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Filter design: a finite dimensional convex optimization approach

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Abstract

We consider the design of transfer functions (filters) satisfying upper and lower bounds on the frequency response magnitude or on phase response, in the continuous and discrete time domains. The paper contribution is to prove that such problems are equivalent to *finite* dimensional *convex* optimization problems involving Linear Matrix Inequality constraints. At now, such optimization problems can be efficiently solved. Note that this filter design problem is usually reduced to a semi infinite dimensional Linear Programming optimization problem under the additional assumption that the filter poles are fixed (for instance, when considering FIR design). Furthermore, the semi infinite dimensional optimization is practically solved, using a gridding approach on the frequency. In addition to be finite dimensional, our formulation allows to set or not the filter poles. These problems were mainly considered in Signal Processing. Our interest is to propose an approach dedicated to Automatic Control problems. In this paper, we focus on the following problems: design of weighting transfers for H_∞ control and design of lead-lag network for control. Numerical applications emphasize the interest of the proposed results.

Keywords Filter design, H_∞ control, Optimization, Linear Matrix Inequalities.

1 Introduction

1.1 Problem interest

The design of filters, in the continuous time or in the discrete time is an important issue in signal processing (see *e.g.* [22, 10]). The design specifications are formulated in the frequency domain. These problems were largely considered and at now some of them are still open.

Nevertheless, in this context, the filter design is of interest in other fields of engineering sciences, for instance, in automatic control. In this field, in many problems such as H_∞ control [30, 4, 31], lead-lag network for classical frequency design [11, 18], arise the design of filters satisfying given constraints on the frequency response (magnitude, phase, magnitude and phase). Some problems can be solved using the approach proposed in this paper.

In these problems, the common feature is the design of a transfer function (or filter) whose magnitude or phase has to satisfy a set of constraints on frequency intervals. The filter design was intensively considered in the literature, mainly in signal processing. Note that, in signal processing, design specifications are slightly different. Our main purpose is to propose an interesting solution for the above automatic control problems. The review in the sequel is presented in this spirit. Possible extensions are nevertheless possible to signal processing problems.

1.2 Review of some previous filter design methods

A first classical approach of the analog filter design satisfying magnitude bounds is based on prototype filters such as Chebyshev filters, Butterworth filters, elliptic filters, etc [22] for achieving given standard types (lowpass, highpass, etc...). On the one hand, the obtained solution is not optimal. Note that this approach, by the use of the bilinear transformation [22, 3], allows to design digital filters. On the other hand, note that the obtained filter order is high and that the underlying formulation is not flexible. For instance, this solution is not adapted to the H_∞ weighting transfer function design. For this particular problem, specific structures were proposed [7]. A more difficult problem was considered by Lanzon and co-authors [13, 14]: their purpose is to design the weighting function and the control law in one shot. Unfortunately, the proposed solutions are not computationally efficient as their problem reduces to μ synthesis which is known difficult.

More recently, a lot of works were dedicated to the design of Finite Impulse Response (FIR) filters, especially in the case of linear phase FIR filters. Note that FIR filter design can be interpreted as the design of a filter whose poles are fixed. For instance, using the Chebychev approximation, several algorithms were developed whose most famous is the Remes exchange algorithm (see *e.g.* [19, 20, 22]). Note that these approaches heavily rely on the linear phase constraint of the designed FIR. Furthermore, the general problem of designing a FIR filter satisfying arbitrary upper and lower bounds on its magnitude is not solved in a clear way.

An alternative approach to FIR filter design is to recast the problem as an linear programming optimization problem. Such a formulation is much more flexible than the previous one. In [26], this approach was applied in order to design linear phase FIR filters. Nevertheless, note that the underlying optimization problem is semi infinite (so called SIP, Semi Infinite Problem) [10]: in [26], this problem was solved in an approximative way, using a frequency gridding in order to obtain a finite dimensional optimization problem. The advantage is that there exist efficient algorithms solving LP optimization problem. But, in general, there is no guarantee that the semi infinite constraints are satisfied between frequency samples. The purpose of this paper is to show that magnitude filter design problem can be equivalently formulated as a convex finite optimization problem.

The case of the FIR design satisfying arbitrary magnitude bounds, without the linear phase constraint[1], using optimization, was considered in [10, 28]. In [28, 29], this design problem is formulated as a semi infinite optimization problem. Some of the constraints can be equivalently replaced by finite dimensional constraints. Actually, their approach is based on a (partial) gridding of the convex optimization problem leading to a large-scale optimization problem involving Linear Matrix Inequality[2] contraints. As a matter of fact, if at now large scale Linear Programming problems can be efficiently solved, it is not the case for LMI programming problems. Furthermore, under strong assumptions on the magnitude bounds, the whole problem can be equivalently formulated as a finite dimensional optimization problem [29]. As a consequence, the proposed solution is not satisfying enough with respect to our objectives.

1.3 Proposed approach and contributions

In this paper, under mild assumptions, we actually prove that filter design under magnitude or phase constraints can be formulated as a finite dimensional convex optimization problems without these strong assumptions. Furthermore, this proof is constructive and of practical interest, that is, we propose a finite dimensional convex optimization problem involving Linear Matrix Inequality constraints.

The practical computation of the solution is performed without any approximation. In our approach, no a priori transfer function structure is fixed (except numerator/denominator order) and no parameters are to be tuned. If a solution exists, the optimization program

[1] Note that, in the context of H_∞ weighting transfer function design, the linear phase overconstraints the problem.

[2] Linear Matrix Inequalities are briefly defined in the sequel.

proposes an adapted structure with the corresponding coefficient values without any gridding. Note that the obtained filters are finite dimensional transfer functions whose order is a priori bounded, allowing an easy implementation[3].

Our result is presented in the case of IIR (Infinite Impulse Response) transfer functions rather that FIR filters as IIR transfer functions satisfy a given constraint, with a less important order. For instance, remember that low order is a strong requirement in the case of H_∞ weighting transfer function design.

Our approach still apply when the FIR filter design is considered (for the sake of shortness, this particular problem is developed in the technical report [24]). In the case of FIR design with magnitude constraints, our approach is closely related to the Wu's one [28] but bypasses her disadvantages. In fact, the Wu and al results are derived from an interesting application of the positive real lemma [21] in the discrete time domain. In contrast, our approach is based on an extension of the Kalman Yakubovitch Popov lemma [21] which is strongly related to the LMI based μ analysis upper bound [6, 16].

Our approach is mainly presented for the continuous time[4] (that is, analog) filter design which is of strong interest *e.g.* in the case of H_∞ weighting function design. Of course, in the field of signal processing, a more important problem is the design of discrete time (that is, digital) filters.

Nevertheless, using our approach, a direct solution can be proposed for the digital filter design. In fact, the solution is a little bit more elegant in the continuous time case than in the discrete time case. In fact, when considering continuous time filters, the corresponding transfer function is naturally a Linear Fractional Transformation (LFT)[5] in ω (or $\frac{1}{\omega}$), that is, in this special case, a rational function of ω (or $\frac{1}{\omega}$). The analysis of the so-called LFT system was intensively investigated in the field of robust control (see *e.g.* [31, 15, 25]. In contrast with the continuous time case, discrete time transfer functions can not be directly written as an LFT (that is a rational function) of ω (or $\frac{1}{\omega}$). This straightforward remark is in fact the fundamental motivation of our choice.

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[3]In fact, the proposed approach can be readily extended in order to design certain classes of non integer order filters. For sake of briefness, this point is not developed.

[4]Using the classical bilinear transformation (see *e.g.* [3, 23]) discrete time formulation can be cast as a continuous time one.

[5]LFT is formally defined in the sequel.

1.4 Paper outline.

The paper is organized as follows. The problem under consideration is presented in the section 2. For the sake of clearness, a simplified magnitude constraint filter design is considered. In section 3.1, a simplified version of the main result is presented: this problem can be recast as a finite dimensional convex optimization problem involving LMI constraints. The general case is then considered in section 3.2. Section 4 focus on the filter design under phase constraints. Numerical applications are developed in the section 5: design of an H_∞ weighting function and design of lead filter satisfying phase constraints. Finally, some concluding remarks are made about the approach. Additional results on model reduction and FIR design are available in the technical report [24].

1.5 Notations

I_r and 0_r denotes the identity matrix and the zero matrices of $\mathbf{R}^{r\times r}$, with I_0 (or 0_0) empty. The subscript is omitted when it is evident from the context. For a real symmetric (complex hermitian) matrix P, $P > 0$ denotes that P is definite positive.

A rational function in δ can be expressed as a Linear Fractional Transformation (LFT), that is, as $D + C\Delta(I - A\Delta)^{-1}B$ with $\Delta = \delta I$ and A, B, C, D are matrices of convenient size. An LFT can be denoted:

$$\Delta \star \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where $\star$ denote the Redheffer star product (for details see [31]).

2 Problem statements

2.1 Filter design under magnitude constraints

Our purpose is to design a fixed order filter whose magnitude is upper (respectively lower) bounded on different N_u (resp. N_l) frequency sub open intervals by different frequency dependent rational functions.

Let us denote the considered N_u open sub interval as $\Omega_1^u, \cdots, \Omega_{N_u}^u$. Let us introduce the positive frequency dependent functions $U_i(\omega)$ which are defined on the sub interval Ω_i^u. In the same way, let us denote the N_l open sub interval as $\Omega_1^l, \cdots, \Omega_{N_l}^l$. Let us introduce the positive frequency dependent functions $L_j(\omega)$ defined on the sub interval Ω_j^l. Furthermore, for all i and j such that $\Omega_i^u \bigcap \Omega_j^l \neq \emptyset$, for all $\omega \in \Omega_i^u \bigcap \Omega_j^l$, $L_j(\omega) < U_i(\omega)$, that is, lower and upper constraints are consistent.

In the sequel, the upper and lower bound functions $L_j(\omega)$ and $U_i(\omega)$ are assumed to be rational (including polynomial functions). This assumption is rather mild due to the special interest of this class of functions in the approximation theory. The important interest of rational functions is that they can be written as a Linear Fractional Transformation [31] of the variable ω or of the variable $\frac{1}{\omega}$, that is, there exist matrices A_i^u, B_i^u, C_i^u, D_i^u, A_j^l, B_j^l, C_j^l et D_j^l such that:

$$\begin{cases} \forall\omega \in \Omega_i^u, \quad U_i(\omega) = D_i^u + C_i^u\Delta_i(\omega)(I - A_i^u\Delta_i(\omega))^{-1}B_i^u \\ \\ \forall\omega \in \Omega_j^l, \quad L_j(\omega) = D_j^l + C_j^l\Delta_j(\omega)(I - A_j^l\Delta_j(\omega))^{-1}B_j^l \end{cases} \tag{1}$$

with $\Delta_i(\omega) = \omega I$ or $\Delta_i(\omega) = \frac{1}{\omega}I$ and $\Delta_j(\omega) = \omega I$ or $\Delta_j(\omega) = \frac{1}{\omega}I$. Furthermore, this rational functions are assumed well-posed, that is, $\forall\omega \in \Omega_i^u$, the matrix $I - A_i^u\Delta_i(\omega)$ is invertible and $\forall\omega \in \Omega_j^l$, the matrix $I - A_j^l\Delta_j(\omega)$ is invertible.

We now present the problem under consideration.

General problem formulation *Given*

1. N_u upper bounds $U_i(\omega)$ defined on sub intervals Ω_i^u defined as above;
2. N_l upper bounds $L_j(\omega)$ defined on sub intervals Ω_j^l defined as above;
3. a positive integer n

find if there exists a minimum phase proper stable filter F of order bounded by n such that :

$$\begin{cases} \forall\ i \in \{1, \cdots, N_u\},\ \forall\ \omega \in \Omega_i^u, \quad |F(j\omega)| < U_i(\omega) \\ \forall\ j \in \{1, \cdots, N_l\},\ \forall\ \omega \in \Omega_j^l, \quad L_j(\omega) < |F(j\omega)| \end{cases} \tag{2}$$

and *compute* it.

Remark For the sake of simplicity, we assume that the numerator order is smaller than the denominator one. Nevertheless, in your approach, if necessary, the numerator order of the filter F can be fixed to any arbitrary value.

Special functions of interest are power functions, that is, functions of the form:

$$U_i(\omega) = E_i^u\omega^{K_i^u} \quad \text{and} \quad L_j(\omega) = E_j^l\omega^{K_j^l} \tag{3}$$

where E_i^u, E_j^l are real positive and K_i^u, K_j^l are integer. In this special case, when magnitudes are expressed in dB, condition (2) boils down to:

$$\begin{cases} \forall\ i \in \{1, \cdots, N_u\},\ \forall\ \omega \in \Omega_i^u, \quad |F(j\omega)|_{dB} < 20\log_{10}(E_i^u) + 20K_i^u\log_{10}(\omega) \\ \forall\ j \in \{1, \cdots, N_l\},\ \forall\ \omega \in \Omega_j^l, \quad 20\log_{10}(E_j^l) + 20K_j^l\log_{10}(\omega) < |F(j\omega)|_{dB} \end{cases} \tag{4}$$

Remark In this case, we can be formulated the bound functions as an LFT in ω or in $\frac{1}{\omega}$, indeed with $K > 0$

$$E\omega^K = \omega I_K \star \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \\ \hline E & 0 & \dots & 0 & 0 \end{array}\right] \quad or \quad E\omega^{-K} = \frac{1}{\omega} I_K \star \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \\ \hline E & 0 & \dots & 0 & 0 \end{array}\right]$$

2.2 Filter design under phase constraints

This problem is symmetric to the filter design under magnitude constraints. As a consequence, we just point out two slight modifications of the formulation: (i) the N_u upper bounds θ_i^u and the N_l lower bounds θ_j^l are constant[6], (ii) the modulus is replaced by the argument. For instance, conditions (2) are replaced by the following conditions:

$$\begin{cases} \forall\ i \in \{1, \cdots, N_u\}, & \forall\, \omega \in \Omega_i^u, & \theta_i^u - \pi & < & \arg F(j\omega) & < & \theta_i^u \\ \forall\ j \in \{1, \cdots, N_l\}, & \forall\, \omega \in \Omega_j^l, & \theta_j^l & < & \arg F(j\omega) & < & \theta_j^l + \pi. \end{cases} \tag{5}$$

3 Proposed approach of filter design under magnitude constraints

In the problem formulation (see *e.g.* conditions (2)), the decision variable is the filter transfer function $F(j\omega)$. Note that, using spectral factorization, an equivalent problem is to consider as decision variable $|F(j\omega)|$. With this change of variable, conditions (2) are linear thus convex constraints in the decision variable. Under this suitable change of variable, solving the General problem formulation is a convex optimization problem but infinite dimensional. As the constraints depend on ω, the number of constraints is infinite. Furthermore, as the decision variable is a real rational function, the decision variable is in an infinite dimensional space.

In the sequel, we prove that both problems can be equivalently formulated as *finite* dimensional optimization problems involving Linear Matrix Inequalities constraints. In this formulation, the number of variables and the number of constraints are finite.

For the sake of clearness, we first present our solution in a special case.

[6]Our approach is not limited to constant constraints. Actually, frequency dependent constraints can be introduced. These constraints have to be expressed as the tangent of real rational functions in ω.

3.1 Simplified formulation

One upper bound constraint ($N_u = 1$) on the frequency domain interval $\Omega = (\underline{\omega};\overline{\omega})$ is considered and denoted in the sequel $U(\omega)$. Note that the subscripts are dropped. There is no lower bound constraint ($N_l = 0$). To sum up, find $F(j\omega)$ such that:

$$\forall\, \omega \in \Omega, \quad |F(j\omega)| < U(\omega) \tag{6}$$

In the two following subsections, our solution is presented. In the section 3.2, the proposed solution is straightforwardly extended to the general case (that is, several upper and lower constraints on different frequency intervals).

Semi infinite dimensional formulation

Note that the filter F has to satisfy a set of constraints involving the magnitude of F, $|F(j\omega)|$. As a consequence, the actual variable is $|F(j\omega)|$. We first propose a process which allows to compute (if it exists) $|F(j\omega)|$ satisfying constraints (2). Using standard results on spectral factorization [8], a stable minimum phase transfer F can be then computed.

In the sequel, the filter F is factorized as follows:

$$F(j\omega) = \frac{N(j\omega)}{D(j\omega)}.$$

Several factorizations are possible. For instance, $N(j\omega)$ and $D(j\omega)$ can be selected as: (coprime) polynomial factors or coprime (rational) factors. In the sequel, without loss of generality, we focus on the first case. With this factorization, condition (6) is equivalent to the following condition:

$$|N(j\omega)| < |D(j\omega)|U(\omega) \tag{7}$$

Note that equation (6) is affine in $|F(j\omega)|$ but to get a finite number of decision variables, we need to set the poles of F, which is conservative. With the previous factorization and with the new variables $|N(j\omega)|$ and $|D(j\omega)|$, the formulation (7) which is equivalent to (6), is still affine in the decision variables $|N(j\omega)|$ and $|D(j\omega)|$. By introducing a suitable parameterization of $|N(j\omega)|^2$ and $|D(j\omega)|^2$, we are able to generate all the possible transfers F of order n. Using these ideas, we derive the following lemma.

Lemma 3.1 *Let $U(\omega)$ be an upper bound magnitude specification defined on Ω and let n be a positive integer. Let us introduced the following polynomial column base*

$$B(j\omega) = \begin{bmatrix} (j\omega)^n & (j\omega)^{n-1} & \cdots & j\omega & 1 \end{bmatrix}^T \tag{8}$$

The two following propositions are equivalent:

(i) *There exists a stable, minimum phase, $F(j\omega)$ filter of order bounded by n such that:*

$$\forall \omega \in \Omega, \quad |F(j\omega)| < U(\omega) \tag{9}$$

(ii) *There exist X_N and X_D symmetric matrices in $R^{(n+1)\times(n+1)}$ such that:*

$$\forall \omega \in \Omega, \qquad \begin{bmatrix} B(j\omega)U(\omega) \\ B(j\omega) \end{bmatrix}^* \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} B(j\omega)U(\omega) \\ B(j\omega) \end{bmatrix} < 0 \tag{10}$$

$$\forall \omega \in [0, +\infty), \qquad B(j\omega)^* X_N B(j\omega) > 0 \quad \text{and} \quad B(j\omega)^* X_D B(j\omega) > 0 \tag{11}$$

Moreover, if there exist X_N and X_D satisfying conditions (10) and (11), the solution F satisfying condition (9) is given by $F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ with

$$N(j\omega)^* N(j\omega) = B(j\omega)^* X_N B(j\omega) \quad \text{and} \quad D(j\omega)^* D(j\omega) = B(j\omega)^* X_D B(j\omega) \tag{12}$$

Proof See the proof in appendix, section A.

Remark Independently of the denominator order (polynomial $D(s)$ order), we can enforce the numerator order (polynomial $N(s)$ order) by setting to zero suitable lines and suitable columns of X_N.

In this section, we present a parameterization of the to-be-obtained filter which is linear and finite dimensional. Nevertheless, the obtained conditions in Lemma 3.1 must be satisfied for frequency intervals, that is, there is an infinite number of inequalities. In the next section, we propose an equivalent formulation where the number of (Linear Matrix) inequalities is finite.

Equivalent finite dimensional formulation

The previously considered polynomial vector

$$B(j\omega) = \begin{bmatrix} (j\omega)^n & (j\omega)^{n-1} & \cdots & j\omega & 1 \end{bmatrix}^T \tag{13}$$

can be easily formulated as an LFT in $j\omega$, that is, $B(j\omega) = D_b + C_b j\omega (I - j\omega A_b)^{-1} B_b$ with

$$A_b = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad B_b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad C_b = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \ddots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \\ 1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad D_b = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{14}$$

Theorem 3.1 *Let $U(\omega) = D_u + C_u\Delta(\omega)(I - \Delta(\omega)A_u)^{-1}B_u$, with $\Delta(\omega) = \omega I$ or $\Delta(\omega) = \frac{1}{\omega}I$ be an upper bound magnitude specification defined on Ω and let n be a positive integer. Let us introduced the following polynomial column base defined by (13). Let be (A, B, C, D) complex matrices defined as:*

$$\forall\delta \in (0,\infty), \quad \delta \star \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \delta \star T_\Omega \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \times \delta \star \widetilde{T}_\Omega \star \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right] \\ \delta \star T_\Omega \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

with

$$T_\Omega = \left[\begin{array}{c|c} -\overline{\omega}I & (\underline{\omega} - \overline{\omega})I \\ \hline \overline{\omega}I & \overline{\omega}I \end{array}\right]$$

and

$$\text{if} \quad \Delta(\omega) = \omega I, \qquad \widetilde{T}_\Omega = T_\Omega$$

$$\text{if} \quad \Delta(\omega) = \tfrac{1}{\omega}I, \qquad \widetilde{T}_\Omega = \left[\begin{array}{c|c} -\underline{\omega}I & (1 - \frac{\underline{\omega}}{\overline{\omega}})I \\ \hline I & (\frac{1}{\overline{\omega}})I \end{array}\right]$$

The two following propositions are equivalent:

(i) *There exists a stable, minimum phase, $F(j\omega)$ filter of order bounded by n such that:*

$$\forall\omega \in \Omega, \quad |F(j\omega)| < U(\omega) \tag{15}$$

(ii) *There exist X_N and X_D symmetric matrices in $R^{(n+1)\times(n+1)}$ such that there exist two symmetric matrices P_N and P_D and a complex matrix P such that:*

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_N \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_N + P_N A & P_N B_b \\ B_b^T P_N & 0 \end{bmatrix} > 0 \tag{16}$$

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_D \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_D + P_D A & P_D B_b \\ B_b^T P_D & 0 \end{bmatrix} > 0 \tag{17}$$

$$P + P^* > 0 \qquad \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^*P^* + PA & PB \\ B^*P^* & 0 \end{bmatrix} < 0 \tag{18}$$

Moreover, if there exist X_N and X_D satisfying conditions (16), (17) and (18), the solution F satisfying condition (15) is given by $F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ with

$$N(j\omega)^* N(j\omega) = B(j\omega)^* X_N B(j\omega) \quad \text{and} \quad D(j\omega)^* D(j\omega) = B(j\omega)^* X_D B(j\omega) \tag{19}$$

Proof See the proof in appendix, section B.

Computation Note that satisfying proposition (ii) conditions of Theorem 3.1 reduces to solving a *convex, finite-dimensional optimization problem* involving complex Linear Matrix Inequality (LMI) constraints. It is equivalent to an optimization problem over twice as big real LMI constraints [1]. As a matter of fact, this latter is a standard LMI optimization problem (actually a feasibility problem, see e.g. [2]) which can be efficiently solved[7] using standard (Matlab) solvers as the free ware code SP [27] and its Matlab/Scilab interface `LMITOOL` [5], or the available commercial Matlab Toolbox, LMI Control Toolbox [9]. See section 5.1 for numerical application.

Interpretation The most sticking fact is that the infinite number of inequalities (10), (11) of Lemma 3.1 are replaced by a finite set of inequalities. To this purpose, Theorem 3.1 is based on an extension of the Kalman Yakubovich Popov (KYP) lemma [21]. Remember that, given a real rational transfer function $H(j\omega) = D + C(j\omega I - A)^{-1}B$, this lemma claims that testing that $\forall \omega$, $H(j\omega) + H(j\omega)^* > 0$ is equivalent to finding a symmetric matrix P such that :

$$\begin{bmatrix} A^T P + PA & PB + C^T \\ B^T P + C & D + D^T \end{bmatrix} > 0.$$

Actually, this classical lemma allows to replace an infinite number of inequalities by one (matrix) inequality by introducing an additional decision variable P. Note that the transfer function $H(j\omega)$ is an LFT of $\frac{1}{j\omega}$ and that the frequency dependent inequality has to be satisfied for any ω. In our problem, the frequency dependent functions can be expressed as LFTs of ω or $\frac{1}{\omega}$ with ω in a bounded interval. Thus the KYP lemma can not be applied. By a suitable change of variable, called loop-shifting [3], these functions are expressed as LFT of δ, with $\delta \in (0, +\infty)$. Actually, the KYP lemma can be extended to the case of LFTs of a real (positive) parameter δ. We thus obtain the following lemma.

Lemma 3.2 *Let us consider $\phi(\delta) = D + C\delta(I - \delta A)^{-1}B$ a well-posed rational function where A, B, C and D are complex matrices and let M be an hermitian matrix. The two following propositions are then equivalent.*
(i)

$$\forall \delta \in (0, +\infty), \qquad \Phi^*(\delta) M \Phi(\delta) < 0 \tag{20}$$

(ii) *There exists a complex matrix P such that $P + P^* > 0$ and:*

$$\begin{bmatrix} C^* \\ D^* \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^* P^* + PA & PB \\ B^* P^* & 0 \end{bmatrix} < 0 \tag{21}$$

Proof: For the proof, see [17].

[7] in polynomial time

Remark A related lemma can be found in [12]. It allows to test if a real rational transfer function, that is an LFT in $\frac{1}{j\omega}$, satisfies a quadratic constraint for all $\omega \in [\omega_1,\ \omega_2]$. Note that, in our problem, as we consider LFI in $\frac{1}{\omega}$ or in ω, we can not apply their result.

A direct application of this theorem allows to derive conditions (17), (16) and (18) from conditions (11) and (10). Indeed, conditions (17) and (16) are equivalent to conditions (11). In addition, condition (18) is equivalent to condition (10).

3.2 General formulation

We consider the general problem formulation with N_u upper bound constraints and N_l lower bound constraints. We first present the solution when the filter poles are free. This solution is then specialized to the case when the filter poles are set.

Theorem 3.2 *Let us defined an upper and lower bound magnitude specification set defined as*

$$\begin{cases} \forall\, i \in \{1,\ \cdots, N_u\}, & \forall \omega \in \Omega_i^u, \quad U_i(\omega) = D_i^u + C_i^u \Delta_i^u(\omega)(I - A_i^u \Delta_i^u(\omega))^{-1} B_i^u \\ \forall\, j \in \{1,\ \cdots, N_l\}, & \forall \omega \in \Omega_j^l, \quad L_j(\omega) = D_j^l + C_j^l \Delta_j^l(\omega)(I - A_j^l \Delta_j^l(\omega))^{-1} B_j^l \end{cases}$$

with $\Delta_i^u(\omega) = \omega I$ *or* $\Delta_i^u(\omega) = \frac{1}{\omega} I$ *and* $\Delta_j^l(\omega) = \omega I$ *or* $\Delta_j^l(\omega) = \frac{1}{\omega} I$. *Let n be a positive integer. Let us introduced the following polynomial column base defined by (13). Let be* $(\overline{A}_i, \overline{B}_i, \overline{C}_i, \overline{D}_i)$ *complex matrices defined as:*

$$\forall\, i \in \{1,\ \cdots, N_u\},$$

$$\forall \delta \in (0,\ +\infty),\quad \delta \star \begin{bmatrix} \overline{A}_i & \overline{B}_i \\ \overline{C}_i & \overline{D}_i \end{bmatrix} = \begin{bmatrix} \delta \star T_{\Omega_i^u} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \times \delta \star \widetilde{T}_{\Omega_i^u} \star \left[\begin{array}{c|c} A_i^u & B_i^u \\ \hline C_i^u & D_i^u \end{array}\right] \\ \delta \star T_{\Omega_i^u} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

with $\Omega_i^u = (\underline{\omega}_i,\ \overline{\omega}_i)$

$$T_{\Omega_i^u} = \left[\begin{array}{c|c} -\overline{\omega}_i I & (\underline{\omega}_i - \overline{\omega}_i) I \\ \hline \overline{\omega}_i I & \overline{\omega}_i I \end{array}\right]$$

and

$$\text{if} \quad \Delta_i^u(\omega) = \omega I, \qquad \widetilde{T}_{\Omega_i^u} = T_{\Omega_i^u}$$

$$\text{if} \quad \Delta_i^u = \frac{1}{\omega} I, \qquad \widetilde{T}_{\Omega_i^u} = \left[\begin{array}{c|c} -\underline{\omega}_i I & (1 - \frac{\underline{\omega}_i}{\overline{\omega}_i}) I \\ \hline I & (\frac{1}{\overline{\omega}_i}) I \end{array}\right]$$

In addition, let be $(\underline{A}_j, \underline{B}_j, \underline{C}_j, \underline{D}_j)$ *complex matrices defined as:*

$\forall\, j \in \{1, \cdots, N_l\}$,

$$\forall \delta \in (0, +\infty), \quad \delta \star \begin{bmatrix} \underline{A}_j & \underline{B}_j \\ \underline{C}_j & \underline{D}_j \end{bmatrix} = \begin{bmatrix} \delta \star T_{\Omega_j^l} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \times \delta \star \widetilde{T}_{\Omega_j^l} \star \left[\begin{array}{c|c} A_j^l & B_j^l \\ \hline C_j^l & D_j^l \end{array}\right] \\ \delta \star T_{\Omega_j^l} \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline C_b & D_b \end{array}\right] \end{bmatrix}$$

with $\Omega_j^l = (\underline{\omega}_j,\ \overline{\omega}_j)$

$$T_{\Omega_j^l} = \left[\begin{array}{c|c} -\overline{\omega}_j I & (\underline{\omega}_j - \overline{\omega}_j) I \\ \hline \overline{\omega}_j I & \overline{\omega}_j I \end{array}\right]$$

and

$$\text{if} \quad \Delta_j^l(\omega) = \omega I, \qquad \widetilde{T}_{\Omega_j^l} = T_{\Omega_j^l}$$

$$\text{if} \quad \Delta_j^l = \frac{1}{\omega} I, \qquad \widetilde{T}_{\Omega_j^l} = \left[\begin{array}{c|c} -\underline{\omega}_j I & (1 - \frac{\underline{\omega}_j}{\overline{\omega}_j}) I \\ \hline I & (\frac{1}{\overline{\omega}_j}) I \end{array}\right]$$

The two following propositions are then equivalent.

(i) *There exists a minimum phase proper stable filter* F *of order bounded by* n *such that:*

$$\begin{cases} \forall\ i \in \{1, \cdots, N_u\},\ \forall\, \omega \in \Omega_i^u, \quad |F(j\omega)| < U_i(\omega) \\ \forall\ j \in \{1, \cdots, N_l\},\ \forall\, \omega \in \Omega_j^l, \quad L_j(\omega) < |F(j\omega)| \end{cases} \tag{22}$$

(ii) *There exist* X_N *and* X_D *symmetric matrices in* $R^{(n+1)\times(n+1)}$ *such that there exist two symmetric matrices* P_N *and* P_D, N_u *complex matrices* P_i^u *and* N_l *complex matrices* P_j^l *such that:*

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_N \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_N + P_N A & P_N B_b \\ B_b^T P_N & 0 \end{bmatrix} > 0 \tag{23}$$

$$\begin{bmatrix} C_b^T \\ D_b^T \end{bmatrix} X_D \begin{bmatrix} C_b & D_b \end{bmatrix} + \begin{bmatrix} A_b^T P_D + P_D A & P_D B_b \\ B_b^T P_D & 0 \end{bmatrix} > 0 \tag{24}$$

$$\forall i, P_i^u + P_i^{u*} > 0, \qquad \begin{bmatrix} \overline{C}_i^* \\ \overline{D}_i^* \end{bmatrix} \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} \overline{C}_i & \overline{D}_i \end{bmatrix} + \begin{bmatrix} \overline{A}_i^* P_i^{u*} + P_i^u \overline{A}_i & P_i^u \overline{B}_i \\ \overline{B}_i^* P_i^{u*} & 0 \end{bmatrix} < 0 \tag{25}$$

$$\forall j, P_j^l + P_j^{l*} > 0, \qquad \begin{bmatrix} \underline{C}_j^* \\ \underline{D}_j^* \end{bmatrix} \begin{bmatrix} X_D & 0 \\ 0 & -X_N \end{bmatrix} \begin{bmatrix} \underline{C}_j & \underline{D}_j \end{bmatrix} + \begin{bmatrix} \underline{A}_j^* P_j^{l*} + P_j^l \underline{A}_j & P_j^l \underline{B}_j \\ \underline{B}_j^* P_j^{l*} & 0 \end{bmatrix} < 0 \tag{26}$$

Moreover, if there exist X_N and X_D satisfying conditions (23), (24), (25) and (26), the solution F of the General problem formulation *is given by* $F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ *with*

$$N(j\omega)^* N(j\omega) = B(j\omega)^* X_N B(j\omega) \quad \text{and} \quad D(j\omega)^* D(j\omega) = B(j\omega)^* X_D B(j\omega) \tag{27}$$

Proof Note that when several upper and lower bounds are considered, the corresponding problem formulation is obtained by stacking the corresponding conditions. As a consequence, this Theorem can be readily proved by a slight modification of Theorem 3.1 proof.

4 Filter design under phase constraints

We just present the result in the case when one constraint is considered, that is:

$$\forall\, \omega \in \Omega, \quad \theta - \pi < \arg F(j\omega) < \theta.$$

Our result relies on the simple fact that this constraint is equivalent to:

$$\forall \omega \in \Omega \qquad \cos\theta\, \frac{F(j\omega) - F^*(j\omega)}{2j} \quad < \quad \sin\theta\, \frac{F(j\omega) + F^*(j\omega)}{2} \tag{28}$$

As in the previous section, the filter F is factorized as follows:

$$F(j\omega) = \frac{N(j\omega)}{D(j\omega)}.$$

Under the mild assumption that $D^*(j\omega)D(j\omega) > 0 \quad \forall \omega \in \Omega$ and by denoting α the complex number $\sin(\theta) + j\cos(\theta)$, the previous condition is equivalent to:

$$\forall \omega \in \Omega, \qquad \alpha N(j\omega) D^*(j\omega) + \alpha^* N^*(j\omega) D(j\omega) > 0. \tag{29}$$

As the constraint (29) is affine in $N(j\omega)D^*(j\omega)$, a natural change of variable with its corresponding parameterization arises:

$$N(j\omega)D^*(j\omega) = X B(j\omega)$$

with $X \in \mathbb{R}^{1\times(n+1)}$ and $B(j\omega) = \begin{bmatrix} (j\omega)^n & (j\omega)^{n-1} & \cdots & j\omega & 1 \end{bmatrix}^T$.

Theorem 4.1 *Let θ be an upper bound phase specification defined on Ω and let n be a positive integer. Let be $\alpha = \sin(\theta) + j\cos(\theta)$.*
Let (A, B, C, D) be complex matrices defined as:

$$\forall \delta \in (0, \infty), \quad \delta \star \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \delta \star T_\Omega \star \left[\begin{array}{c|c} jA_b & jB_b \\ \hline \alpha C_b & \alpha D_b \\ 0 & 1 \end{array}\right]$$

with

$$T_\Omega = \left[\begin{array}{c|c} -\overline{\omega} I & (\underline{\omega} - \overline{\omega}) I \\ \hline \overline{\omega} I & \overline{\omega} I \end{array}\right]$$

The two following propositions are equivalent:

(i) *There exists a stable, minimum phase, $F(j\omega)$ filter of order bounded by n such that:*

$$\forall \omega \in \Omega, \quad \theta - \pi < \arg F(j\omega) < \theta \tag{30}$$

(ii) *There exists a matrix X in $R^{1\times(n+1)}$ such that there exists a complex matrix P such that:*

$$P + P^* > 0 \qquad \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^* P^* + PA & PB \\ B^* P^* & 0 \end{bmatrix} < 0 \tag{31}$$

Moreover, if there exists X satisfying condition (31), the solution F satisfying condition (30) is given by $\arg F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ *with*

$$N(j\omega) D^*(j\omega) = X B(j\omega) \tag{32}$$

Remark The choice of F from the obtained solution $N(j\omega)D^*(j\omega)$ is clearly not unique. Note that the roots of $N(s)D^*(s)$ are the zeros and the conjugate of poles of $F(s)$. In order to obtain a stable and minimum phase filter, the roots of the polynomial $N(s)D^*(s)$ with a positive (negative) real part have to be considered as the poles (zeros) of F. The main drawback of this approach is to lead, in some cases, to non proper transfer functions. Nevertheless, a proper transfer function can be obtained by augmenting the obtained filter F with high frequency poles.

Remark Our solution can be readily extended to the case of several upper and lower phase constraints on different frequency intervals. For the sake of clearness, the corresponding theorem is omitted.

Remark This result can be extended to the phase matching problem, that is, the model is reduced by approximating the phase (for an explicit formulation see the technical report [24]).

5 Applications

5.1 H_∞ weighting function design

For this numerical application, a Matlab 6 script handling arbitrary numbers of upper and lower constraints of the form (3) was developed. The optimization problems are solved using the Matlab toolbox, LMI Control Toolbox [9].

We want to find a weighting function W whose magnitude satisfies upper and lower constraints given in table 1.

Upper bound functions	**Lower bound functions**
$U_1(\omega) = 3.4\omega$ defined on $(0.002,\ 0.0965)$	$L_1(\omega) = 0.97\omega$ defined on $(0.01,\ 0.1)$
$U_2(\omega) = 0.0267$ defined on $(8.9189,\ 10.14)$	$L_2(\omega) = 1.0212$ defined on $(0.8244,\ 1.19)$
$U_3(\omega) = 1.9440$ defined on $(0.2,\ 978)$	$L_3(\omega) = 1.0212$ defined on $(130,\ 958)$

Table 1: An example of set of upper and lower bound functions

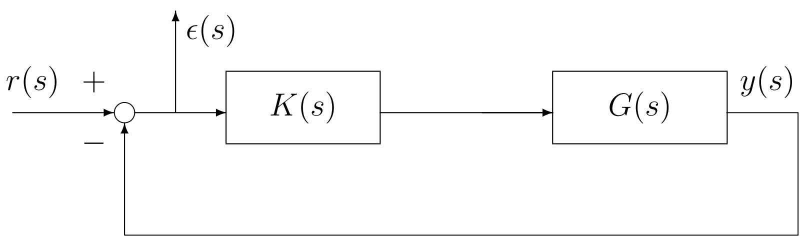

Figure 1: Plant $G(s)$ with the control law $K(s)$

Consider a system $G(s)$ controlled by a one degree of freedom control law $K(s)$ (see figure 1). We want to design a control law ensuring that the closed loop system output y is able to track with a small error, step and sinusoïdal reference signals r. Such a property is obtained if the function transfer from the input r to the tracking error $\epsilon = r - y$ has its magnitude bounded by a weighting function magnitude satisfying the considered set of constraints.
By applying our approach, we obtain a 3^{d} order filter W:

$$W(s) = \frac{1.1367(s + 0.002689)(s^2 + 0.1976s + 90.44)}{(s + 53.38)(s^2 + 1.494s + 1.391)}.$$

Our programme was unable to compute a 2^{d} order filter. As a consequence, there does not exist a 2^{d} order filter that meets the specification.

5.2 Lead filter design

In the sequel, a continuous time lead filter is designed in order to satisfy phase constraints. To the purpose, we apply the proposed approach in Theorem 4.1. As an example, a set of constraints is given in the figure 3. For the numerical application, we choice to design filter satisfying a lead phase of $\phi_m = 70°$ at $\pm 1°$ over a frequency interval larger than one decade.

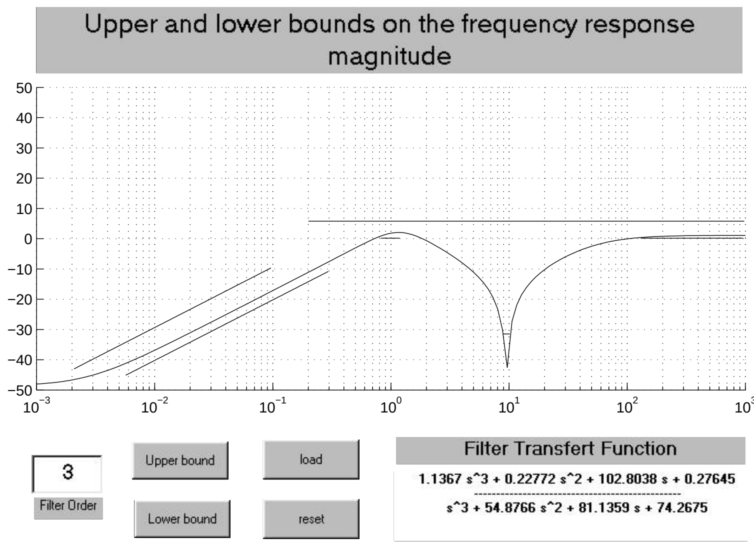

Figure 2: H_∞ weighting function

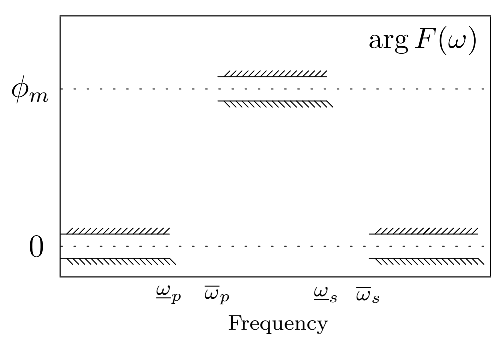

Figure 3: Example of phase constraints

The constraints have been defined as follows:

$$\begin{array}{ccccll} -1^\circ & < & \arg F(j\omega) & < & 7^\circ & \text{for all } \omega \in (0, 3.10^{-3}) \\ 69^\circ & < & \arg F(j\omega) & < & 71^\circ & \text{for all } \omega \in (0.1, 2) \\ -1^\circ & < & \arg F(j\omega) & < & 7^\circ & \text{for all } \omega \in (10^2, +\infty) \end{array}$$

We succeed in computing a filter whose the sum of the numerator order and the denominator order is less than 4. In fact, we can choice a 2^{d} order, stable, minimum phase filter F:

$$F(s) = \frac{(s + 0.5818)(s + 0.02485)}{(s + 8.141)(s + 0.3495)}.$$

The designed filter is presented Figure 4.

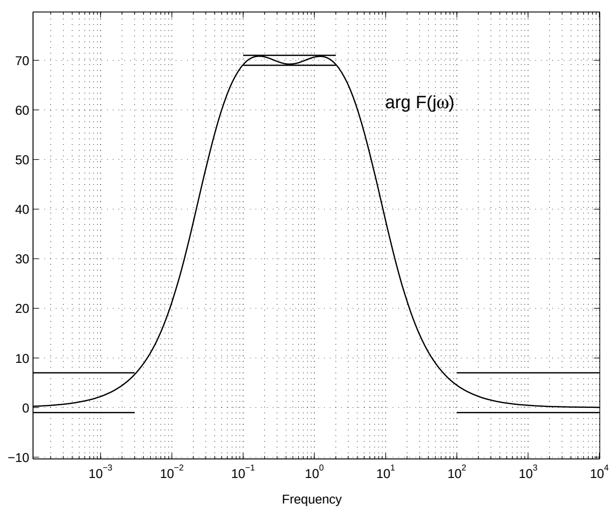

Figure 4: Phase lead filter

6 Conclusion

The proposed approach was presented in the case of the design of filters satisfying hard upper and lower magnitude bound.

The salient features of the approach are that:

1. the problem is equivalently cast as a *finite dimensional convex* optimization problem involving Linear Matrix Inequality constraints, which can be efficiently solved using the available softwares;

2. the proposed formulation is really flexible, allowing to solve problems using a natural form;
3. no extra assumption of the obtained filter (as the filter poles are fixed or the phase filter is linear) are introduced;
4. the minimum order filter satisfying upper and lower magnitude bounds is obtained.

This approach has be extended to the case of filter design satisfying phase constraints. Additional results are in the technical report [24].

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A Proof of Lemma 3.1

$\boxed{(i) \Rightarrow (ii)}$

Let us rewrite the filter transfer function F as $F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ where $N(j\omega)$ and $D(j\omega)$ are two polynomials of order p and of order n with $n \geq p$, that is:

$$\begin{aligned} N(j\omega) &= b_p(j\omega)^p + b_{p-1}(j\omega)^{p-1} + \ldots + b_1 j\omega + b_0 \qquad (33)\\ D(j\omega) &= a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1 j\omega + a_0 \qquad (34) \end{aligned}$$

Consider the polynomial base $B(j\omega)$ of order n defined in (13). Equations (33) and (34) can be written as:

$$\begin{aligned} N(j\omega) &= \begin{bmatrix} 0 & \ldots & 0 & b_p & \ldots & b_0 \end{bmatrix} B(j\omega) \qquad (35)\\ D(j\omega) &= \begin{bmatrix} a_n & \ldots & \ldots & \ldots & a_0 \end{bmatrix} B(j\omega) \qquad (36) \end{aligned}$$

It is then straightforward that by introducing X_N and X_D symmetric matrices in $R^{(n+1)\times(n+1)}$ such that

$$\begin{aligned} X_N &= \begin{bmatrix} 0 & \ldots & 0 & b_p & \ldots & b_0 \end{bmatrix}^T \begin{bmatrix} 0 & \ldots & 0 & b_p & \ldots & b_0 \end{bmatrix} \qquad (37)\\ X_D &= \begin{bmatrix} a_n & \ldots & \ldots & \ldots & a_0 \end{bmatrix}^T \begin{bmatrix} a_n & \ldots & \ldots & \ldots & a_0 \end{bmatrix} \qquad (38) \end{aligned}$$

we have

$$\begin{aligned} N(j\omega)^* N(j\omega) &= B(j\omega)^* X_N B(j\omega) > 0 \qquad (39)\\ D(j\omega)^* D(j\omega) &= B(j\omega)^* X_D B(j\omega) > 0 \qquad (40) \end{aligned}$$

Thus, we obtain conditions (11).

Furthermore, the following condition

$$|F(j\omega)| < U(\omega) \tag{41}$$

leads to

$$\begin{array}{ll} \Longrightarrow & |N(j\omega)| < |D(j\omega)|U(\omega) \\ \Longrightarrow & N(j\omega)^* N(j\omega) < U(\omega)^* D(j\omega)^* D(j\omega) U(\omega) \\ \Longrightarrow & B(j\omega)^* X_N B(j\omega) - U(\omega)^* B(j\omega)^* X_D B(j\omega) U(\omega) < 0 \end{array}$$

(from equations (39) and (40)). We thus obtain condition (10).

$\boxed{(ii) \Rightarrow (i)}$

Let us consider X_N and X_D two real symmetric matrices such that conditions (11) hold. Using spectral factorization, there exist two Hurwitz polynomials $N(j\omega)$ and $D(j\omega)$ such that

$$N(j\omega)^* N(j\omega) = B(j\omega)^* X_N B(j\omega) \tag{42}$$
$$D(j\omega)^* D(j\omega) = B(j\omega)^* X_D B(j\omega) \tag{43}$$

Furthermore, we assume that, in addition, X_N and X_D satisfy equation (10).

$$\begin{bmatrix} B(j\omega)U(\omega) \\ U(\omega) \end{bmatrix}^* \begin{bmatrix} -X_D & 0 \\ 0 & X_N \end{bmatrix} \begin{bmatrix} B(j\omega)U(\omega) \\ U(\omega) \end{bmatrix} < 0 \tag{44}$$

$$\begin{array}{ll} \Longrightarrow & B(j\omega)^* X_N B(j\omega) - U(\omega)^* B(j\omega)^* X_D B(j\omega) U(\omega) < 0 \\ \Longrightarrow & N(j\omega)^* N(j\omega) < U(\omega)^* D(j\omega)^* D(j\omega) U(\omega) \\ \Longrightarrow & |N(j\omega)| < |D(j\omega)|U(\omega) \end{array}$$

So there exist a filter

$$F(j\omega) = \frac{N(j\omega)}{D(j\omega)}$$

proper, stable and minimum phase such that condition (9) holds.

QED

B Proof of Theorem 3.1

From Lemma 3.1, there exists a stable minimum phase filter $F(j\omega)$ of order bounded by n such that:

$$\forall \omega \in \Omega, |F(j\omega)| < U(\omega)$$

if and only if there exist two real symmetric matrices X_N and X_D such that condition (11) and condition (10) hold. From the Kalman Yakubovic Popov Lemma, conditions (11) are equivalent to condition (16) and condition (17).

Note that in equation (10), the parameter ω is restricted to the bounded interval $\Omega = (\overline{\omega},\ \underline{\omega})$. In order to apply Lemma 3.1, we introduce a change of variable, called in the Automatic Control context loop-shifting [3]. The interest of such change of variable is to be written as an LFT. The new variable δ is defined in order that $\omega \in \Omega \iff \delta \in (0,\ +\infty)$. We straightforwardly obtain:

$$\left\{ \begin{array}{lcl} \omega & = & \overline{\omega}\dfrac{1+\delta\underline{\omega}}{1+\delta\overline{\omega}} \\ \dfrac{1}{\omega} & = & \dfrac{1}{\overline{\omega}}\dfrac{1+\delta\overline{\omega}}{1+\delta\underline{\omega}} \end{array} \right. \iff \left\{ \begin{array}{lcl} \omega & = & \delta \star T_{\Omega} \\ \frac{1}{\omega} & = & \delta \star \widetilde{T}_{\Omega} \end{array} \right. \tag{45}$$

with T_{Ω} and $\widetilde{T}_{\Omega}$ defined in Theorem 3.1. With this change of variable, equation (10) can be written as

$$\forall \delta \in (0,\ +\infty), \qquad \Phi(\delta)^* \left[\begin{array}{cc} -X_D & 0 \\ 0 & X_N \end{array} \right] \Phi(\delta) < 0 \tag{46}$$

with

$$\Phi(\delta) = \delta \star \left[\begin{array}{cc} A & B \\ C & D \end{array} \right]$$

with A, B, C and D complex matrices defined as in Theorem 3.1. From the Lemma 3.2, condition (18) is obtained from condition (46).

QED