The weighted incremental norm approach: from linear to nonlinear H_∞ control ⋆

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Abstract

Weighted induced norms can be used to handle robustness and sensitivity requirements in the linear context as shown by Zames in its seminal work. This paper deals with an attempt to extend linear H_∞ control concepts to the nonlinear context making use of weighted incremental norms. It is shown how these concepts make it possible to handle basic requirements such as robust stability, disturbance attenuation and steady state behaviors.

Key words: Nonlinear systems analysis, robust control, sensitivity function, nonlinear H_∞ control, gain-scheduling.

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1 Introduction

H_∞ control is issued from the effort of formalizing in mathematical terms performance and robustness requirements. In the linear context a significant part of the activities is devoted to reformulate and generalize the classical control concepts introduced by Black, Bode and Horowitz such as phase and gain margins and the sensitivity concept. Zames (1981) shows how weighted induced norms can be used to handle both robustness and sensitivity requirements. Is it possible to extend this idea to the nonlinear context ? That is the question studied hereafter.

The nonlinear extension of the H_∞ optimization problem is investigated in the recent literature in the $\mathcal{L}_2$ framework through polynomial expansions (Foias and Tannenbaum, 1989), dissipativity techniques / nonlinear differential game arguments (Basar and Bernhard, 1991), linear H_∞ methods applied to systems perturbed by nonlinear uncertainties (Becker et al., 1993). A robustness theory for nonlinear systems with general unstructured uncertainties (graph perturbation, gap metric) was presented in the framework of differentiable/incremental norms by Georgiou (1993).

In this paper, providing a unified view of results stated by the authors (see Fromion, 1995; Fromion et al., 1995, 1996; Fromion, 1997; Fromion et al., 1999), we show that a possible way to extend the H_∞ approach to the nonlinear context can be the weighted incremental norms. For this purpose, we investigate some of the aspects considered by Zames (1981). The result is that the incremental framework allows us to take into account not only the classical linear requirements, such as robust stability with respect to unstructured uncertainties, attenuation with respect to output perturbations and sensitivity, but also specific problems associated with the nonlinear nature of the plant such as initial condition uncertainties and steady state properties with respect to specific classes of inputs (constant or periodic).

The paper is organized as follows. The robustness problem is addressed in section 2 where robustness against unstructured uncertainties is restated in terms of an incremental test. In section 3, it is shown how the attenuation problem can be reduced to the minimization of a weighted incremental norm. In section 4, weighted induced norms, namely $\mathcal{L}_2$ or incremental gains, are used to handle, in a weak sense, constraints on the behavior of the closed-loop system. A simple counter-example illustrates the limitation of the $\mathcal{L}_2$ gain approach. Moreover it is shown how, under suitable assumptions, incremental boundedness ensures the Lyapunov stability of any motion, guarantees the existence of specific permanent behaviors and ensures their asymptotic attractiveness. Section 5 studies the connection of the proposed approach with linear time-varying H_∞ control. The classical gain scheduling (Shamma, 1988; Rugh, 91)

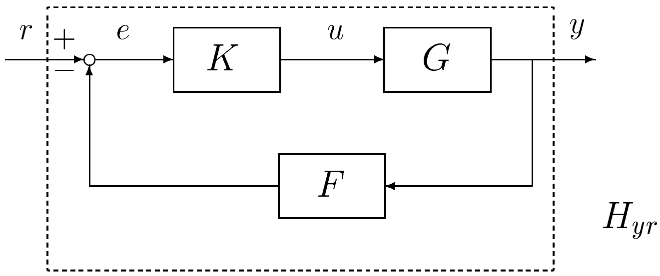

Fig. 1. The nonlinear feedback system

is also re-interpreted as an approximation of an incremental objective.

Notations and definitions The notations and terminology, here used, are classical in the input-output context (see Willems, 1969; Desoer and Vidyasagar, 1975). The $\mathcal{L}_2$-norm of $f : [t_0, \infty) \mapsto \mathbb{R}^n$ is $\|f\|_2 = \sqrt{\int_{t_0}^{\infty} \|f(t)\|^2 dt}$. The *causal truncation* at $T \in [t_0, \infty)$, denoted by $P_T f$ gives $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise. The *extended space*, $\mathcal{L}_2^e$ is composed with the functions whose causal truncations belong to $\mathcal{L}_2$. For convenience, $\|P_T u\|_2$ is denoted by $\|u\|_{2,T}$.

In the sequel, we consider systems exhibiting the differential representation:

$$\Sigma \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, and $u(t) \in \mathbb{R}^p$. f and h, defined from $\mathbb{R}^n \times \mathbb{R}^p$ into $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, are assumed to be C^1 and uniformly Lipschitz. The unique solution $x(t) = \phi(t, t_0, x_0, u)$ belongs to $\mathcal{L}_2^e$ for all $x_0 \in \mathbb{R}^n$ and for any $u \in \mathcal{L}_2^e$. It is assumed that there exists x_{0e} such that $f(x_{0e}, 0) = 0$ and $h(x_{0e}, 0) = 0$, *i.e.* the system initialized at x_{0e} is unbiased, $\Sigma(0) = 0$. The notion of incremental $\mathcal{L}_2$-gain can now be recalled.

Definition 1 *Σ is said to be a weakly finite gain stable system if there exists $\gamma \geq 0$, $\beta \geq 0$ such that $\|\Sigma(u)\|_2 \leq \gamma\|u\|_2 + \beta$ for all $u \in \mathcal{L}_2$. Σ is said to be finite gain stable when $\beta = 0$. The gain of Σ coincides with the minimum value of γ and is denoted by $\|\Sigma\|_i$.*

Definition 2 *Σ has a finite incremental gain if there exists $\eta \geq 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta\|u_1 - u_2\|_2$ for all $u_1, u_2 \in \mathcal{L}_2$. The incremental gain of Σ coincides with the minimum value of η and is denoted by $\|\Sigma\|_\Delta$. Σ is said to be incrementally stable if it is stable, i.e. it maps $\mathcal{L}_2$ to $\mathcal{L}_2$, and has a finite incremental gain.*

We consider in the sequel, the nonlinear feedback system depicted in figure 1, where G, K, F are nonlinear causal operators from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, representing

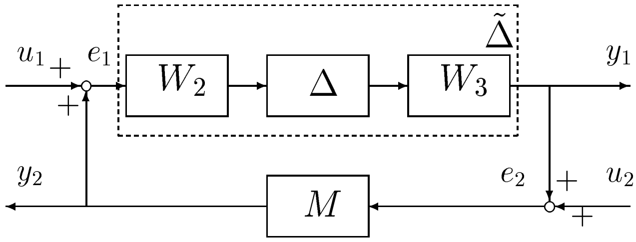

Fig. 2. A perturbed closed-loop system

respectively the plant, the compensator and the feedback, and where r, e, u and y, which belong to $\mathcal{L}_2^e$, denote respectively the system input, the error signal, the plant input, and the system output. The closed-loop system is assumed to be well-posed and the input-output map between the system input and the system output is denoted by H_{yr} and is given by $GK(I + FGK)^{-1}$.

2 Robust stability against unstructured uncertainties

The description of unstructured uncertainties through the use of suitable weights is classical in the linear context (Safonov, 1980; Zames, 1981; Doyle et al., 1982). This is for example the usual way to take into account uncertainties due to actuator dynamics, output sensor errors, high-frequency neglected dynamics (bending modes) or some limitations of the system such as gain margin requirements through the use of an input multiplicative error. Such a description, not depending on the nature of the nominal plant, can be assumed to hold in the nonlinear context too.

With this in mind, we will consider additive uncertainties, *i.e.* $\tilde{G} = G + \tilde{\Delta}$), multiplicative uncertainties, *i.e.* $\tilde{G} = G(I + \tilde{\Delta})$, or some other types as in (Doyle et al., 1982) and we will assume that $\tilde{\Delta}$ belongs to Ω_Δ defined by:

$$\Omega_\Delta \equiv \left\{ \tilde{\Delta} = W_3 \Delta W_2 \mid \|\Delta\|_\Delta < 1 \right\} \tag{2}$$

where W_2 and W_3 are two causal and incrementally stable operators.

We can now formulate the robustness problem as the property of the induced norm of the system augmented with the weighting functions W_2 and W_3. The assumption set on uncertainty allows us to represent the perturbed system as depicted in figure 2 where M is the generic nominal closed-loop system. The following theorem represents a first extension of a known linear result.

Theorem 3 *If M is incrementally stable and if the following inequality holds true:*

$$\|W_2 M W_3\|_\Delta \leq 1, \tag{3}$$

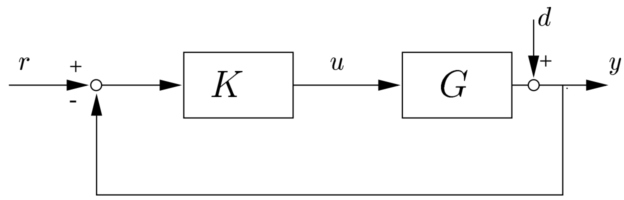

Fig. 3. The perturbed closed-loop system

then, the closed-loop system of figure 2 is incrementally stable for any $\tilde{\Delta}$ belonging to Ω_Δ.

Sketch of proof The proof will show that the map between u_1 and e_1 (see figure 2) has a finite incremental gain. Let us consider the interconnection between W_2 and $MW_3\Delta$. The definition of Ω_Δ and the inequality (3) ensure:

$$\|W_2MW_3\Delta\|_\Delta \leq \|W_2MW_3\|_\Delta\|\Delta\|_\Delta < 1$$

which allows us to prove (Willems, 1969, Theorem 4.6) that the closed-loop system associated with W_2 and $MW_3\Delta$ is incrementally stable. The map between u_1 and e_1 is incrementally stable thus concluding the proof.

3 Disturbance attenuation problem

The use of feedback control schemes is mainly linked to their ability to reduce the effect of non measurable perturbations or to shrink model uncertainties: the desensitivity property (see *e.g.* Desoer and Wang, 1980; Zames, 1981). Two major types of desensitivities are classically considered: the infinitesimal desensitivity, related to small perturbations, and the comparison desensitivity when no restrictions concerning the size of perturbations are present. As it is pointed out by Desoer and Wang (1980), on the basis of Taylor type expansion arguments, it is possible to link one to the other.
In the following it is shown that the desensitivity requirement can be reformulated as the minimization of the incremental norm of a suitable weighted map. The reader is referred to Desoer and Wang (1980) for a complete presentation of the desensitivity problem in a nonlinear context. In the sequel, we will just consider the output disturbance problem (the other cases presented by Desoer and Wang (1980) can be worked out as well).

As Desoer and Wang (1980), we assume without loss of generality that the feedback map is the identity, *i.e.* $F = I$, and we associate with the closed-loop system depicted in figure 3 an "equivalent" open-loop map, Ho_{yr} depicted in figure 4. If the open-loop controller is given by $K_o = K(I + GK)^{-1}$, then the open-loop system in figure 4, which maps the inputs (r, d) in $\mathcal{L}_2^e \times \mathcal{L}_2^e$ to the

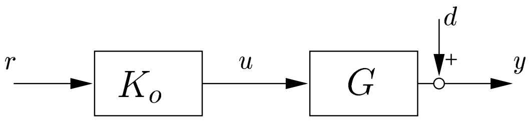

Fig. 4. The perturbed equivalent open-loop system

output y in $\mathcal{L}_2^e$, satisfies for all $r \in \mathcal{L}_2^e$ and for $d = 0$, the following equality:

$$Ho_{yr}(r,0) \triangleq H_{yr}(r,0)$$

where H_{yr} is the system in figure 3 which maps inputs (r,d) which belong to $\mathcal{L}_2^e \times \mathcal{L}_2^e$ to the output y which also belongs to $\mathcal{L}_2^e$.

We now compute the effect induced by the output perturbations on the open-loop system:

$$\delta Ho_{yr}(r,d) = GK(I+GK)^{-1}(r) + d - GK(I+GK)^{-1}(r) = d. \qquad (4)$$

whereas the closed-loop configuration gives:

$$\begin{aligned} \delta H_{yr}(r,d) &= GK(I+GK)^{-1}(r-d) + d - GK(I+GK)^{-1}(r) && (5)\\ &= (I+GK)^{-1}(r) - (I+GK)^{-1}(r-d) && (6) \end{aligned}$$

since $GK(I+GK)^{-1} = I - (I+GK)^{-1}$.
The main interest of the feedback control strategy stands in its capability of reducing the effect of disturbances. In mathematical terms, the feedback has a desensisitivity effect if the following inequality is satisfied:

$$\|\delta H_{yr}(r,d)\|_{2,T} < \|\delta Ho_{yr}(r,d)\|_{2,T}.$$

Unfortunately, for realistic systems, this inequality cannot be satisfied for any inputs and disturbances belonging to $\mathcal{L}_2^e$. Indeed, like in the linear context (Zames, 1981), one has the following theorem:

Theorem 4 *Consider the closed-loop system in figure 1 with $F = I$. If the open-loop operator GK is strictly causal then $\|(I+GK)^{-1}\|_\Delta \geq 1$.*

PROOF. The proof works by contradiction. Given $S = (I+GK)^{-1}$, the small incremental gain theorem ensures the existence of $(I-S)^{-1}$, if $\|S\|_\Delta < 1$. Moreover, from $I = (I+GK)(I+GK)^{-1} = (I+GK)^{-1} + GK(I+GK)^{-1}$, one deduces that $(I-S) = GK(I+GK)^{-1}$. Because of the strict causality of GK, then $GK(I+GK)^{-1}$ is strictly causal too so that $I-S$ is not invertible (Willems, 1969), thus obtaining a contradiction.

This theorem implies that there exists $r, d \in \mathcal{L}_2^e$ such that $\|S(r-d) - S(r)\|_{2,T} \geq \|d\|_{2,T}$, *i.e.* a disturbance with respect to which the feedback law does not behave better than the open loop strategy, $\|\delta H_{yr}(r,d)\|_{2,T} \geq \|\delta H o_{yr}(r,d)\|_{2,T}$.

Following this preliminary remark, the interest of feedback law is necessarily limited to a specific class of perturbations, say $P^e \subset \mathcal{L}_2^e$. The use of the feedback control law could be justified if (and only if) there exist $\epsilon \ll 1$ such that

$$\|\delta H_{yr}(r,d)\|_{2,T} \leq \epsilon \|\delta H o_{yr}(r,d)\|_{2,T} \tag{7}$$

for any $d \in P^e \subset \mathcal{L}_2^e$ and any $r \in \mathcal{L}_2^e$.
We now show how this requirement can be formulated in terms of a weighted incremental criteria. To this purpose, as in the H_∞ approach (see Zames, 1981), we assume that the set of possible disturbances where desensitivity must be achieved can be defined by:

$$P^e = \{d \in \mathcal{L}_2^e | \|W_p^{-1}(d) - W_p^{-1}(r+d)\| \leq \epsilon \|d\| \text{ for any } r \in \mathcal{L}_2^e\}$$

where W_p and W_p^{-1} are two causal and incrementally stable operators.

Theorem 5 *Consider the nonlinear feedback system depicted in figure 3. If* $\|(I+GK)^{-1}W_p\|_\Delta \leq 1$ *then* $\|\delta H_{yr}(r,d)\|_{2,T} \leq \epsilon \|\delta H o_{yr}(r,d)\|_{2,T}$ *for any* $d \in P^e \subset \mathcal{L}_2^e$ *and any* $r \in \mathcal{L}_2^e$.

PROOF. From the hypothesis, for all $w_1, w_2 \in \mathcal{L}_2^e$, one has:

$$\|(I+GK)^{-1}W_p(w_1) - (I+GK)^{-1}W_p(w_2)\|_{2,T} \leq \|w_1 - w_2\|_{2,T}$$

and, since W_p is assumed invertible, we deduce that

$$\|(I+GK)^{-1}(r) - (I+GK)^{-1}(r-d)\|_{2,T} \leq \|W_p^{-1}(r) - W_p^{-1}(r-d)\|_{2,T}.$$

This implies from the definition of W_p that

$$\|\delta H_{yr}(r,d)\|_{2,T} \leq \epsilon \|d\|_{2,T}$$

for any $r \in \mathcal{L}_2^e$ and $d \in P^e$ which concludes the proof.

Note that the use of nonlinear weighting functions for specifying attenuation requirements allows to take into account operating points defined by the values of r and d (a remark in section 5, after proposition 17, clarifies this aspect).

The ability of the feedback to reject the effect of uncertainties can be studied in the same way. As a matter of fact, the uncertainties generate signal perturbations which should be rejected as much as possible. Since the uncertainties modify the input-output properties, the rejection is obtained only if the perturbed maps possesses suitable properties.

4 Input-output performance

Input-output performance is related to suitable properties of the input-output maps together with requirements on the outputs corresponding to prefixed input signals. In this context a classical requirement concerns the steady-state behaviors associated to constant or periodic references together with their maintenance under perturbations. Such a property directly follows from the internal stability when dealing with linear systems: as a matter of fact the stability of the null trajectory ensures the asymptotic rejection of any perturbation acting in finite time or vanishing at the infinity. This is no longer the case in a nonlinear context where external and internal stability on any output trajectory does not follow from the stability with respect to a particular one.

4.1 Black formulae

Desoer and Wang (1980) describe the performance as the ability of a closed-loop system to asymptotically minimize the gain between references and error signals. This approach is recalled in the sequel. Denoting by $R_d^e \subset \mathcal{L}_2^e$ the set of inputs of interest, (*e.g.* sinusoids, steps, ramps,...), one sets:

Definition 6 *Asymptotic performance of the system depicted in figure 1 is satisfied on R_d^e if for all $r \in R_d^e$, there exists $T_0 \geq t_0$ such that for all $T \geq T_0$, one has: $\|(I + FGK)^{-1} r\|_{2,T} \ll \|r\|_{2,T}$.*

Definition 6 ensures that the relation $FH_{yr} \approx I$ is asymptotically satisfied on R_d^e and indicates that H_{yr}, restricted to the domain of interest, is essentially specified by F and is quite independent of G. This is the nonlinear equivalent of the well-known Black formulae (Desoer and Wang, 1980).

The asymptotic performance can now be specified in terms of a weighted $\mathcal{L}_2$ norm as pointed out below. We assume the existence of an invertible causal and $\mathcal{L}_2$ stable operator W_I and $T_0 \geq t_0$ such that, for all $r \in R_d^e$ and all $T \geq T_0$, one has:

$$\|W_I^{-1}(r)\|_{2,T} \ll \|r\|_{2,T}. \tag{8}$$

Theorem 7 *When the weighting function satisfies condition (8) and*

$$\|(I + FGK)^{-1} W_I\|_i \leq 1, \tag{9}$$

then, the closed-loop system in figure 1 has the asymptotic performance property on R_d^e.

PROOF. From (9) it follows that $\|(I+FGK)^{-1}W_I(w)\|_2 \leq \|w\|_2$ for any $w \in \mathcal{L}_2$. Because of theorem 2.1 in (Willems, 1969), the gains of $(I+FGK)^{-1}W_I$ on $\mathcal{L}_2$ and $\mathcal{L}_2^e$ are equal so that for all $T \geq t_0$, one has:

$$\|(I+FGK)^{-1}W_I(w)\|_{2,T} \leq \|w\|_{2,T}.$$

Condition (8) implies that for all $r \in R_d^e$, there exists $T_0 \geq t_0$ such that for all $T \geq T_0$, one has:

$$\|(I+FGK)^{-1}r\|_{2,T} \leq \|W_I^{-1}(r)\|_{2,T} \ll \|r\|_{2,T}$$

which concludes the proof.

The minimization of a $\mathcal{L}_2$ gain is not enough. A first aspect which limits the validity of this approach is the unbiasedness assumption. A nominal system can always be assumed to be unbiased setting $\tilde{H}(u) = H(u) - H(0)$ but this unbiased assumption cannot be maintained when the initial condition is modified. Indeed, as pointed out by Hill and Moylan (1980), a change in the initial condition makes the system weakly $\mathcal{L}_2$-gain stable which implies that (9) can thus be rewritten, for a suitable $\beta \geq 0$, as:

$$\|(I+FGK)^{-1}W_I(w)\|_2 \leq \|w\|_2 + \beta$$

thus ensuring that for all $r \in R_d^e$, there exists $T \geq t_0$ sufficiently large such that for all $T \geq t_0$

$$\|(I+FGK)^{-1}r\|_{2,T} \leq \|W_I^{-1}(r)\|_{2,T} + \beta.$$

Therefore, the value of β limits the performance of the system. This means that the validity of (9) for the nominal system does not guarantee the robustness with respect to the initial condition of the asymptotic performance.

This first limitation is overcome when an incremental type criteria is used. As matter of fact, under the weak assumption that the "perturbed" initial condition is reachable from the initial one, *i.e.* there exits an input u which allows to reach the perturbed initial condition from the nominal one under a finite time, it is possible to claim (see Fromion et al., 1996, lemma 1) that the nonlinear operator associated to the perturbed initial condition satisfies the same weighted criteria and thus it has the asymptotic performance property on R_d^e.

4.2 Steady state properties

The existence of a unique constant steady state behavior associated to any constant input together with the Lyapunov stability of the unperturbed tra-

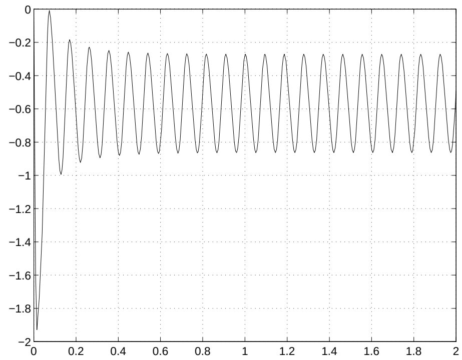

Fig. 5. The response of system (10) to a step input

jectory represents in many cases a minimal requirement.

In this context it is possible to prove that $\mathcal{L}_2$ gain type criteria do not allow to guarantee such properties other than for the null input. To convince the reader of this last assertion, we provide a simple example. Consider

$$\begin{cases} \dot{x_1}(t) = -f(x_1(t)) - 70x_2(t) + u(t) \\ \dot{x_2}(t) = 70x_1(t) - 14x_2(t) \\ y(t) = x_1(t) \\ x_1(0) = x_2(0) = 0 \end{cases} \tag{10}$$

where $u \in \mathcal{L}_2^e$ and

$$f(x) = \begin{cases} -400x - 640 & \text{for } x \leq -2 \\ -90x^3 + 200x|x| - 120x & \text{for } x \in [-2, 2] \\ -400x + 640 & \text{for } x \geq 2 \end{cases}$$

(10) can be rewritten as the interconnection between a strictly passive linear system and $f(x)$, a passive memoryless nonlinearity such that $< f(x)|x >> 0$, then the given system is $\mathcal{L}_2$-gain stable using the passivity theorem (Desoer and Vidyasagar, 1975). It is well known that the steady state behavior of this system for the null control is zero. Nevertheless, it is not difficult to prove that an oscillatory behavior is obtained if a constant input acts on this system (figure 5). Also in this case, the limitation of the approach can be overcome by referring to an incremental criterium. From (Fromion, 1997) incrementally bounded systems possess a unique steady-state behavior if the output and the state of the system are linked according to the following definition.

Definition 8 *The unperturbed motion of* Σ*, associated with an initial condition* $x_{0r} \in \mathbb{R}^n$ *and an input* $u_r \in \mathcal{L}_2^e$*, i.e.* $x_r(t) = \phi(t, t_0, x_{0r}, u_r)$*, is said to be uniformly observable if there exist* β*, a function of class* $\mathcal{K}_\infty$*, and a constant*

$T_o \geq 0$ *such that*

$$\int_t^{t+T_o} \|h(\phi(\tau, t, x_r(t), u_r), u_r(\tau)) - h(\phi(\tau, t, x, u_r), u_r(\tau))\|^2 d\tau \geq \beta(\|x_r(t) - x\|)$$

for all $x \in \mathbb{R}^n$ *and* $t \geq t_0$.

Theorem 9 (Fromion, 1997) *Let* Σ *be a dynamical system with a finite incremental gain. If the unperturbed motion, associated with* $x_{0r} \in \mathbb{R}^n$ *and* $u_r \in \mathcal{L}_2^e$, *is uniformly observable, then for any* $\tilde{u}_r \in \mathcal{L}_2^e$ *such that* $u_r - \tilde{u}_r$ *belongs to* $\mathcal{L}_2$, *one has:*

$$\lim_{t\to\infty} \|\phi(t, t_0, x_{0r}, u_r) - \phi(t, t_0, x_{0r}, \tilde{u}_r)\| = 0.$$

We note that contrarily to $\mathcal{L}_2$-gain stable systems, incrementally bounded systems possess suitable properties for a large class of inputs since the previous result holds true for any input in $\mathcal{L}_2^e$ and thus, as an example, for the constant inputs.

A second interesting result which is pointed out concerns the state evolution under specific inputs. To do so the next definition is necessary.

Definition 10 *The state space of* Σ *is said to be reachable from* $x_0 \in \mathbb{R}^n$ *if, given any* $x \in \mathbb{R}^n$ *there exist* $u \in \mathcal{L}_2^e$ *and a finite time* T_r *such that* $x = \phi(t + T_r, t_0, x_0, u)$.

The state space is said to be uniformly and isotropically reachable from x_0 *if in addition there exist* α_r, *a function of class* $\mathcal{K}$, *and* $T_r \geq 0$ *satisfying*

$$\int_{t-T_r}^{t} \|u_1(\tau) - u_2(\tau)\|^2 d\tau \leq \alpha_r(\|x_1 - x_2\|)$$

for all $x_1, x_2 \in \mathbb{R}^n$ *and* $t \geq t_0 + T_r$ *where* $u_1, u_2 \in \mathcal{L}_2$ *and* $x_i = \phi(t, t - T_r, x_0, u_i)$ *with* $i \in \{1, 2\}$.

Theorem 11 (Fromion, 1997) *Let* Σ *be a stationary dynamical system with a finite incremental gain. Assume that the unperturbed motion of* Σ, *associated with* $x_{0r} \in \mathbb{R}^n$ *and* $u_r \in \mathcal{L}_2^e$, *is uniformly observable and its state space is uniformly isotropically reachable from* x_{0r}. *Then, if* u_r *is a periodic input, the unperturbed motion is asymptotically periodic. Moreover, there exists at least one initial condition such that the motion and the output associated with this input are periodic functions.*

Theorem 11 implies that it is possible to associate an equilibrium point with each constant input.

We conclude this section by pointing out that the incremental criterium under some assumptions concerning the state space realization of the closed-loop system, ensures Lyapunov stability of unperturbed motions. This property provides a better characterization of the robustness with respect to the initial state or for specifying the effect of past inputs over the future system behaviors.

Definition 12 *An unperturbed motion of system Σ, associated with $x_{0r} \in \mathbb{R}^n$ and $u_r \in \mathcal{L}_2^e$, is said to be uniformly asymptotically stable in the sense of Lyapunov, if for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $t_1 \geq t_0$ and $\|x_r(t_1) - x_{0p}\| < \delta(\epsilon)$, one has for all $t \geq t_1$:*

$$\|\phi(t, t_1, x_r(t_1), u_r) - \phi(t, t_1, x_{0p}, u_r)\| \leq \epsilon$$

and

$$\lim_{t\to\infty} \|\phi(t, t_1, x_r(t_1), u_r) - \phi(t, t_1, x_{0p}, u_r)\| = 0.$$

If this last property holds true for any initial perturbed state x_{0p}, then the unperturbed motion is said to be uniformly globally asymptotically stable.

Theorem 13 (Fromion, 1997) *Let Σ be a dynamical system with a finite incremental gain. If the unperturbed motion of Σ, associated with $x_{0r} \in \mathbb{R}^n$ and $u_r \in \mathcal{L}_2^e$, is uniformly observable and if the state space of Σ is uniformly isotropically reachable from x_{0r} then this unperturbed motion is uniformly globally asymptotically stable.*

Theorem 13 claims, under some assumptions on the state space realization of the system, that the perturbed behavior asymptotically goes (in sense of Lyapunov) to the unperturbed one.

5 Connections with non-stationary and stationary H_∞ control

We illustrate in this section the connection between requirements set in terms of weighted incremental norms and some local ones associated with the linearizations of the operator. For this purpose, we first recall an important result in the context of incrementally bounded systems, which links the incremental gain of a nonlinear operator to the norm of its derivatives. We show that satisfying a weighted incremental type criterium is equivalent to satisfy an infinity of non-stationary suboptimal H_∞ criteria. Then, the type of stationary and of non-stationary H_∞ criteria associated with the linear approximation is discussed.

For the sake of clarity, we first recall known results about the differentiability of nonlinear operators defined over functional spaces.

Definition 14 *Given an operator Σ, defined from $\mathcal{L}_2$ into $\mathcal{L}_2$, let $u_0 \in \mathcal{L}_2$ and assume the existence for any $h \in \mathcal{L}_2$ of a continuous linear operator $D\Sigma_G[u_0]$*

from $\mathcal{L}_2$ into $\mathcal{L}_2$ such that

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma(u_0 + \lambda h) - \Sigma(u_0)}{\lambda} - D\Sigma_G[u_0](h) \right\|_2 = 0$$

then $D\Sigma_G[u_0]$ is called the Gâteaux derivative (the linearization) of Σ at u_0.

This definition of the derivative on $\mathcal{L}_2$ is restrictive. Indeed, since the derivative is by definition a continuous linear operator on $\mathcal{L}_2$, it is bounded. Consequently, this implies that the operator has a derivative only if it is a finite gain stable operator.

We can avoid this strong restriction by defining the derivative of a causal nonlinear system on the extended space $\mathcal{L}_2^e$. In this case, the existence of the derivative follows from the existence of a finite gain on a finite support. With reference to the definition given by Willems (1969), we introduce:

Definition 15 *$D\Sigma_G[u_0]$ from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ is said to be the Gâteaux derivative of the causal operator Σ, defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, at u_0 if it is linear and if for all $T \in [t_0, \infty)$, $P_T D\Sigma_G[u_0]$ is the Gâteaux derivative of $P_T\Sigma$ at $P_T u_0$.*

When the system is generated by differential equations, definition 15 corresponds to the usual linearization concept. Under the assumption made on f and h in equation (1), *i.e.* uniformly Lipschitz and C^1, $y = \Sigma(u)$ has a Gâteaux derivative [1] for all $u \in \mathcal{L}_2^e$. Moreover, its linearization along the input $u_r(t)$, denoted by $\bar{y} = D\Sigma_G[u_r](\bar{u})$, satisfies the differential equations

$$\begin{cases} \dot{\bar{x}}(t) = \frac{\partial f}{\partial x}(x_r(t), u_r(t))\bar{x}(t) + \frac{\partial f}{\partial u}(x_r(t), u_r(t))\bar{u}(t) \\ \bar{y}(t) = \frac{\partial h}{\partial x}(x_r(t), u_r(t))\bar{x}(t) + \frac{\partial h}{\partial u}(x_r(t), u_r(t))\bar{u}(t) \\ \bar{x}(t_0) = 0 \end{cases} \tag{11}$$

where $x_r(t)$ is the solution of (1) under input $u_r(t)$.

The theorem recalled below is a key result in the context of nonlinear control. It sets a strong connection between the incremental norm and the local properties associated with the derivative of a nonlinear system.

Theorem 16 (Willems, 1969) [2] *Let us assume that a causal operator Σ defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ has a Gâteaux derivative at each point u_0 of $\mathcal{L}_2^e$. Σ has a finite incremental gain if and only if there exists a finite constant η such that*

[1] It is possible to prove that if f and g are not linear functions of their arguments that the system is not Fréchet differentiable on $\mathcal{L}_2^e$.

[2] The proof provided by Willems (1969), even if Σ is Gâteaux differentiable and not Fréchet differentiable, can be easily extended to our case.

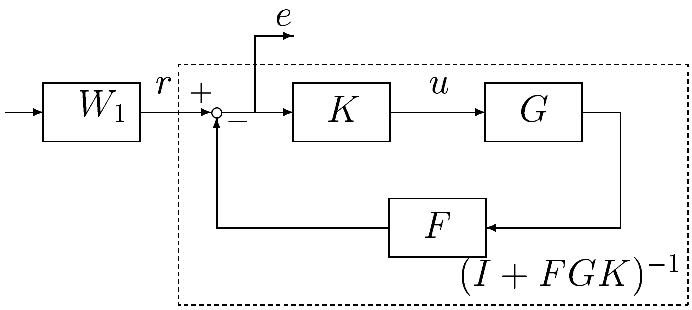

Fig. 6. The augmented plant

for any $u_0 \in \mathcal{L}_2^e$ *and any* $T \geq t_0$, *one has*

$$\|P_T D\Sigma_G[u_0]\|_i \leq \eta.$$

Moreover $\|\Sigma(u_1) - \Sigma(u_2)\|_\Delta = \sup_{u_0} \|P_T D\Sigma_G[u_0]\|_i$.

With this in mind we can now point out the connections between the weighted incremental approach and H_∞ control. Let $M_{zw} = W_o H W_i$ be the augmented plant where W_i and W_o are the input and output weighting functions associated with robustness and performance requirements. We assume that the augmented system is described by a differential equation with C^1 and globally Lipschitz drift and output functions (this ensures the existence of the Gâteaux derivative of the augmented system). From theorem 16, one deduces:

Proposition 17 *If the augmented system,* $M_{zw} = W_o H W_i$, *possesses a Gâteaux derivative for every input in* $\mathcal{L}_2^e$ *then* $\|M_{zw}\|_\Delta \leq 1$, *if and only if*

$$\|DW_{oG}[H(W_i(w_0))]DH_G[W_i(w_0)]DW_{iG}[w_0]\|_i \leq 1 \ \ \forall w_0. \in \mathcal{L}_2^e \tag{12}$$

Recalling that $DM_{zwG}[w_0]$ is a linear time-varying operator, proposition 17 shows that solving a weighted incremental problem is equivalent to solving an infinite number of linear time-varying weighted induced norm problems. It is worth noting that the constraints (12) are satisfied if (and only if) an infinite number of linear time-varying weighted H_∞ constraints are satisfied.

The non-stationary characteristics of the induced norm criterium of proposition 17 is discussed through a simple example described by figure 6. Given a small variation $\delta r(t)$ of the system input $r(t)$, the tracking error variation can then be approximated on a finite time interval

$$e = S(r + \lambda\delta r) - S(r) \approx DS_G[r](\lambda\delta r)$$

where $S = (I + FGK)^{-1}$. Let us here assume that the performance requirements are taken into account by using a linear weighting input operator, $W_i = W_I$. This weighting function, assumed to be causal and invertible, sat-

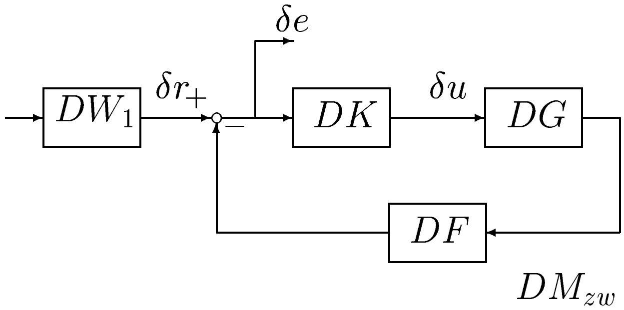

Fig. 7. Linearization of the augmented plant

isfies the following relation (see section 4):

$$\|P_T W_I^{-1}(r)\|_2 \ll \|P_T r\|_2.$$

If the condition of proposition 17 is satisfied, it can be claimed that

$$\|DS_G[W_I(w_0)]W_I\|_i \leq 1.$$

The above relation represents an H_∞ time-varying constraint (see figure 7), which ensures for any $\delta r \in R_d^e$ that there exists a time, $T_0 \geq t_0$, such that for all $T \geq T_0$, one has:

$$\|P_T \delta e\|_2 \ll \|P_T \delta r\|_2.$$

Note that

$$\|P_T DS_G[W_I(w_0)](\delta r)\|_2 \leq \|P_T\ W_I^{-1}(\delta r)\|_2 \ll \|P_T\ \delta r\|_2.$$

In a performance context, proposition 17 can be interpreted in two different ways:

- as a constraint on the linearizations of the system along the trajectory defined by $W_I(w_0)$. Consequently, this guarantees a good behavior of the nonlinear system along this trajectory despite small perturbations belonging to R_d^e.
- as a constraint on the output variations with respect to small input variations. For example, the output associated with a step input can be interpreted as the succession of responses to small step-inputs associated with each linearization of the nonlinear system along the trajectory generated by this step. The quality of this output is directly linked to the linearizations of the weighting functions W_I.

In the approach proposed by Shamma (1988), one has to check whether the gain scheduling system satisfies a criterion of the same type as in equation (12). Our approach from a different point of view highlihts the interest of the study of Linear Parameter Varying (LPV) plants in a nonlinear context.

In the rest of this paragraph, we will show a close connection between the incremental approach and the classical gain-scheduling technique. For this purpose, we restrict our attention to a specific class of linearizations, namely the time invariant ones. We then define Z_e, the set of equilibrium points associated with any constant input:

$$Z_e = \{(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p \mid \phi(t, t_0, x_e, u_e) = x_e \ \forall t \geq t_0\}$$

where ϕ is the state transition map of Σ.

Theorem 18 (Fromion et al., 1996) *Let Σ be the system given by (1) with a finite incremental gain η. Let u_e any constant input and x_e its associated equilibrium point. If x_e is reachable from x_0 then the linearization of Σ, given by the following linear time invariant system:*

$$D\Sigma_G(u_e) \begin{cases} \dot{\bar{x}}(t) = F\bar{x}(t) + G\bar{u}(t) \\ \bar{y}(t) = H\bar{x}(t) + J\bar{u}(t) \\ \bar{x}(t_0) = 0 \end{cases}$$

$F = \frac{\partial f}{\partial x}(x_e, u_e), G = \frac{\partial f}{\partial u}(x_e, u_e), H = \frac{\partial h}{\partial x}(x_e, u_e), J = \frac{\partial h}{\partial u}(x_e, u_e)$, *has a finite $\mathcal{L}_2$ gain less than or equal to η, i.e. $\|D\Sigma_G[u_e]\|_i \leq \eta$.*

This result sets a direct connection between our nonlinear framework and the classical gain scheduling techniques, especially with the approaches based on the extended linearization (Rugh, 91), where some properties are imposed to the linear time-invariant linearizations of the system associated with constant inputs. The result of theorem 18 renews the interest of incremental norm versus $\mathcal{L}_2$ since, with reference to the example in section 4.2, we demonstrate that $\mathcal{L}_2$-gain stability does not necessarily guarantee the stability of the linearizations associated with constant inputs. Furthermore, with respect to the weighted incremental norm approach and with reference to the augmented system previously defined, which has norm less than 1, *i.e.* $\|M_{zw}\|_\Delta \leq 1$), theorem 18 ensures that all the linearizations satisfy an H_∞ criterion. This criterion is specified at each equilibrium point by the stationary linearization of the nonlinear weighting functions, *i.e.*

$$\|DW_{oG}[H(W_i(w_0))]DH_G[W_i(w_0)]DW_{iG}[w_0]\|_i \leq 1$$

where $DW_{oG}[H(W_i(w_0))]$, $DH_G[W_i(w_0)]$ and, $DW_{iG}[w_0]$ are linear time invariant systems. This last fact has interesting connections with the work presented by Hyde and Glover (1993).

6 Conclusion

It has been shown how weighted incremental norms can be used to handle, in a non linear context, basic requirements such as robust stability, disturbance attenuation and steady state behaviors. The strong connections between requirements set in terms of weighted incremental norms and local requirements, given with reference to the linear approximations of the plant, are also pointed out.

The practical interest of this approach is illustrated in (Fromion et al., 1999) where the case study of a PI controlled missile is investigated.

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