{-1}B.$$

Elementary operations on LFT (addition, product, etc..) are defined, see *e.g.* [49].

2   Parameter dependent controller design for a parameter dependent plant

2.1   Problem formulation

Let us consider the LTI system $P(s, \theta)$ defined by a parameter dependent state space representation:

$$\begin{cases} \dot{x}(t) &= A(\theta)x(t) &+ B_w(\theta)w(t) &+ B_u(\theta)u(t) \\ z(t) &= C_z(\theta)x(t) &+ D_{zw}(\theta)w(t) &+ D_{zu}(\theta)u(t) \\ y(t) &= C_y(\theta)x(t) &+ D_{yw}(\theta)w(t) \end{cases} \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ the command input, $y(t) \in \mathbb{R}^{n_y}$ the measured output, $z(t) \in \mathbb{R}^{n_z}$ the controlled output, $w(t) \in \mathbb{R}^{n_w}$ the disturbance input and θ a time-invariant scalar parameter (conventionally

$\theta \in [0, 1]$). The state space matrices of $P(s, \theta)$ are assumed to be rational functions of θ, well-posed on $[0, 1]$. We then consider the following problem.

EXTENDED H_∞ CONTROL PROBLEM   Given $P(s, \theta)$ as defined in (1) and $\gamma > 0$ find, if there exist, a parameter dependent controller

$$K(s, \theta) = \frac{1}{s} I_n \star \left[\begin{array}{c|c} A_K(\theta) & B_K(\theta) \\ \hline C_K(\theta) & D_K(\theta) \end{array} \right] \tag{2}$$

where $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ are rational functions of θ, of limited degree and well-posed on $[0, 1]$, such that, for any $\theta \in [0, 1]$:

1. the closed loop system $P(s, \theta) \star K(s, \theta)$ is asymptotically stable;

2. $\|P(s, \theta) \star K(s, \theta)\|_\infty < \gamma$.

The state space matrices $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ of the controller (2) are required to be rational in θ of limited degree in order to obtain a controller implementation of reasonable complexity. A more complex dependence on θ is useless since it would be necessary to approximate these functions by, for example, rational or polynomial ones of limited degree for a practical implementation. For the sake of briefness, the control objective is defined using the H_∞ norm. Nevertheless, the proposed approach can be readily applied *e.g.* to the H_2 control problem [16] or to the multiobjective control problem [38].

2.2 Proposed approach

In the following theorem, the design of a parameter dependent H_∞ controller is formulated as an optimization problem. It is obtained by a direct extension of the standard H_∞ control solution proposed in [38].

Theorem 2.1 *Given $\gamma > 0$, there exists a parameter dependent controller*

$$K(s, \theta) = \frac{1}{s} I_n \star \left[\begin{array}{c|c} A_K(\theta) & B_K(\theta) \\ \hline C_K(\theta) & D_K(\theta) \end{array} \right]$$

such that, for any $\theta \in [0, 1]$:

1. *the closed loop system $P(s, \theta) \star K(s, \theta)$ is asymptotically stable;*

2. *$\|P(s, \theta) \star K(s, \theta)\|_\infty < \gamma$*

if and only if there exist

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- *symmetric parameter dependent matrices $\mathcal{X}(\theta) \in \mathbb{R}^{n \times n}$ and $\mathcal{Y}(\theta) \in \mathbb{R}^{n \times n}$ well-posed on $[0,1]$;*

- *parameter dependent matrices $\mathcal{A}(\theta) \in \mathbb{R}^{n \times n}$, $\mathcal{B}(\theta) \in \mathbb{R}^{n \times n_y}$, $\mathcal{C}(\theta) \in \mathbb{R}^{n_u \times n}$ and $\mathcal{D}(\theta) \in \mathbb{R}^{n_u \times n_y}$ well-posed on $[0,1]$*

satisfying (3) and (4) for any $\theta \in [0,1]$:

$$
\begin{bmatrix} \mathcal{X}(\theta) & I \\ I & \mathcal{Y}(\theta) \end{bmatrix} > 0 \tag{3}
$$

$$
\begin{bmatrix}
\begin{array}{c} A(\theta)\mathcal{X}(\theta) + \mathcal{X}(\theta)A(\theta)^T + \ldots \\ B_u(\theta)\mathcal{C}(\theta) + (B_u(\theta)\mathcal{C}(\theta))^T \end{array} & (.)^T & (.)^T & (.)^T \\
\begin{array}{c} \mathcal{A}(\theta) + \ldots \\ (A(\theta) + B_u(\theta)\mathcal{D}(\theta)C_y(\theta))^T \end{array} & \begin{array}{c} A(\theta)^T\mathcal{Y}(\theta) + \mathcal{Y}(\theta)A(\theta) + \ldots \\ \mathcal{B}(\theta)C_y(\theta) + (\mathcal{B}(\theta)C_y(\theta))^T \end{array} & (.)^T & (.)^T \\
(B_w(\theta) + B_u(\theta)\mathcal{D}(\theta)D_{yw}(\theta))^T & (\mathcal{Y}(\theta)B_w(\theta) + \mathcal{B}(\theta)D_{yw}(\theta))^T & -\gamma I & (.)^T \\
C_z(\theta)\mathcal{X}(\theta) + D_{zu}(\theta)\mathcal{C}(\theta) & C_z(\theta) + D_{zu}(\theta)\mathcal{D}(\theta)C_y(\theta) & D_{zw}(\theta) + D_{zu}(\theta)\mathcal{D}(\theta)D_{yw}(\theta) & -\gamma I
\end{bmatrix} < 0 \tag{4}
$$

where $(.)^T$ denotes the transpose of the symmetric block.

A state space representation of a parameter dependent controller is then obtained with

$$
\begin{bmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{bmatrix} = \begin{bmatrix} L(\theta) & -J(\theta) & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix} \times \ldots
$$
$$
\left(\begin{bmatrix} I_n & 0 \\ 0 & B_u(\theta) \\ 0 & I_{n_u} \end{bmatrix} \mathcal{V}(\theta) \begin{bmatrix} \mathcal{X}(\theta)^{-1} & 0 \\ -C_y(\theta) & I_{n_y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A(\theta) & 0 \\ 0 & 0 \end{bmatrix} \right) \tag{5}
$$

where

$$
\begin{bmatrix} L(\theta) & -J(\theta) \end{bmatrix} = \left(\begin{bmatrix} I_n \\ I_n \end{bmatrix} \mathcal{X}(\theta) \begin{bmatrix} I_n & \mathcal{Y}(\theta) \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & -I_n & I_n \end{bmatrix} \right) \star I_n
$$

and where

$$
\mathcal{V}(\theta) = \begin{bmatrix} \mathcal{A}(\theta) & \mathcal{B}(\theta) \\ \mathcal{C}(\theta) & \mathcal{D}(\theta) \end{bmatrix}.
$$

The optimization problem involving constraint (3) and constraint (4) is convex in the decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$, which is a desirable feature. Unfortunately, it is also infinite dimensional. As parameterized by θ, there is an infinite number of constraints. As functions of θ, the decision variables are in an infinite dimensional space. In this form, this prevents an efficient computation of the solution.

However, the problem considered in Theorem 2.1 is a little bit more general than the one considered in EXTENDED H_∞ CONTROL PROBLEM. Remind that the controller state space matrices $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ of the controller (2) are required to be rational functions of θ of limited degree. From equation (5), the decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ are then enforced to be rational functions in θ of limited degree. To this purpose, the optimization problem introduced in Theorem 2.1 is modified with:

$$
\mathcal{X}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{X}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \quad \mathcal{Y}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{Y}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \quad \mathcal{V}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{V}_i}{1 + \sum_{i=1}^{N} \theta^i d_i},
$$
(6)

where for $i = 0, \ldots, N$, $\mathcal{X}_i = \mathcal{X}_i^T \in \mathbb{R}^{n \times n}$, $\mathcal{Y}_i = \mathcal{Y}_i^T \in \mathbb{R}^{n \times n}$, $\mathcal{V}_i \in \mathbb{R}^{(n+n_u) \times (n+n_y)}$, and for $i = 1, \ldots, N$, $d_i \in \mathbb{R}$. In (6), only N is *a priori* chosen[2]. The integer N is a trade-off parameter. A small N allows to obtain a low complexity controller, that is, a controller whose state space matrices are rational functions of small degree, with the possible drawback of a poor performance. Performance can be improved by increasing N, with the possible drawback of a large complexity controller. The example presented in section 3.2 illustrate that good performance can be obtained with a small N.

With respect to EXTENDED H_∞ CONTROL PROBLEM, an interesting optimization problem is thus :

Given N, find the decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ defined by (6) such that for any $\theta \in [0, 1]$, constraint (3) and constraint (4) are satisfied.

In the next section, this infinite dimensional optimization problem is equivalently recast as a finite dimensional convex optimization problem involving Linear Matrix Inequalities which can be efficiently computed.

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[2] N is the degree of the denominator when the rational function is written as the fraction of two polynomials in $\frac{1}{\theta}$. Thus, it is not necessary that $d_N \neq 0$ since the higher term coefficient is 1.

2.3 Finite dimensional solution

Before presenting the result, let us first associate to (6):

$$\mathcal{R}_{\mathcal{X}} = \begin{bmatrix} \mathcal{X}_N & \cdots & \mathcal{X}_1 & \mathcal{X}_0 \end{bmatrix}$$

$$\mathcal{R}_{\mathcal{Y}} = \begin{bmatrix} \mathcal{Y}_N & \cdots & \mathcal{Y}_1 & \mathcal{Y}_0 \end{bmatrix}$$

$$\mathcal{R}_{\mathcal{V}} = \begin{bmatrix} \mathcal{V}_N & \cdots & \mathcal{V}_1 & \mathcal{V}_0 \end{bmatrix}$$

$$\mathcal{R}_{d,p} = \begin{bmatrix} d_N I_p & \cdots & d_1 I_p & I_p \end{bmatrix}$$

$$J_p(c_i) = \left[\begin{array}{ccccc|c} 0 & I_p & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & I_p & 0 \\ -c_N I_p & \cdots & \cdots & \cdots & -c_1 I_p & I_p \\ \hline I_p & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & I_p & 0 \\ -c_N I_p & \cdots & \cdots & \cdots & -c_1 I_p & I_p \end{array} \right] \tag{7}$$

where c_i, $i = 1, \ldots, N$ are real scalars such that, for any $\theta \in [0,1]$, $1 + \sum_{i=1}^{N} \theta^i c_i \neq 0$. Let us introduce $\mathcal{L}(A_\Phi, B_\Phi, C_\Phi, D_\Phi, M, \mathcal{S}, \mathcal{G})$, with $\dim(\mathcal{S}) = \dim(\mathcal{G}) = \dim(A_\Phi)$, defined by:

$$\begin{bmatrix} C_\Phi^T \\ D_\Phi^T \end{bmatrix} M \begin{bmatrix} C_\Phi & D_\Phi \end{bmatrix} + \begin{bmatrix} A_\Phi^T(\mathcal{S} - \mathcal{G}) + (\mathcal{S} + \mathcal{G})A_\Phi - 2\mathcal{S} & (\mathcal{S} + \mathcal{G})B_\Phi \\ B_\Phi^T(\mathcal{S} - \mathcal{G}) & 0 \end{bmatrix}.$$

Theorem 2.2 *Given N, there exist decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ defined by (6) and well-posed on $[0,1]$ such that, for any $\theta \in [0,1]$, constraint (3) and constraint (4) are satisfied if and only if there exist*

- *symmetric matrices $\mathcal{X}_i \in \mathbb{R}^{n \times n}$ and $\mathcal{Y}_i \in \mathbb{R}^{n \times n}$, and matrices $\mathcal{V}_i \in \mathbb{R}^{(n+n_u) \times (n+n_y)}$, $i = 0, \ldots, N$;*

- *scalars d_i, $i = 1, \ldots, N$*

such that

(i) there exist a symmetric positive definite matrix $\mathcal{S}_0$ and a skew-symmetric matrix $\mathcal{G}_0$ such that

$$\mathcal{L}\left(A_{\Omega_0}, B_{\Omega_0}, C_{\Omega_0}, D_{\Omega_0}, \begin{bmatrix} 0 & -\mathcal{W} \\ -\mathcal{W}^T & 0 \end{bmatrix}, \mathcal{S}_0, \mathcal{G}_0 \right) < 0 \tag{8}$$

with

$$\mathcal{W} \triangleq \begin{bmatrix} \mathcal{R}_\mathcal{X} & 2\mathcal{R}_{d,n} \\ 0 & \mathcal{R}_\mathcal{Y} \end{bmatrix} \; and \; \theta I \star \left[\begin{array}{c|c} A_{\Omega_0} & B_{\Omega_0} \\ \hline C_{\Omega_0} & D_{\Omega_0} \end{array} \right] \triangleq \left[\begin{array}{c} I_{2n} \\ \hline \begin{array}{cc} \theta I \star J_n(c_i) & 0 \\ 0 & \theta I \star J_n(c_i) \end{array} \end{array} \right].$$

(ii) *there exist a symmetric positive definite matrix* $\mathcal{S}$ *and a skew-symmetric matrix* $\mathcal{G}$ *such that*

$$\mathcal{L}\left(A_\Omega, B_\Omega, C_\Omega, D_\Omega, \begin{bmatrix} 0 & \mathcal{Z}(\gamma) \\ \mathcal{Z}(\gamma)^T & 0 \end{bmatrix}, \mathcal{S}, \mathcal{G} \right) < 0 \qquad (9)$$

with

$$\mathcal{Z}(\gamma) = \left[\begin{array}{c|cc|ccc} \mathcal{R}_\mathcal{V} & 0 & 0 & 0 & 0 \\ \hline 0 & \mathcal{R}_\mathcal{X} & \mathcal{R}_{d,n} & 0 & 0 \\ 0 & 0 & \mathcal{R}_\mathcal{Y} & 0 & 0 \\ \hline 0 & \mathcal{R}_{d,n} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{d,n_w} & 0 \\ 0 & 0 & 0 & \gamma\mathcal{R}_{d,n_w} & 0 \\ 0 & 0 & 0 & 0 & \gamma\mathcal{R}_{d,n_z} \end{array} \right]$$

and with

$$\theta I \star \left[\begin{array}{c|c} A_\Omega & B_\Omega \\ \hline C_\Omega & D_\Omega \end{array} \right] \triangleq \begin{bmatrix} F_1(\theta)^T \\ F_2(\theta)F_3(\theta) \end{bmatrix}$$

where

$$F_1(\theta) = \left[\begin{array}{cc|cc|cccc} 0 & B_u(\theta) & A(\theta) & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & A(\theta)^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_w(\theta)^T & B_w(\theta)^T & 0 & -\frac{1}{2}I_{n_w} & 0 \\ 0 & D_{zu}(\theta) & C_z(\theta) & 0 & 0 & D_{zw}(\theta) & 0 & -\frac{1}{2}I_{n_z} \end{array} \right]$$

$$F_2(\theta) = \left[\begin{array}{c|cccc} \theta I \star J_{n+n_y}(c_i) & 0 & 0 & 0 & 0 \\ 0 & \theta I \star J_n(c_i) & 0 & 0 & 0 \\ 0 & 0 & \theta I \star J_n(c_i) & 0 & 0 \\ 0 & 0 & 0 & \theta I \star J_{n_w}(c_i) & 0 \\ 0 & 0 & 0 & 0 & \theta I \star J_{n_z}(c_i) \end{array} \right]$$

$$F_3(\theta) = \left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & C_y(\theta) & D_{yw}(\theta) & 0 \\ \hline \multicolumn{4}{c}{I_{n+n+n_w+n_z}} \end{array} \right]$$

The state space representation of a parameter dependent controller is then obtained using (5) with

$$\mathcal{X}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{X}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \quad \mathcal{Y}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{Y}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \quad \mathcal{V}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{V}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}.$$

Computation: for a given value of γ, the optimization problem defined by (8) and (9) is an LMI feasibility problem since $\mathcal{W}$ and $\mathcal{Z}(\gamma)$ are affine in the decision variables $\mathcal{R}_{\mathcal{X}}$, $\mathcal{R}_{\mathcal{Y}}$, $\mathcal{R}_{\mathcal{V}}$ and d_i, $i = 1, \ldots, N$. Another interesting problem is to minimize γ over LMI constraints (8) and (9). This minimization is a quasi convex optimization problem[3], the minimum value of γ can be found by performing a dichotomy on γ.

Remark The results presented in Theorem 2.2 do not depend on the choice of the scalars c_i. They can be chosen in order to improve the numerical resolution of the optimization problem defined by (8) and (9). For instance, it is chosen in order to limit the order of an LFT realization of $\Omega(\theta)$, thus reducing the computational burden of (9). As the obtained result is insensitive to this choice as long as $1 + \sum_{i=1}^{N} c_i \theta^i$ does not vanish on $[0, 1]$, the introduction of the scalars c_i comes from purely computational considerations.

2.4   Proof of Theorem 2.2

The problem considered in Theorem 2.1 is a particular case of an *infinite* dimensional convex optimization problem involving *parameter dependent* LMI constraints. Following [36], the finite dimensional optimization problem of Theorem 2.2 is derived from Theorem 2.1 along two steps. The first step is the introduction of a finite parameterization of the decision variables. From equation (6), the decision variables $\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ are naturally parameterized by a finite number of coefficients: the matrices $\mathcal{X}_i$, $\mathcal{Y}_i$ $\mathcal{V}_i$, $i = 0, \cdots, N$ and the scalars d_i, $i = 1, \cdots, N$. In order to obtain a finite number of optimization constraints, the second step is the application of the following lemma.

Lemma 2.1 *Let $H_1(\theta)$ and $H_2(\theta)$ be two matrices of rational functions of θ, well-posed on $[0, 1]$. Let C be a matrix and N an integer.*

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[3]Quasi convexity can be proved by a simple adaptation of the proof of the (LMI) Generalized Eignevalue Problems, see [9].

There exists $\Upsilon(\theta)$ a (possibly structured) matrix of rational functions of θ of degree N, well-posed on $[0,1]$:

$$\Upsilon(\theta) = \frac{\sum_{i=0}^{N} \theta^i \Upsilon_i}{1 + \sum_{i=1}^{N} \theta^i d_i}$$

such that

$$\forall \theta \in [0,1], \quad H_1(\theta)(C + \Upsilon(\theta))H_2(\theta) + (H_1(\theta)(C + \Upsilon(\theta))H_2(\theta))^T < 0 \quad (10)$$

if and only if there exist $N+1$ matrices Υ_i, $i = 0, \ldots, N$, and N scalar d_i, $i = 1, \ldots, N$, such that the two following conditions are satisfied:

(i) there exist a symmetric positive definite matrix $\mathcal{S}_d$ and a skew symmetric matrix $\mathcal{G}_d$ such that:

$$\mathcal{L}\left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -c_N & \cdots & \cdots & \cdots & -c_1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \left[\begin{array}{ccccc} & & 0 & & \\ \hline 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -c_N & \cdots & \cdots & \cdots & -c_1 \end{array}\right], \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \cdots\right.$$
$$\left. \cdots \begin{bmatrix} 0 & -\mathcal{R}_{d,1} \\ -\mathcal{R}_{d,1}^T & 0 \end{bmatrix}, \mathcal{S}_d, \mathcal{G}_d\right) < 0$$
$$(11)$$

(ii) there exist a symmetric positive definite matrix $\mathcal{S}$ and a skew symmetric matrix $\mathcal{G}$ such that

$$\mathcal{L}\left(A_H, B_H, C_H, D_H, \begin{bmatrix} 0 & \mathcal{U}(\Upsilon_i, d_i) \\ \mathcal{U}(\Upsilon_i, d_i)^T & 0 \end{bmatrix}, \mathcal{S}, \mathcal{G}\right) < 0 \quad (12)$$

where

$$\theta I \star \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] \triangleq \left[\begin{array}{c} H_1(\theta)^T \\ \bar{H}(\theta)H_2(\theta) \end{array}\right]$$

and where $\mathcal{U}(\Upsilon_i, d_i)$ is an affine function of Υ_i and of d_i such that

$$\mathcal{U}(\Upsilon_i, d_i)\bar{H}(\theta) = \frac{(\Upsilon_0 + C) + \sum_{i=1}^{N} \theta^i (\Upsilon_i + d_i C)}{1 + \sum_{i=1}^{N} \theta^i c_i} \quad (13)$$

12

Remark The factorization (13) is always possible. Such a factorization is not unique. For instance, a factorization is given by $\bar{H}(\theta) = \theta I \star J_p(c_i)$, with p the number of columns of $\Upsilon(\theta)$, $J_p(c_i)$ defined by (7) and $\mathcal{U}(\Upsilon_i, d_i) = \begin{bmatrix} \Upsilon_N + d_N C & \cdots & \Upsilon_1 + d_1 C & \Upsilon_0 + C \end{bmatrix}$.

Interpretation Lemma 2.1 conditions are obtained through an extension of the Kalman-Yakubovich-Popov Lemma. The first point is that any matrix $\Phi(\theta)$ of rational functions of θ, well-posed for $\theta = 0$, has an LFT representation, that is, there exists four matrices A_Φ, B_Φ, C_Φ and D_Φ such that[49]:

$$\Phi(\theta) = \theta I \star \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & D_\Phi \end{array} \right]. \tag{14}$$

Lemma 2.2 (Extended Kalman-Yakubovich-Popov Lemma) *Let $\Phi(\theta)$ be a rational matrix function of θ, well-posed on $[0,1]$, defined by its LFT realization as in (14). Let M be a matrix.*
Then the condition

$$\forall \theta \in [0,1], \qquad \Phi(\theta)^T M \Phi(\theta) < 0$$

holds if and only if there exist a symmetric positive definite matrix $\mathcal{S}$ and a skew-symmetric matrix $\mathcal{G}$ such that

$$\mathcal{L}(A_\Phi, B_\Phi, C_\Phi, D_\Phi, M, \mathcal{S}, \mathcal{G}) < 0.$$

Let us discuss the application of Lemma 2.2 for proving Lemma 2.1 :

- Condition (11) is equivalent to the strict positivity of the rational function $\dfrac{1 + \sum_{i=1}^{N} \theta^i d_i}{1 + \sum_{i=1}^{N} \theta^i c_i}$ for any $\theta \in [0, 1]$. The polynomial $1 + \sum_{i=1}^{N} \theta^i c_i$ has a constant sign on $[0, 1]$ as it does not vanish on $[0, 1]$. The polynomial $1 + \sum_{i=1}^{N} \theta^i d_i$ is enforced to get a constant sign on $[0, 1]$: well posed of $\Upsilon(\theta)$ is thus ensured.

- Condition (12) is equivalent to:

$$\forall \theta \in [0,1], \; H_1(\theta) \left(\frac{(\Upsilon_0 + C) + \sum_{i=1}^{N} \theta^i (\Upsilon_i + d_i C)}{1 + \sum_{i=1}^{N} \theta^i c_i} \right) H_2(\theta) + \cdots$$

$$\cdots + \left(H_1(\theta) \left(\frac{(\Upsilon_0 + C) + \sum_{i=1}^{N} \theta^i (\Upsilon_i + d_i C)}{1 + \sum_{i=1}^{N} \theta^i c_i} \right) H_2(\theta) \right)^T < 0. \tag{15}$$

Thus, combining both items, condition (10) is obtained. Note that the contribution of Lemma 2.1 is allowed by the following fact: the sign of a real valued polynomial with no roots on an interval is constant on this interval. This elementary property was intensively used for formulating the approximation of real valued functions by real valued rational functions as a convex optimization problem (see *e.g.* [10]). It explains why the extension to the rational approximation of complex valued functions (and the model reduction problem) is a much more difficult problem.

Proof of Lemma 2.1 From the previous discussion, condition (i) is a necessary and sufficient condition for the well-posedness of $\Upsilon(\theta)$ on $[0, 1]$. Condition (ii) ensures that condition (10) is satisfied.

Let us focus on condition (i). $\Upsilon(\theta)$ is well-posed on $[0, 1]$ if and only if the polynomial $1 + \sum_{i=1}^{N} d_i \theta^i$ does not vanish on $[0, 1]$. As the polynomial is real valued, with real coefficients, its sign is then constant for any $\theta \in [0, 1]$. The sign is positive since for $\theta = 0$, the value of the polynomial is 1. Let us introduce the polynomial $1 + \sum_{i=1}^{N} c_i \theta^i$ that does not vanish on $[0, 1]$. Then, the polynomial $1 + \sum_{i=1}^{N} d_i \theta^i$ does not vanish on $[0, 1]$ if and only if

$$\forall \theta \in [0, 1], \quad \frac{1 + \sum_{i=1}^{N} d_i \theta^i}{1 + \sum_{i=1}^{N} c_i \theta^i} > 0. \tag{16}$$

Since $\dfrac{1 + \sum_{i=1}^{N} d_i \theta^i}{1 + \sum_{i=1}^{N} c_i \theta^i} = \mathcal{R}_{d,1} \times \theta \star J_1(c_i)$, condition (16) is then equivalent to

$$\forall \theta \in [0, 1], \quad \begin{bmatrix} 1 \\ \theta \star J_1(c_i) \end{bmatrix}^T \begin{bmatrix} 0 & -\mathcal{R}_{d,1} \\ -\mathcal{R}_{d,1}^T & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \theta \star J_1(c_i) \end{bmatrix} < 0 \tag{17}$$

Lemma 2.2 is now applied with

$$M = \begin{bmatrix} 0 & -\mathcal{R}_{d,1} \\ -\mathcal{R}_{d,1}^T & 0 \end{bmatrix} \quad \text{and} \quad \Phi(\theta) = \begin{bmatrix} 1 \\ \theta \star J_1(c_i) \end{bmatrix}.$$

Condition (17) is satisfied if and only if there exist a symmetric positive definite matrix $\mathcal{S}_d$ and a skew symmetric matrices $\mathcal{G}_d$ such that condition (11) is satisfied.

Let us now consider condition (ii). Using (16), condition (10) is equivalent to condition (15). Since

$$\frac{(\Upsilon_0 + C) + \sum_{i=1}^{N} \theta^i (\Upsilon_i + d_i C)}{1 + \sum_{i=1}^{N} \theta^i c_i} = \mathcal{U}(\Upsilon_i, d_i) \bar{H}(\theta)$$

condition (15) is equivalent to

$$\forall \theta \in [0,1], \quad \left[\begin{array}{c} H_1(\theta)^T \\ \bar{H}(\theta) H_2(\theta) \end{array} \right]^T \left[\begin{array}{cc} 0 & \mathcal{U}(\Upsilon_i, d_i) \\ \mathcal{U}(\Upsilon_i, d_i)^T & 0 \end{array} \right] \left[\begin{array}{c} H_1(\theta)^T \\ \bar{H}(\theta) H_2(\theta) \end{array} \right] < 0.$$
(18)

Lemma 2.2 is now applied with

$$M = \left[\begin{array}{cc} 0 & \mathcal{U}(\Upsilon_i, d_i) \\ \mathcal{U}(\Upsilon_i, d_i)^T & 0 \end{array} \right] \quad \text{and} \quad \Phi(\theta) = \left[\begin{array}{c} H_1(\theta)^T \\ \bar{H}(\theta) H_2(\theta) \end{array} \right]$$

Condition (18) is satisfied if and only if there exist a symmetric positive definite matrix $\mathcal{S}$ and a skew symmetric matrices $\mathcal{G}$ such that condition (12) is satisfied. □

Remark Lemma 2.1 can be extended to the case when H_1, H_2 and Υ are rational functions of *several* parameters $\theta_1, \cdots, \theta_m$, see Appendix. In this case, the conditions corresponding to (11) and (12) are no longer necessary.

The interest of Lemma 2.1 with respect to previously published results will be discussed in section 2.5. Theorem 2.2 is now proved by applying Lemma 2.1.

Proof of Theorem 2.2 Lemma 2.1 is first applied to condition (4). Condition (4) can be factorized in the form of (10) with $H_1(\theta) = F_1(\theta)$, $H_2(\theta) = F_3(\theta)$ and

$$C + \Upsilon(\theta) = \left[\begin{array}{c|cccc} \mathcal{V}(\theta) & & & 0 & \\ \hline & \mathcal{X}(\theta) & I_n & 0 & 0 \\ 0 & 0 & \mathcal{Y}(\theta) & 0 & 0 \\ \hline & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_w} & 0 \\ & 0 & 0 & \gamma I_{n_w} & 0 \\ & 0 & 0 & 0 & \gamma I_{n_z} \end{array} \right].$$

Lemma 2.1 is then applied with $\bar{H}(\theta) = F_2(\theta)$ and $\mathcal{U}(\Upsilon_i, d_i) = \mathcal{Z}(\gamma)$. Note that, in the special case of (4), condition (16) is implied by condition (15) and thus can be dropped. It only remains condition (12) in Lemma 2.1, which leads to condition (9).

Condition (3) can be factorized in the form of (10) with $H_1(\theta) = I$, $H_2(\theta) = I$ and

$$C + \Upsilon(\theta) = \left[\begin{array}{cc} -\mathcal{X}(\theta) & -2I_n \\ 0 & -\mathcal{Y}(\theta) \end{array} \right].$$

Lemma 2.1 is then applied with $\bar{H}(\theta) = \begin{bmatrix} \theta I \star J_n(c_i) & 0 \\ 0 & \theta I \star J_n(c_i) \end{bmatrix}$ and $\mathcal{U}(\Upsilon_i, d_i) = -\mathcal{W}$. Note that we use the same scalars c_i for both conditions (3) and (4). Here again, it only remains (12) in Lemma 2.1, which is (8). $\square$

2.5   Discussion of Lemma 2.1

We discuss Lemma 2.1 with respect to existing results on infinite dimensional optimization problems. (A more detailed discussion is presented in [14].) Infinite dimensional optimization problems also arise in robust/LPV analysis and control. First, general parameter dependent LMI constraints are considered in (10). In robust/LPV analysis and control, existing results usually consider special cases of parameter dependent LMI constraints, *e.g.* homogeneous in the decision variable, see [13, 18, 6]. Second, the set of rational functions with free denominator is considered for the decision variables. In robust/LPV analysis and control, existing results consider subset of ours, such as the set of functions independent of θ (see [34, 2]), or the set of affine functions (see [3, 23, 18]), or the set of polynomial functions of a given degree (see [6]) or the set of rational functions with *a priori* fixed denominator and of a given degree (see [13, 29]). Thus, for a given degree, the set considered in this paper includes previous ones, leading to less conservative conditions.

Third, the transformation of an infinite number of constraints into a finite one does not introduce conservatism for one parameter. The case of several parameters can also be treated with a reasonable computation burden. Other approaches can be used, such as gridding ones [46] or polytopic-like ones [5, 3, 23, 47, 44]. The gridding class does not ensure the LMI constraint to hold over its whole range, unless at the price of an extremely high computation burden. The polytopic-like class generally suffers of conservatism. Moreover, it cannot be directly applied to rational dependent LMI constraints; generally, it is applied to quadratic parameter dependent LMI constraints. Finally, for several parameters, the computation burden explodes with the exponential growth in the number of parameter independent constraints.

3 Application to the design of a parameter dependent controller for a set of parameterized trade-offs

In this section, the solution to the parameter dependent controller design presented in section 2.3 is applied to the design of a trade-off dependent controller.

3.1 Problem formulation

In the H_∞ control approach, the design of a controller K is recast as an optimization problem on weighted closed loop transfer functions. The considered closed loop functions are defined by P^w (which depends on the plant):

$$\begin{cases} \dot{x}^w(t) &= A^w x^w(t) &+ B_p^w p(t) &+ B_u^w u(t) \\ q(t) &= C_q^w x^w(t) &+ D_{qp}^w p(t) &+ D_{zu}^w u(t) \\ y(t) &= C_y^w x^w(t) &+ D_{yp}^w p(t) \end{cases} .$$

The desired performance specifications are introduced through the choice of the weighting functions W_i and W_o.
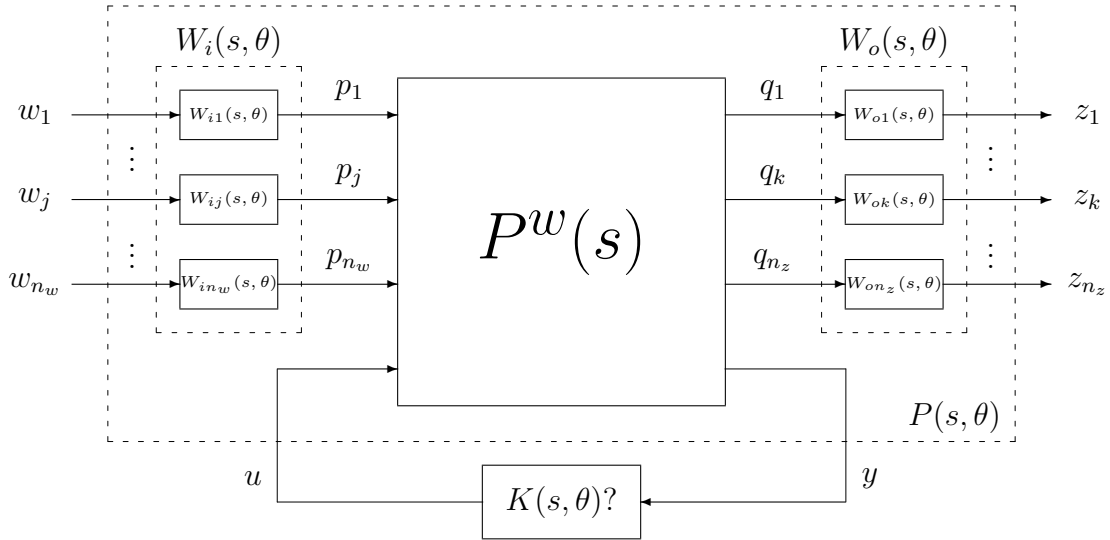

Figure 1: Trade-off dependent controller design problem

A set of performance trade-offs parameterized by a scalar $\theta \in [0, 1]$ is

then defined by weighting functions that are depending on θ:

$$W_i(s, \theta) = \frac{1}{s}I \star \left[\begin{array}{c|c} A_{W_i}(\theta) & B_{W_i}(\theta) \\ \hline C_{W_i}(\theta) & D_{W_i}(\theta) \end{array} \right], \quad \text{and} \quad W_o(s, \theta) = \frac{1}{s}I \star \left[\begin{array}{c|c} A_{W_o}(\theta) & B_{W_o}(\theta) \\ \hline C_{W_o}(\theta) & D_{W_o}(\theta) \end{array} \right]$$

whose state space representation are assumed rational functions of θ, well-posed on $[0, 1]$. The generalized plant is then defined as (see Figure 1):

$$P(s, \theta) = \left[\begin{array}{cc} W_o(s, \theta) & 0 \\ 0 & I \end{array} \right] P^w(s) \left[\begin{array}{cc} W_i(s, \theta) & 0 \\ 0 & I \end{array} \right]. \tag{19}$$

The problem is, given $\gamma > 0$, to compute a trade-off dependent controller, that is, a controller $K(s, \theta)$ whose state space representation are (explicit) rational functions of θ such that

$$\forall \theta \in [0, 1], \quad \|P(s, \theta) \star K(s, \theta)\|_\infty < \gamma \quad \text{with} \quad P(s, \theta) \text{ given by (19)}. \tag{20}$$

As the state space matrices of $P(s, \theta)$ are rational in θ, the trade-off dependent control problem is a subcase of the parameter dependent control problem, problem considered in section 2. The solution presented in Theorem 2.2 can then be applied.

Remind that in order to get a controller whose state space matrices are reasonably complex functions of θ, the decision variables in Theorem 2.2 are enforced to be rational functions of θ of limited degree. The question of the performance loss introduced by this constraint arises. A possible evaluation can be obtained by (i) finding the smallest γ, denoted γ_r, such that there exists $K(s, \theta)$ of the considered structure satisfying (20), (ii) comparing γ_r with the obtained γ, denoted γ_{best} ("best achievable performance"), by considering a controller without any constraint on its state space matrices (except well-posedness). Of course, the effective computation of γ_{best} with its corresponding controller is an open problem. Nevertheless, a lower bound of γ_{best} can be straightforwardly obtained by computing γ_{θ_i} for a "lot" of values θ_i where γ_{θ_i} is the smallest γ such that there exists $K_{\theta_i}(s)$ with $\|P(s, \theta_i) \star K_{\theta_i}(s)\|_\infty < \gamma$. A lower bound on γ_{best} is then $\max_{\theta_i} \gamma_{\theta_i}$. In the sequel, the obtained controller for a given θ_i is denoted $K_{\theta_i}(s)$ and it is referred to as a *"pointwise"* controller. For purpose of comparison, a criterion is given in percent: $100\frac{\gamma_r - \gamma_{best}}{\gamma_{best}}$. Moreover, it is important that γ_{θ_i} is approximately constant for any θ_i.

In the next subsection, our approach is evaluated through a numerical example. The example focuses on the DC motor control with quite realistic

specifications. We evaluate the obtained performance with our approach with respect to the best achievable performance. In addition, we evaluate the obtained performance with our approach with respect to the performance obtained by the same approach but with an *a priori* fixed denominator for the rational decision variables. Remind that a contribution of this paper is to allow to optimize on the denominator of the rational decision variables. Another example which emphasizes the benefit of the use of rational decision variables with the extended Kalman Yakubovich Popov Lemma with respect to other approaches is developed in the technical report [14].

3.2 DC motor control

The considered plant is a DC motor which can be modeled by

$$G(s) = \frac{235}{s(\frac{s}{66}+1)} = \frac{1}{s}I \star \left[\begin{array}{cc|c} -66 & 0 & 32 \\ 32 & 0 & 0 \\ \hline 0 & 15 & 0 \end{array} \right].$$

It is controlled by a one degree of freedom controller. The purpose is to design a trade-off dependent controller ensuring that the closed loop system output is able to track, with a small error, step and low frequency sinusoidal reference signals with different transient times (from 0.02 s for $\theta = 1$ up to 0.06 s for $\theta = 0$) and with the most limited possible control input energy. The closed loop system has to reject step and low frequency sinusoidal input disturbance signals. For different trade-offs between transient time and control input energy, such a problem is addressed by the weighted H_∞ problem presented in Figure 2 [42]. The usual H_∞ problem is for a given trade-off, that is for a given $\theta_i \in [0,1]$: find $K_{\theta_i}(s)$ such that

$$\left\| \begin{array}{cc} W_1(s,\theta_i)S_{\theta_i}(s) & W_1(s,\theta_i)G(s)S_{\theta_i}(s)W_3(s,\theta_i) \\ W_2(s,\theta_i)K_{\theta_i}(s)S_{\theta_i}(s) & W_2(s,\theta_i)T_{\theta_i}(s)W_3(s,\theta_i) \end{array} \right\|_\infty < \gamma \qquad (21)$$

with $S_{\theta_i}(s) = \dfrac{1}{1+G(s)K_{\theta_i}(s)}$ and $T_{\theta_i}(s) = \dfrac{G(s)K_{\theta_i}(s)}{1+G(s)K_{\theta_i}(s)}$.

Choice of the weighting functions The weighting functions $W_i(s,\theta)$, $i \in \{1,2\}$ have the following form [21]:

$$\frac{1}{s} \star \left[\begin{array}{c|c} -\omega_{ci}(\theta)\sqrt{\dfrac{|G_{\infty i}^2 - 1|}{|G_{0i}^2 - 1|}} & (G_{0i} - G_{\infty i})\sqrt{\dfrac{|G_{\infty i}^2 - 1|}{|G_{0i}^2 - 1|}} \\ \hline \omega_{ci}(\theta) & G_{\infty i} \end{array} \right]$$

where $G_{0i} = |W_i(0,\theta)|$, $G_{\infty i} = \lim_{\omega \to \infty} |W_i(j\omega,\theta)|$ (with $(G_{0i}-1)(G_{\infty i}-1) < 0$) and $\omega_{ci}(\theta) > 0$, the crossover frequency, such that $|W_i(j\omega_{ci}(\theta),\theta)| = 1$.
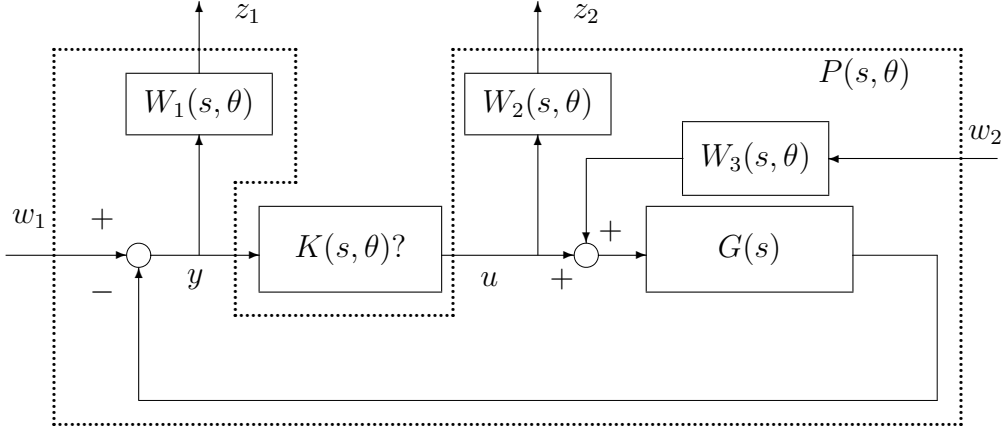
Figure 2: Weighted H_∞ problem

1. W_1 is chosen for ensuring tracking performance and modulus margin.

   (a) The considered trade-off can be defined by ω_{c1}, as it can be related to the transient time response (20 rad/s for 0.06 s up to 80 rad/s for 0.02 s). Thus, ω_{c1} is directly related to θ by $\omega_{c1}(\theta) = 20 + 60\theta$. The parameter θ can be then interpreted as the crossover frequency of $W_1(s, \theta)$, up to an affine transformation.

   (b) G_{01} is an upper bound on the static error: we set $-40dB$.

   (c) $G_{\infty 1}$ is a lower bound on the modulus margin: we set $-6dB$.

2. W_2 is chosen for ensuring control energy limitation: the smallest ω_{c2} the smallest the control energy. We first obtain the smallest possible value of ω_{c2} for three values of θ (23.33 rad/s and $\gamma_0 = 0.991$ for $\theta = 0$, 180 rad/s and $\gamma_{0.5} = 0.986$ for $\theta = 0.5$, 700 rad/s and $\gamma_1 = 0.992$ for $\theta = 1$). Using a least square method, we then obtain $\omega_{c2}(\theta) = 23.33 + \dfrac{204\theta}{1 - 0.7\theta}$. In addition, we choose $20\log(G_{02}) = 10dB$ and $20\log(G_{\infty 2}) = -60dB$.

3. $W_3(s, \theta)$ is chosen in order to specify the input disturbance rejection. For simplicity, W_3 is chosen as a constant gain: $W_3(s, \theta) = 0.05$.

$P(s, \theta)$ is then obtained with the parameter dependent matrices $A(\theta)$ and $C_z(\theta)$ rational functions with the denominator $1 - 0.7\theta + 0\theta^2$.

With these weighting functions, γ_{θ_i} is computed for several values of $\theta_i \in [0, 1]$ with a step of 0.01: we have $\gamma_{\theta_i} \approx 1$. An estimation of γ_{best} is 0.998.

Computation of the trade-off dependent controllers Trade-off dependent controllers are obtained by applying Theorem 2.2 along three way:

1. with $N = 2$ and the denominator of the decision variables a priori chosen. A natural choice for the denominator is the $A(\theta)$ and $C_z(\theta)$ ones, that is, $1 - 0.7\theta + 0\theta^2$;

2. with $N = 2$ to evaluate the effect of optimizing on the decision variable denominator;

3. with $N = 3$ for improving the previous result.

In order to improve the numerical resolution, we choose $1 + c_1\theta = 1 - 0.7\theta$, $c_2 = 0$ for $N = 2$ and $1 + c_1\theta + c_2\theta^2 = (1 - 0.7\theta)(1 + 3\theta)$, $c_3 = 0$ for $N = 3$ with $(1 - 0.7\theta)$ for limiting the size of the matrices A_Ω, B_Ω, C_Ω, D_Ω and $(1 + 3\theta)$ arbitrary. The optimization problems are solved using `Matlab` 6.5 with the LMI control toolbox [24].

Table 1: Obtained results

	Theorem 2.2 $N = 2$ a priori chosen denominator	Theorem 2.2 $N = 2$	Theorem 2.2 $N = 3$
γ_r	1.105	1.06	1
$100\dfrac{\gamma_r - \gamma_{best}}{\gamma_{best}}$ (upper bound)	$\approx 11\%$	$\approx 6\%$	$< 1\%$

The obtained results are presented in Table 1. Note that with the a priori chosen denominator, the obtained result is quite correct with a lower bound on the criterion of 11%. Nevertheless, with the same degree, this result is strongly improved with a smaller value: 6%. Note that, in this case, the denominator of the decision variables is $1 - 1.12\theta + 3.37\theta^2$, that is, a polynomial with complex roots, really different of the denominator of $A(\theta)$ and $C_z(\theta)$. Its *a priori* selection would be difficult. Theorem 2.2 with $N = 3$ allows to obtain a trade-off dependent controller whose performance is dramatically close to the best achievable performance.

Performance analysis Let us now compare the trade-off dependent controller obtained with Theorem 2.2 and $N = 3$ to the pointwise controllers by inspecting the Bode magnitude of $S(s, \theta_i)$, $K(s, \theta_i)S(s, \theta_i)$, $G(s, \theta_i)S(s, \theta_i)$ and $T(s, \theta_i)$ (see Figure 3), the tracking of a step reference and the rejection of a step disturbance (see Figure 4) and the Bode diagram of $K(s, \theta_i)$ (see Figure 5) (for each figure $\theta_i \in \{0, 0.5, 1\}$, thick line for the trade-off depen-
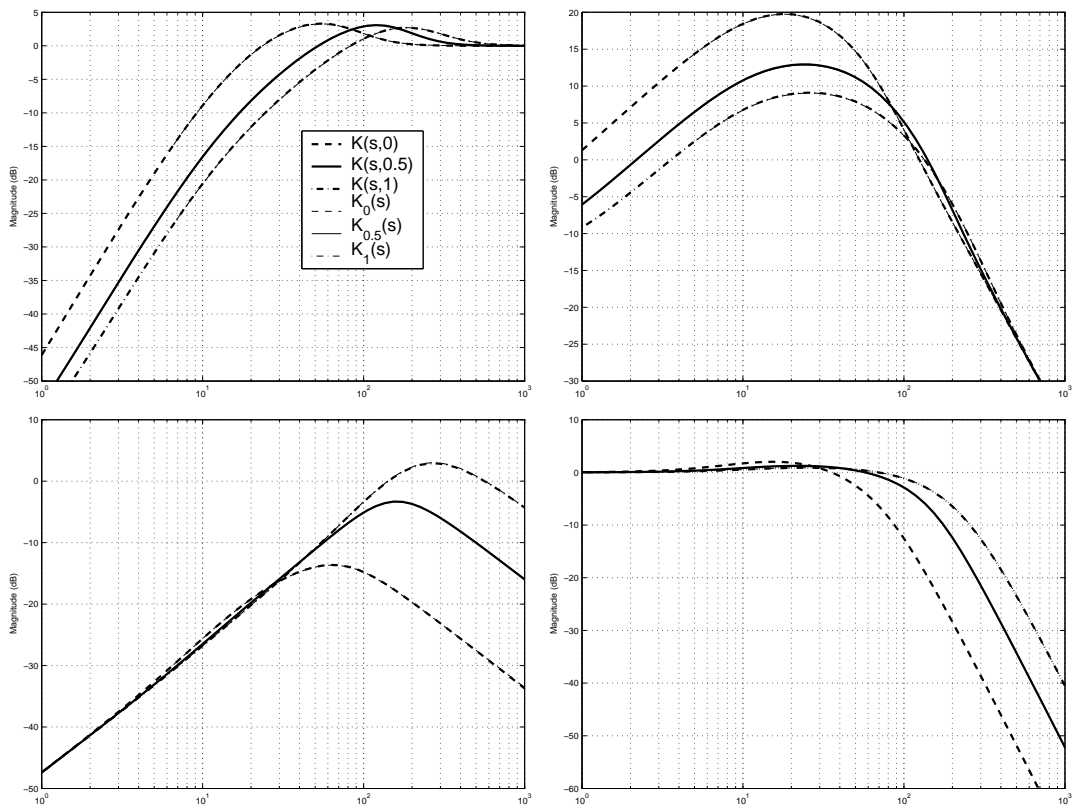
21

Figure 3: Bode magnitude of S (top left), GS (top right), KS (bottom left) and T (bottom right) for $\theta_i \in \{0, 0.5, 1\}$
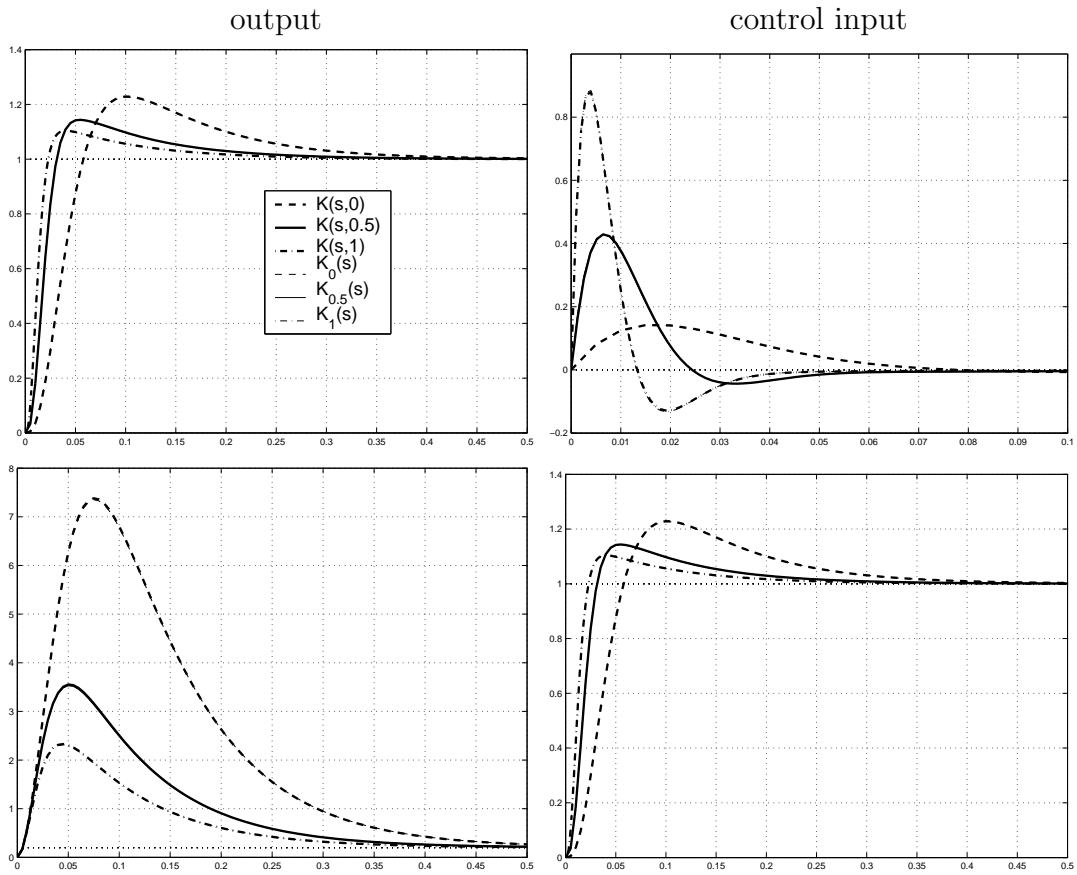
Figure 4: Transient responses to a unit step reference signal (top) and to a unit step disturbance signal (bottom) for $\theta_i \in \{0, 0.5, 1\}$ (top right plot has a different scale)
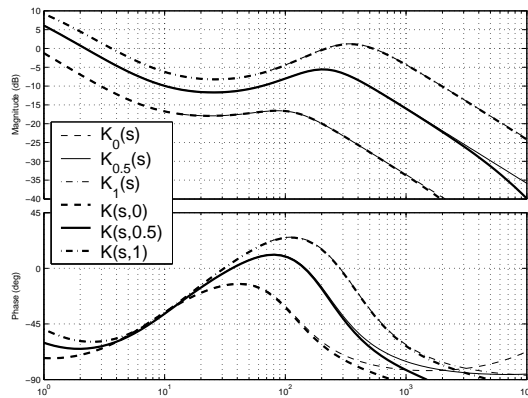

Figure 5: Bode plots of controllers (optimization with $N = 3$) for $\theta_i \in \{0, 0.5, 1\}$

dent controller and thin line for the pointwise controllers). The obtained trade-off dependent controller satisfies the design (rise time) specification. It actually recovers the performance obtained with the pointwise controllers using low degree rational functions, which is really good.

An interesting feature has to be noticed from Figure 5. The structure of the controllers is a Proportional Integral (PI) with a lead effect (and a low pass filter). However, for $\theta = 0$, the lead effect is small and can be neglected; the trade-off dependent controller can be reduced to a PI controller (with a low pass filter). Whereas, for $\theta = 1$, the lead effect is important and cannot be neglected; the trade-off dependent controller is a PI plus a lead transfer function (and a low pass filter). It is a well-known fact that a DC motor can be controlled by a PI if the desired transient response is slow enough. Faster transient response involves a PI plus lead effect controller. Using classical rules of automatic control, know-how... a qualitative link between the performance specifications and the controller gains can be established. Our approach explicitly express the controller structure and the controller gains as analytic functions of the performance specifications, that is, a quantitative link.

4   Conclusion

In this paper, the design of a parameter dependent controller for a parameter dependent plant was recast as a (convex) finite dimensional optimization problem involving LMI constraints. We proved that a parameter (rational) dependent LMI constraint can be equivalently transformed in a parameter independent LMI constraint. An interesting contribution is that the denominator of the decision variables is optimized, which dramatically improves the existing results. This fact was emphasized through an example. The obtained result can be extended to several parameters (see Appendix). But, in this case, conservative is introduced.

The obtained result was applied to the design of a trade-off dependent controller. An example emphasizes the interest of our approach based on rational functions. Using low order rational functions, we recover the performance obtained using pointwise controllers, that is, the best performance possible.

The solution of the parameter dependent controller design has a broader application including parameter dependent plants. One of the most interesting application is probably the (classical) gain scheduled control [43, 22]. Traditional solutions are based on gridding and interpolation, with well-known important difficulties. Our proposed solution is an alternative ap-

proach where the parameter dependent controller is readily obtained avoiding gridding and interpolation.

From a more general point of view, the proposed approach can be applied to control problems involving parameter dependent LMI conditions. Such formulations were proposed, for instance, for the control of nonlinear systems [37, 31, 26, 25], control of saturated systems [32] and control of spatially invariant (distributed) systems [11, 4]. Under some technicalities, the proposed approach can be also adapted to control problems involving parameter dependent Riccati equations. Parameter dependent Riccati equations have been widely studied. Results on the existence and the analyticity of a solution have been given (see [12, 35] and the references therein). But to the authors best knowledge, no efficient method to compute such a solution has been proposed. We hope that our proposed approach paves the way to an efficient solution to these problems.

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A Extension of Lemma 2.1

Lemma 2.1 is here extended to parameter dependent LMI with several parameters:

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix}$$

with $\theta_i \in [0,1]$, $i = 1, \ldots, m$. Let us first introduced the following set of structured block diagonal symmetric matrices:

$$\mathcal{S}(r_i) = \left\{ \mathcal{S} = \begin{bmatrix} \mathcal{S}_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \mathcal{S}_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mathcal{S}_m \end{bmatrix} \middle| \mathcal{S}_i = \mathcal{S}_i^T > 0, \dim(\mathcal{S}_i) = r_i, \ i = 1, \ldots, m \right\},$$

and the following set of structured block diagonal skew-symmetric matrices:

$$\mathcal{G}(r_i) = \left\{ \mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \mathcal{G}_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mathcal{G}_m \end{bmatrix} \middle| \mathcal{G}_i = -\mathcal{G}_i^T, \dim(\mathcal{G}_i) = r_i, \ i = 1, \ldots, m \right\}.$$

The extension of Lemma 2.1 is now presented.

Lemma A.1 *Let $H_1(\theta)$ and $H_2(\theta)$ be matrices of rational functions of θ, well-posed on $[0,1] \times \cdots \times [0,1]$. Let C be a constant matrix and N_j be m positive integers. Let $c_{i_1,\ldots,i_m}$, $i_j = 0, \ldots, N_j$, $j = 0, \ldots, m$, $c_{0,\ldots,0} = 1$ be scalars such that for any $\theta \in [0,1] \times \cdots \times [0,1]$, $\sum_{j=1}^m \sum_{i_j=0}^{N_j} c_{i_1,\ldots,i_m} \theta_1^{i_1} \ldots \theta_m^{i_m} \neq 0$.*

Then there exists $\Upsilon(\theta)$ a (possibly structured) matrix of rational functions of θ, well-posed on $[0,1] \times \cdots \times [0,1]$:

$$\Upsilon(\theta) = \frac{\sum_{j=1}^m \sum_{i_j=0}^{N_j} \Upsilon_{i_1,\ldots,i_m} \theta_1^{i_1} \ldots \theta_m^{i_m}}{\sum_{j=1}^m \sum_{i_j=0}^{N_j} d_{i_1,\ldots,i_m} \theta_1^{i_1} \ldots \theta_m^{i_m}}$$

with $d_{0,\ldots,0} = 1$ such that

$$\forall \theta \in [0,1] \times \cdots \times [0,1], \quad H_1(\theta)(C + \Upsilon(\theta))H_2(\theta) + (H_1(\theta)(C + \Upsilon(\theta))H_2(\theta))^T < 0$$

if there exist matrices $\Upsilon_{i_1,\ldots,i_m}$ and scalars $d_{i_1,\ldots,i_m}$, $i_j = 0,\ldots,N_j$, $j = 0,\ldots,m$ with $d_{0,\ldots,0} = 1$ such that the two following conditions are satisfied:

(i) there exist $\mathcal{S}_d \in \mathcal{S}(k_i)$ and $\mathcal{G}_d \in \mathcal{G}(k_i)$ such that:

$$\mathcal{L}\left(A_P, B_P, \begin{bmatrix} 0 \\ \hline C_P \end{bmatrix}, \begin{bmatrix} 1 \\ \hline D_P \end{bmatrix}, \begin{bmatrix} 0 & -\mathcal{T}(d_{i_1,\ldots,i_m}) \\ -\mathcal{T}(d_{i_1,\ldots,i_m})^T & 0 \end{bmatrix}, \mathcal{S}_d, \mathcal{G}_d\right) < 0$$

where $\mathcal{T}(d_{i_1,\ldots,i_m})$ is an affine function of $d_{i_1,\ldots,i_m}$ such that for some positive k_i, $i = 1,\ldots,m$

$$\mathcal{T}(d_{i_1,\ldots,i_m}) \times \begin{bmatrix} \theta_1 I_{k_1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \theta_i I_{k_i} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \theta_m I_{k_m} \end{bmatrix} \star \begin{bmatrix} A_P & B_P \\ \hline C_P & D_P \end{bmatrix} \triangleq \frac{\sum_{j=1}^{m} \sum_{i_j=0}^{N_j} d_{i_1,\ldots,i_m} \theta_1^{i_1} \ldots \theta_m^{i_m}}{\sum_{j=1}^{m} \sum_{i_j=0}^{N_j} c_{i_1,\ldots,i_m} \theta_1^{i_1} \ldots \theta_m^{i_m}}$$

$$(22)$$

(ii) there exist $\mathcal{S} \in \mathcal{S}(l_i)$ and $\mathcal{G} \in \mathcal{G}(l_i)$ such that

$$\mathcal{L}\left(A_H, B_H, C_H, D_H, \begin{bmatrix} 0 & \mathcal{U}(\Upsilon_{i_1,\ldots,i_m}, d_{i_1,\ldots,i_m}) \\ \mathcal{U}(\Upsilon_{i_1,\ldots,i_m}, d_{i_1,\ldots,i_m})^T & 0 \end{bmatrix}, \mathcal{S}, \mathcal{G}\right) < 0$$

where

$$\begin{bmatrix} \theta_1 I_{l_1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \theta_i I_{l_i} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \theta_m I_{l_m} \end{bmatrix} \star \begin{bmatrix} A_H & B_H \\ \hline C_H & D_H \end{bmatrix} \triangleq \begin{bmatrix} H_1(\theta)^T \\ \bar{H}(\theta) H_2(\theta) \end{bmatrix}$$

and where $\mathcal{U}(\Upsilon_{i_1,\ldots,i_m}, d_{i_1,\ldots,i_m})$ is an affine function of $\Upsilon_{i_1,\ldots,i_m}$ and $d_{i_1,\ldots,i_m}$, $i_j = 0,\ldots,N_j$, $j = 0,\ldots,m$ such that for some positive l_i, $i = 1,\ldots,m$

$$\mathcal{U}(\Upsilon_{i_1,\ldots,i_m}, d_{i_1,\ldots,i_m}) \bar{H}(\theta) = \frac{\sum_{j=1}^{m} \sum_{i_j=0}^{N_j} (\Upsilon_{i_1,\ldots,i_m} + d_{i_1,\ldots,i_m} C) \theta_1^{i_1} \ldots \theta_m^{i_m}}{\sum_{j=1}^{m} \sum_{i_j=0}^{N_j} c_{i_1,\ldots,i_m} \theta_1^{i_1} \ldots \theta_m^{i_m}}.$$

$$(23)$$

Remark The factorizations (22) and (23) are always possible, although not unique.

Remark In Lemma A.1, we obtain only sufficient conditions since Lemma 2.2 is no longer necessary and sufficient in the case of several parameters.