A POSSIBLE EXTENSION OF H_∞ CONTROL TO THE NONLINEAR CONTEXT*

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ABSTRACT

The aim of this paper is to show how some basic notions of robust control can be set in a nonlinear context making use of the concept of incremental norm. The approach here proposed provides a natural framework for extending well-known H_∞ linear control concepts and to take into account robust stability, desensitivity, input-output and internal stability requirements. Moreover, it further justifies classical techniques such as gain scheduling and extended linearization. Finally an algebraic type condition ensuring incremental stability of a nonlinear system is given.

1. INTRODUCTION

H_∞ control optimization is issued from a lot of work formalizing in mathematical terms linear robustness and performances requirements. Except more recent results developed in a state space formalism ([5]), most of the work takes reference to an input-output approach ([26,2,21]). A significant part of the activities in the linear context were devoted to reformulate and generalize the classical control concepts introduced by Black, Bode and Horowitz such as phase and gain margins and desensitivity ([6,29]). Quite recently in [29], it has been shown that weighted induced norms are suitable to take into account both robustness and desensitivity requirements

A natural question is thus to discuss the nonlinear problem, revisiting the basic nonlinear control concepts pointed out for example in [3]. Doing so, one proposes to extend the H_∞ approach to the nonlinear context in terms of weighted incremental norms. This approach was recently developed in [8] and [10] where it is shown how the incremental approach enables to handle robustness and desensitivity problems.

The present paper discusses such an approach by summarizing results proposed in [7-11] where complete proofs are given.

If robustness can be set in terms of weighted L_2-induced norm or incremental norm, performance requirements suggest the use of incremental norm. We will show how weighted incremental norms can be used to handle important requirements of feedback such as desensitivity and attenuation with respect to exogenous perturbations, suitable steady state behaviors associated to step inputs and Lyapunov stability.

A first point concerns attenuation and desensitivity. A characterization in terms of linearization [17,3] combined with a classical result from functional analysis recalled in [25], which links the gain of the linearizations of a nonlinear operator with its incremental norm, shows how the problem can be reduced to the minimization of a weighted incremental norm.

Another interesting point is the link between incremental and Lyapunov stability. It can be shown that, under minimality of the state space representation of a given operator, all the trajectories associated to an incrementally stable system are Lyapunov stable. This corresponds to a robust behavior with respect to initial state perturbations. Moreover, it can be proved that incremental stability ensures the desired steady state behaviors i.e. the global asymptotic stability of equilibrium associated to constant intputs.

The paper is organized as follows. Section 2 recalls some usual notations and definitions. Section 3 visits robustness and performance problems in terms of incremental norm along 6 subsections showing the interest of such an approach. Firstly, robustness with respect to unstructured uncertainties is characterized in terms of weighted induced norm so generalizing a classical linear

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result. A first step towards performance is to restate input - output requirements under feedback in these terms. Connections between desensitivity and incremental norms are pointed out. Attention is focussed on the link between weighed incremental objectives and linear H_∞ time varying objectives set on the linearizations of the nonlinear operator. Necessity of the weights with reference to desensitivity problems is proved so extending a classical linear result. The links between input-output properties and Lyapunov stability are discussed at the end of Section 3. Finally, we point out the steady state properties of incrementally stable systems in face of step inputs. Section 4 gives an algebraic type condition characterizing incremental stability for affine nonlinear systems.

2. NOTATIONS AND DEFINITIONS

The notations and terminology recalled hereafter are classical in an input-output context ([2]). Denoting by E, the set of real measurable n vector valued functions of the real variable t on R^+, one defines $L_2^n = \{x \in E | \|x\|_2 < \infty\}$ where $\|x\|_2 = \sqrt{\int_0^\infty x(t)^T x(t)\ dt}$ and the associated extended space $L_2^{n,e} = \{x \in E | P_\tau x \in L_2^n, \quad \forall \tau \in R^+\}$ where P_τ is the causal operator which truncates a signal at time τ. For convenience, one sets $\|u\|_{2,\tau} \overset{\Delta}{=} \|P_\tau u\|_2$.

Definition 2.1 : An operator H from $L_2^{m,e}$ into $L_2^{p,e}$, is *weakly L_2-gain* stable if there exist finite non-negative constants γ and β such that

$$\|H(u)\|_2 \le \gamma \|u\|_2 + \beta \text{ for all } u \in L_2^m.$$

Its *gain* coincides with the minimum value of γ. When $\beta = 0$, the system is said to be *L_2-gain stable* and $\|H\|_{i_2}$ classically denotes its L_2- gain.

Definition 2.2 : An operator H from $L_2^{m,e}$ into $L_2^{p,e}$, has a *finite incremental gain* if there exists a finite non-negative constant η such that

$$\|H(u_1) - H(u_2)\|_2 \le \eta \|u_1 - u_2\|_2 \text{ for all } u_1, u_2 \in L_2^m.$$

Its *incremental gain* coincides with the minimum value of η and is denoted $\|H\|_\Delta$.

Definition 2.3 : An operator H from $L_2^{m,e}$ into $L_2^{p,e}$, is *incrementally stable* if it is stable, i.e., it maps L_2^m to L_2^p, with a finite incremental gain.

Remark : A finite gain stable linear operator H is incrementally stable and $\|H\|_{i_2} = \|H\|_\Delta$.

Definition 2.4 : Given a causal operator H, defined from L_2^m into L_2^p, let $u_0 \in L_2^m$ and assume there exists a bounded linear operator $DH|_{u_0}$ from L_2^m into L_2^p such that

$$H(u_0 + h) = H(u_0) + DH|_{u_0} h + \alpha(h)\|h\|_2$$

with $\lim_{\|h\|_2 \to 0} \|\alpha(h)\|_2 = 0$

or equivalently such that

$$\lim_{\|h\|_2 \to 0} \frac{\left\|H(u_0 + h) - H(u_0) - DH|_{u_0} h\right\|_2}{\|h\|_2} = \lim_{\|h\|_2 \to 0} \|\alpha(h)\|_2 = 0$$

$DH|_{u_0}$ is called the *Fréchet derivative* ([16]) or the *linearization* ([25]) of H at u_0.

Definition 2.5 : Consider the feedback system depicted in fig. 1, where H_1 and H_2 are causal input-output operators.

The feedback system is said *well-posed* if for any pair of inputs $(u_1, u_2) \in L_2^{m,e} \times L_2^{p,e}$, there exist unique e_1, y_2 and e_2, y_1 belonging to $L_2^{m,e}$ and $L_2^{p,e}$ respectively such that $e_1 = u_1 + y_2$, $e_2 = u_2 + y_1$, $y_1 = H_1 e_1$ and $y_2 = H_2 e_2$ and such that the following mapping

$$H: (u_1, u_2) \mapsto (y_1, y_2, e_1, e_2)$$

is causal (see [26] for a complete discussion about well-posedness).

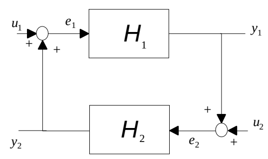

figure 1

Definition 2.6 : The feedback system depicted in fig. 1 is *internally incrementally stable* if the following mapping

$$H: (u_1, u_2) \mapsto (y_1, y_2, e_1, e_2)$$

is well-posed and incrementally stable.

3. AN INCREMENTAL APPROACH FOR NONLINEAR CONTROL

We illustrate in this section the fact that some nonlinear robustness or performance problems can be discussed, as in the linear context, in terms of the induced norm of a certain augmented system. All the results are extracted from [8] and [10].

To simplify the notations, we drop out the dimensions of the input and output vectors and assume all the operators to be unbiased (i.e. $H(0)=0$).

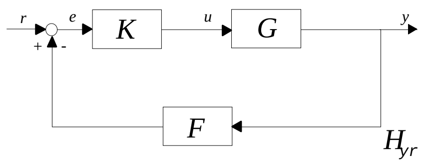

figure 2

We first consider the nonlinear feedback system of fig. 2, where G ,K, F are nonlinear causal operators from L_2^e to L_2^e. The closed loop input-output map, denoted as H_{yr}, is given by $GK(I+FGK)^{-1}$ and is assumed to be well-posed.

3.1.Robust stability against unstructured uncertainties

We first consider unstructured uncertainties $\tilde{\Delta}$ placed at the input or output of the plant model. In this way, one can represent uncertainties in the actuator dynamics, high frequency neglected dynamics (e.g. bending modes) or output sensor errors (...) [6,29,4]. We assume that $\tilde{\Delta}$ belongs to a set Ω_Δ defined as

$$\Omega_\Delta \equiv \{ W_3 \Delta W_2 \mid \|\Delta\|_\Delta < 1 \}$$

where $\Delta : L_2 \rightarrow L_2$ is a (possibly nonlinear) causal operator and W_2 and W_3 are causal, incrementally stable input-output maps from L_2 to L_2.

A stability result is proposed for the interconnected system of fig. 3 (M is the generic nominal closed loop system).

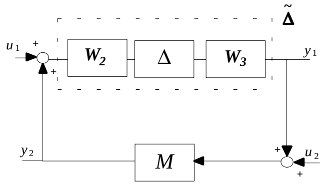

figure 3

Proposition 1 : *If M is incrementally stable and if the following inequality holds*

$$\|W_2 M W_3\|_\Delta \leq 1$$

then the closed loop system of fig. 3 is incrementally stable for any uncertainty belonging to Ω_Δ.

Proof : The proof is based on a result in [25, Th. 4.6] which claims that the interconnection of incrementally stable systems, depicted on fig. 1, is incrementally stable if $(I - H_1 H_2)^{-1}$ is incrementally stable. In our case, setting $H_1 = M W_3 \Delta$ and $H_2 = W_2$, the assumptions ensure that for all $\tilde{\Delta} \in \Omega_\Delta$, one has

$$\|W_2 M W_3 \Delta\|_\Delta \leq \|W_2 M W_3\|_\Delta \|\Delta\|_\Delta < 1$$

and thus, invoking incremental small gain arguments, we get

$$\|(I - W_2 M W_3 \Delta)^{-1}\|_\Delta < \infty$$

A simple reasoning, based on the definition of internal stability, enables to prove that $(I - M W_3 \Delta W_2)^{-1}$ is necessary incrementally stable. Then, one applying again Theorem 4.6 in [24], one concludes

Remark : A Similar result can be obtained for L_2 gain stability [10].

3.2. Tracking and asymptotic properties

Let us recall the definition proposed in [3], which defines performance as the ability for a closed loop system to "asymptotically" minimize the gain between the inputs of interest and the error signals. More precisely, denoting as $R_d^e \subset L_2^e$ the set of inputs of interest (i.e., sinusoids, steps, ramps...), one sets

Definition 3.1 : A*symptotic performance* of the system depicted in fig. 2 is satisfied on R_d^e if for all $r \in R_d^e$ and for T sufficiently large, one has

$$\|(I + FGK)^{-1} r\|_{2,T} << \|r\|_{2,T}$$

This definition ensures that the relation $FH_{yr} \cong I$ is asymptotically verified on R_d^e and indicates that H_{yr}, restricted to the domain of interest, is essentially specified by F and is quite independent from G. This is the nonlinear equivalent of the well known Black‘s formulae ([3]).

Let us now specify asymptotic performance in terms of a weighted incremental norm. As in the linear H_∞ context [29,18], we assume the existence of an invertible causal operator such that, for all $r \in R_d^e$ and for T sufficiently large, one has

$$\|W_I^{-1} r\|_{2,T} << \|r\|_{2,T}$$

Proposition 2 : *The asymptotic performance of the closed loop system of fig. 2 is guaranteed on* R_d^e *if*

$$\|(I + FGK)^{-1} W_I\|_\Delta \leq 1$$

Sketch of proof : The proof of Proposition 2 is based on a result in [25, Th. 3.2] and classical manipulations on the inequalities between norms of signals

Remark : In the linear context (see [29]), one assumes the condition

$$\|W_I^{-1} r\|_{2,T} \cong \|r\|_{2,T} \quad r \in R_d^e$$

and the existence of ε <<1 such that

$$\|(I + FGK)^{-1} W_I\|_{i_2} \leq \varepsilon$$

In fact, our condition is the nonlinear analog but it is formulated in a normalized sense to be consistent with Proposition 1. This comment implies, under some minor modifications, that the weighted incremental norm is a weighted semi-norm in the terminology of Zames ([29]).

3.3. Weighted incremental norm and time-varying H∞ control

On the basis of a classical functional analysis result linking the incremental gain of a nonlinear operator with the gain of its derivatives, we deduce the following proposition.

Proposition 3 : *If the augmented system,* $M_{zw} = W_o H W_i$, *is linearizable for all inputs and if* $\|M_{zw}\|_\Delta \leq 1$ *then*

$$\left\| DW_o\big|_{HW_i(w_0)} DH\big|_{W_i(w_0)} DW_i\big|_{w_0} \right\|_{i_2} \leq 1 \quad \forall w_0 \in L_2$$

Sketch of proof : To achieve the proof, we use Lemma 7.1 in [25] and simply recall that the derivative of the composition of nonlinear operators works out like the composition of functions

The interest of Proposition 3 stands in the fact that solving a weighted incremental problem is equivalent to solve an infinite number of linear time-varying weighted H_∞ problems.

We illustrate this fact by a simple example. Let us consider the input system $r(t)$ modified by a small variation $\delta r(t)$, so that the tracking error variation can be approximated as follows :

$$\delta e \overset{\Delta}{=} H(r+\delta r) - H(r) \cong DH|_r \, \delta r$$

where $H = (I + FGK)^{-1}$.

Now, if we assume that weighting functions are linear and just take into account performance objectives i.e. $W_i = W_I$ and $W_o = I$ then, in this particular case, Proposition 3 can be rewritten as

$$\left\| DH|_u W_I \right\|_{i_2} \leq 1$$

which represents a H_∞ constraint ensuring that for all $\delta r \in R_d^e$, one has

$$\delta e << \delta r$$

because

$$\left\| DH|_u \delta r \right\| \leq \left\| W_I^{-1} \delta r \right\| << \delta r$$

In the performance context, the result of Proposition 3 can be interpreted in two different following ways in the sense that it guarantees

- A constraint on the linearization of the system along the trajectory defined by W_I (w_o). Actually, a good behavior of the nonlinear system along this trajectory is guaranteed with respect to small perturbations belonging to R_d^e.
- A constraint on the output variations with respect to small input variations. For example, the output associated to a step input can be interpreted as the succession of responses to small step-inputs associated to each linearization of the nonlinear system along the trajectory generated by this step. The quality of this output is directly linked to the linearizations of the weighting functions W_I.

Remarks :

(*i*) - If we relax the linearity assumption on W_I, the local performance is consequently specified around w_0 by $DW_I\big|_{w_0}$.

(*ii*) In [24], the properties of gain scheduled control for nonlinear plants are characterized through the properties associated to their linearizations. The incremental approach justifies such an approach.

Proposition 3 has a more interesting consequence for non linear control. Let us remember that it is possible to use the linearizations to characterize the desensitization or attenuation properties of a nonlinear plant [17, 3]. Combining Th. III.1 [3] and Proposition 3, it can be claimed that the problem of desensitization by feedback can be transformed into a problem of minimization of a weighted incremental norm [8,10].

3.4. Necessity of weighted incremental norm for desensitivity problem

One can extend the result obtained in [29], where the necessity of the weights is proved in a linear context. More precisely, we show that the minimization of an incremental norm without weights does not allows to satisfy desensitivity objectives.

Proposition 4 : *If the open loop operator FGK is strictly causal then* $\left\| (I+FGK)^{-1} \right\|_\Delta \geq 1$.

Proof : we prove this fact by contradiction**.**Let $S = (I+FGK)^{-1}$. The small incremental gain Theorem ensures the existence of $(I-S)^{-1}$ if $\|S\|_\Delta < 1$. So that

$$I = S^{-1}S = (I+FGK)^{-1} + FGK\,(I+FGK)^{-1}$$

Because of the strict causality of *FGK*, $FGK\,(I+FGK)^{-1}$ is strictly causal. Consequently the operator

$$FGK\,(I+FGK)^{-1} = (I-S)$$

is not invertible ([25]), so obtaining the announced contradiction

3.5. Lyapunov and incremental stability

In this section we discuss the link between Lyapunov and incremental stability, which clarifies the notion of incremental stability.

For, given the input-output operator H_{x_0}, let the associated state-space representation

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(0) = x_0 \\ y(t) &= h(x) \end{aligned} \tag{3.1}$$

where $u(t) \in R^m$, $y(t) \in R^P$, $x(t) \in R^n$. f ,g and h are C^1, with bounded first order derivatives. In addition, we assume that $x = 0$ is an equilibrium point, i.e., $f(0) = 0$ and without loss of generality we assume $h(0) = 0$.

a. Lyapunov stability of unperturbed motions

We recall that the perturbed input-output behavior of an incrementally stable system converges to the unperturbed one when an assumption is made on its state space representation [11].

Before this, let us recall some definitions about Lyapunov stability.

Let $x_r(t)$ the non perturbed motion of the differential equation (3.1) initialized in x_{0r}, under the input $u_r(t)$ (i.e. $x_r(t) = \varphi(t, x_{0r}, u_r(t))$).

Definition 3.1 **[**30**]** ***:*** The unperturbed motion, is said uniformly asymptotically stable in the sense of Lyapunov if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $t \geq 0$ and $\|x_0 - x_{0r}\| \leq \delta$, one has

$$\left\| \varphi(t, x_0, u_r(t)) - x_r(t) \right\| \leq \varepsilon$$

and

$$\lim_{t\to\infty} \left\| \varphi(t, x_0, u_r(t)) - x_r(t) \right\| = 0$$

If these two properties hold for all $x_0 \in R^n$, the unperturbed motion is said globally uniformly asymptotic stable.

To establish the result, let us associate to the input $u_r(t) \in L_2^{m,e}$, the following system :

$$y_G = G_{x_0}[u_r, y_r](\bar{u}) \overset{\Delta}{=} H_{x_0}(u_r + \bar{u}) - y_r$$

where $y_r = H_{x_{0r}}(u_r)$.

Referring to [26,13] for the notion of uniform observability and reachability, one can state the following proposition.

Proposition 5 : *If* $H_{x_{0r}}$ *has finite incremental gain and if for a given input* $u_r(t) \in L_2^{m,e}$, $G_{x_{0r}}[u_r, y_r]$ *has an uniformly observable equilibrium point and an uniformly reachable state space from the equilibrium point then, the unperturbed motion,* $x_r(t)$, *and the output,* $y_r(t)$, *associated to* $u_r(t)$ *are globally uniformly asymptotically stable.*

Sketch of proof : The proof works out showing that if $H_{x_{0r}}$ is incrementally stable then the operator $G_{x_{0r}}[u_r, y_r]$ is L_2 gain stable. Using classical results concerning the link between L_2-gain and Lyapunov stability, the proof can be achieved (see [26])

This result is interesting in the context of asymptotic performance since its ensures the robustness of asymptotic performance against an initial condition perturbation.

b. Steady state and finite incremental gain

In this section, we recall that the incremental boundedness of the input-output operator ensures the asymptotic stability of the equilibrium points associated to constant controls [9].

Let Z_e denotes the set of these equilibrium points :

$Z_e = \{(x_e, u_e) \in R^n \times R^m | \varphi(t, x_e, u_e) = x_e \ \forall t \in R^+\}$

The following operator is associated to the equilibrium pair (x_e, u_e) of H_{x_0} :

$$G_{x_0}[u_e, y_e](u) \overset{\Delta}{=} H_{x_0}(u + u_e) - y_e$$

where $y_e = H_{x_e}(u_e)$, so that $G_{x_e}[u_e, y_e](0) = 0$.

Corollary : *If* H_{x_0} *has finite incremental gain and if for any pair* (x_e, u_e) $\in Z_e$, $G_{x_0}[u_e, y_e]$ *has an uniformly observable equilibrium point and an uniformly reachable state space from the equilibrium point then this equilibrium point is globally asymptotically stable.*

Remarks *:*

(*i*) - The result of Corollary holds for a general operator without any restriction concerning the state space generator. This is developed in [8].

(*ii*) - Corollary proves that the feedback achieved servoaction. In particular, if we assume that the nonlinear compensator contains an "integral control" on the tracking error (i.e. $\dot{e}(t) = K(u_0 - y_m(t))$ then, for any constant input reference u_0,under assumptions of Corollary, the tracking error asymptotically converges to zero (i.e., $\lim_{t\to\infty} y_m(t) \to u_0$).

3.6. Scheduled gain and incrementally stable system

Let us now point out how incremental stability induces suitable properties with respect to pseudo linearization or gain scheduling [19,20].

Let us denote by $\bar{y} = DG|_0 \bar{u}$ the linearization of (3.1) around $(x_e, u_e) \in Z_e$, with the state representation

$$\dot{\bar{x}} = F\bar{x} + G\bar{u}, \qquad \bar{x}(0) = 0$$
$$\bar{y} = H\bar{x}, \tag{3.2}$$

where $\bar{u} \in R^m$, $\bar{x} \in R^n$ and $\bar{y} \in R^p$,

$$F = \frac{\partial f}{\partial x}(x_e) + \frac{\partial g}{\partial x}(x_e)u_e,\ G = g(x_e) \text{ and } H = \frac{\partial h}{\partial x}(x_e).$$

Proposition 6 [9] ***:*** *If the nonlinear operator* H_{x_0} *has a finite incremental gain less or equal to* η *and if any state in* Z_e *is reachable from* x_0, *then for any fixed pair* (x_e, u_e) $\in Z_e$, *the associated linearization,* $DG|_0$, *has* L_2 *-gain* $\gamma \le \eta$ *, i.e.* $\|DG|_0\|_\infty \le \eta$.

If, in addition, (*F*,*G*) *is controllable and* (*H*,*F*) *is observable, then there exists a symmetric and strictly positive definite matrix,* $P \in R^{n\times n}$ *such that*

$$PF + F^T P + \tilde{\eta}^{-2} PGG^T P + H^T H = 0 \qquad \forall \tilde{\eta} > \eta .$$

Remarks :

(*i*) - Proposition 6 proves that system (3.1) is locally exponentially stable in a neighborhood of x_e.

(*ii*) - Proposition 6 can be linked to extended linearization or gain scheduling techniques [19,20].

4. AN ALGEBRAIC CONDITION FOR INCREMENTAL STABILITY

The object of this section is to give an algebraic type condition for incremental stability making use of the dissipativity properties of an extended system.

Proposition 7 : *Let* η *be a fixed non negative constant. The nonlinear system* (*3.1*) *is incrementally stable and has incremental gain less or equal to* η *if there exists a real function,* $V: R^n \times R^n \to R^+$, *continuously differentiable in these arguments* (*i.e.* C^1) *and such that for all* $x_1, x_2 \in R^n$, *one has*

(*i*) $V(0,0) = 0$

(*ii*) $V(x_1, x_2) \ge 0$

(*iii*) $\dfrac{\partial V(x_1,x_2)}{\partial x_1} g(x_1) + \dfrac{\partial V(x_1,x_2)}{\partial x_2} g(x_2) = 0$

$$(iv)\ \frac{\partial V(x_1,x_2)}{\partial x_1} f(x_1) + \frac{\partial V(x_1,x_2)}{\partial x_2} f(x_2) + \tfrac{1}{4}\eta^{-2} \frac{\partial V(x_1,x_2)}{\partial x_1} g(x_1) g^T(x_1) \frac{\partial V^T(x_1,x_2)}{\partial x_1} + (h(x_1) - h(x_2))^T (h(x_1) - h(x_2)) \le 0$$

Sketch of proof : Associating with the system (3.1) the augmented system defined as

$$\begin{cases} \dot{x}_1 = f(x_1) + g(x_1)u_1, & x_1(0) = x_{10} \\ \dot{x}_2 = f(x_2) + g(x_2)u_2, & x_2(0) = x_{10} \\ y_1 = h(x_1) \\ y_2 = h(x_2) \end{cases} \quad (4.1)$$

we denote $\bar{x}^T = (x_1^T, x_2^T)$ and we consider the new variables

$$p = u_1 - u_2$$
$$z = y_1 - y_2$$

System (4.1) is incrementally stable if and only if for all $u_1, u_2 \in L_2^m$, one has

$$\|z\|_2 \leq \eta \|p\|_2 \quad (4.2)$$

when $\bar{x}(0) = 0$.

Using classical arguments about dissipativity ([27]), we claim that (4.2) is satisfied if the system is dissipative with respect to the *supply rate function*

$$w(t) = \eta^2 \|p(t)\|^2 - \|z(t)\|^2 \quad (4.3)$$

For, we recall Theorem 1 in [27] which claims that system (4.1) is dissipative with respect to $w(t)$ if and only if the following integral is finite for all $\bar{x} \in R^{2n}$

$$V_a(\bar{x}) = \sup_{\substack{\bar{x}\rightarrow \\ T>0}} \int_0^T w(t)dt \quad (4.4)$$

where the sup. is taken on $u_1, u_2 \in L_2^m$, and $z(t)$ corresponds to the output of system (4.1), initialized in $\bar{x}$ under inputs $u_1(t)$ and $u_2(t)$.

Thus, on the basis, of references [27], we obtain the announced condition

Remark : Necessity can be obtained. This is developed in [15] arguing as in [1] in terms of viscosity solutions to Hamilton-Jacobi type equations.

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