MTNS 2004

Robust Control: from linear to nonlinear

Performance and robustness analysis and design for LTI systems: a quick overview from a robust control point of view

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Supélec

France

Outline

- Considered problem
- H_∞
- μ analysis
- Conclusion

Considered problem

Considered problem

- What is a good method in automatic control ?

- For:
  - Identification
  - Synthesis
  - Analysis

- Looking for methods who are:
  - General
  - Easy to use
  - CAD (computer-aided design)

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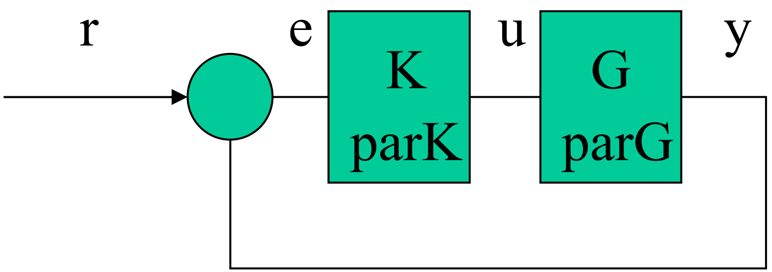
Considered problem

- **A robustness definition**
  - Let consider a property **P** (stability, raising time, overshoot ...)
  - And a set of models **F**
  - A controller K is P-robust, if the connexion K***F** ensures the property P for all the models in the set **F**.
- Note that :
  - *Robustness is defined for **a given property P** and **a given family F**.*
  - ***Robustness was proposed long time again before 1990!... The major advance was to formalize and explicitly integrate this notion in controller design methods.***

Considered problem

- Some optimisation problem for control



- LQG $\Rightarrow \min_{K} \int e(t)^2 dt + \int u(t)^2 dt$

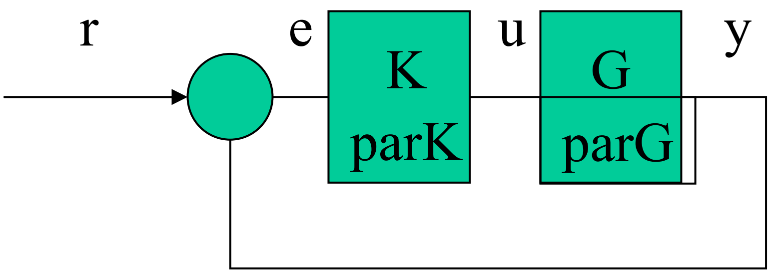
Spec. formulation: partial; Resolution: easy

- Structured LQG $\Rightarrow \min_{parK} \int e(t)^2 dt + \int u(t)^2 dt$

Nice problem … but very difficult

Considered problem

- Some optimisation problem for control



- H_∞ optimisation ⇨ $$\min_{K} template\left(\frac{e(j\omega)}{r(j\omega)}\right)$$

Spec. formulation: partial but classical (shaping);
Resolution: easy

- Structured robust Control ⇨ $$\min_{parK}\max_{parG} J(K,G)$$

Spec. formulation: ideal … just a dream

- Many others (control, analysis, identification…)

Considered problem

- General problems are often **non polynomial problem (NP)**

↓

No well suited for engineering approach

- Even some simple and classical problems. For example, the multivariable gain margin

$$\min_{parG/BoucleStable} \|parG\|$$

  - The smallest vector of parameters that destabilize the clossed loop.
  - NP hard problem

Considered problem

What can we do with the **NP engineer questions** ?

↕

General criterion

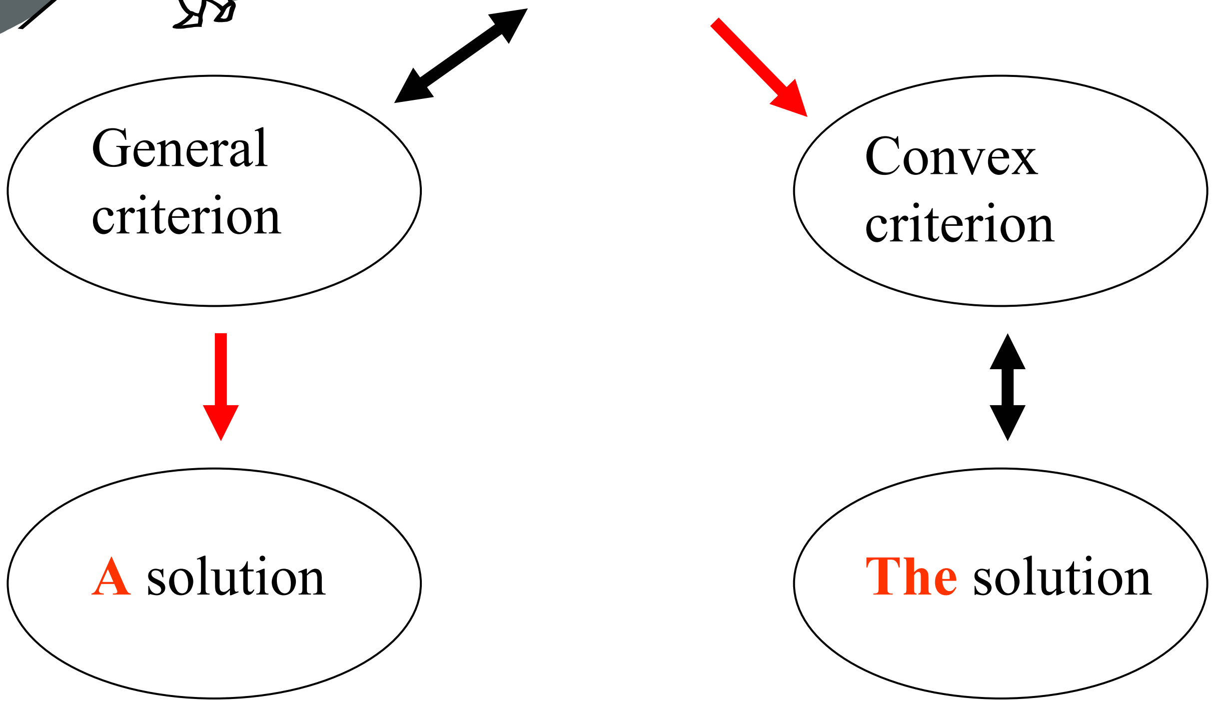
↓

A solution

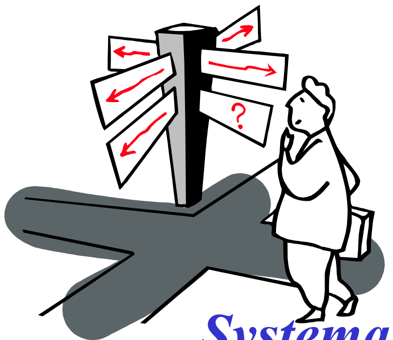
Considered problem

What can we do with the

NP engineer questions ?



Considered problem



What can we do with the

NP engineer questions ?

Systematic translation

General criterion

General optimisation
+ a priori information
Xstart, bounds, variable transformation…

A solution

Local minima

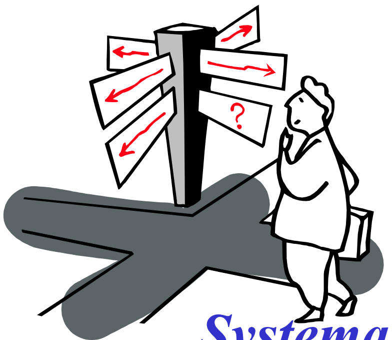
Convex criterion

The solution

These approaches are not mutually exclusive

Supélec

Considered problem



What can we do with the **NP engineer questions** ?

Systematic translation

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A solution

Local minima

Rewrite the specs

Convex criterion

Immersion
Multiplier
Relaxation

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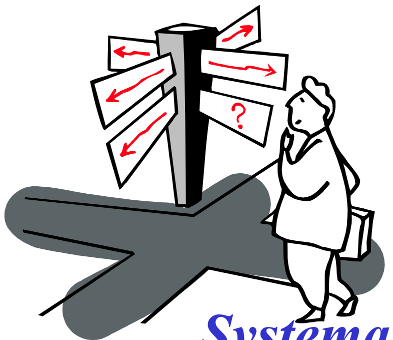
The solution

Global minimum

These approaches are not mutually exclusive

Supélec

Considered problem



What can we do w

NP engineer ques

Enlarge the field of robust contr methods to nonlinear systems while preserving:
- global properties for stability
- global performance specs
- direct consideration of robustness specs

Systematic translation

Rew

General criterion

C
cı

General optimisation
+ a priori information Xstart, bounds, variable transformation…

Convex optimisation

A solution

Local minima

The solution

Global minima

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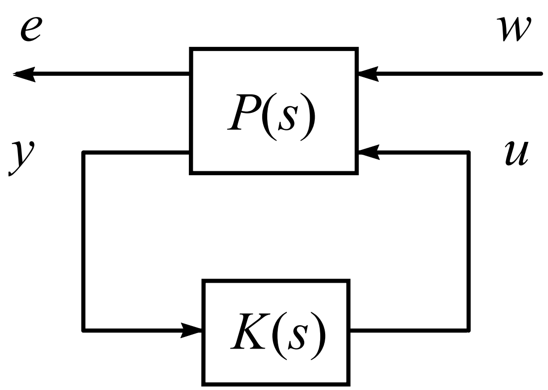
H_∞

Supélec

H_∞ – Standard problem

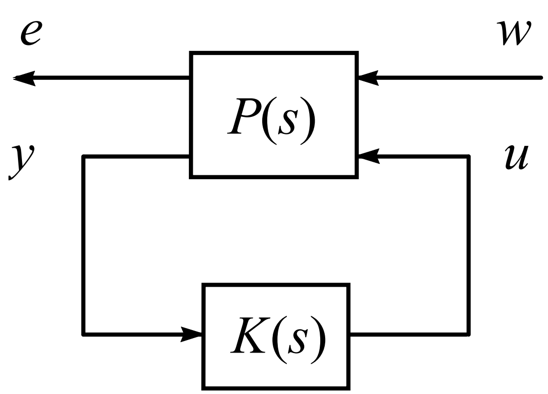
Supélec

H_∞ – Standard problem



- w: input signals (perturbations, noises, …)
- u: control signals
- e: output signals that allows characterization of a 'good' behaviour (typically look like an 'error' signal)
- y: measurement signals

H_∞ – Standard problem



- **standard problem**
  - P(s) and γ
  - Find K(s) such as
    - closed loop is stable
    - and $\|Te/w\|_\infty \leq \gamma$

$$\| \mathbf{F}_l \left(P(s), K(s) \right) \|_\infty < \gamma$$

- Controllers that allow to obtain the smallest value γ* are optimal.

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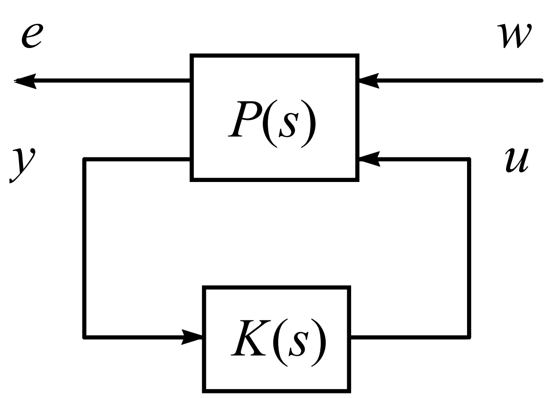
H_∞ – Standard problem



- Solutions:
  - Hankel operator (decomposition, projection,…)
  - Glover-Doyle solution at the end of the 80's using Riccati equations
  - Kwakernak polynomial approach give nice insight on the behaviour of central solution and rank deficiency of the optimal controller
  - Gahinet, Apkarian method (1994) using LMI

- K. Glover, J.C. Doyle. "State-Space Formulae for all Stabilizing Controllers That Satisfy an *H*∞-Norm Bound and Relations to Risk Sensitivity". *Systems & Control Letters*, vol. 11, pp. 167-172, 1988.
- J.C. Doyle, K. Glover, P.K. Khargonekar, B.A. Francis. "State-Space Solutions to Standard *H*2 and *H*∞ Control Problems". *IEEE Trans. Autom. Control*, AC 34 n° 8, pp. 831-846, 1989.
- P. Gahinet, P. Apkarian. "A Linear Matrix Inequality Approach to H∞ Control". Int. J. of Robust & Nonlinear Contr., vol. 4, pp. 421-448, 1994.
- T. Iwasaki, R.E. Skelton. "All Controllers for the General H∞ Control Problem: LMI Existence Conditions and State-Space Formulas". Automatica, vol. 30 n° 8, pp. 1307-1317, 1994.

Supélec

H_∞ – Typical 4 block problem

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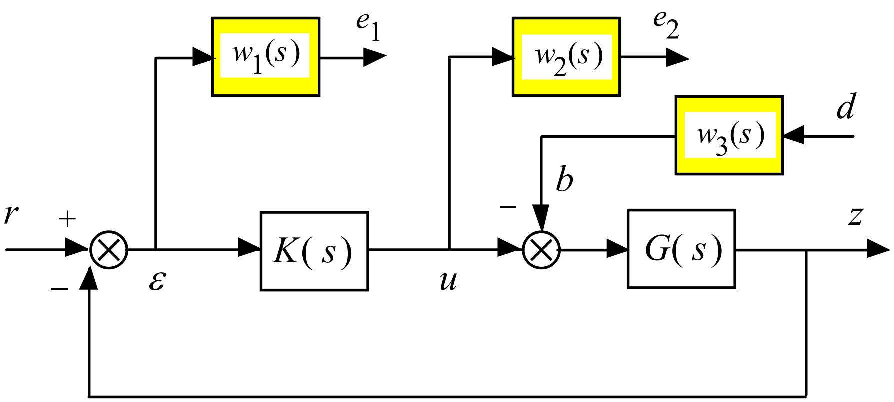
H_∞ – Typical 4 block problem



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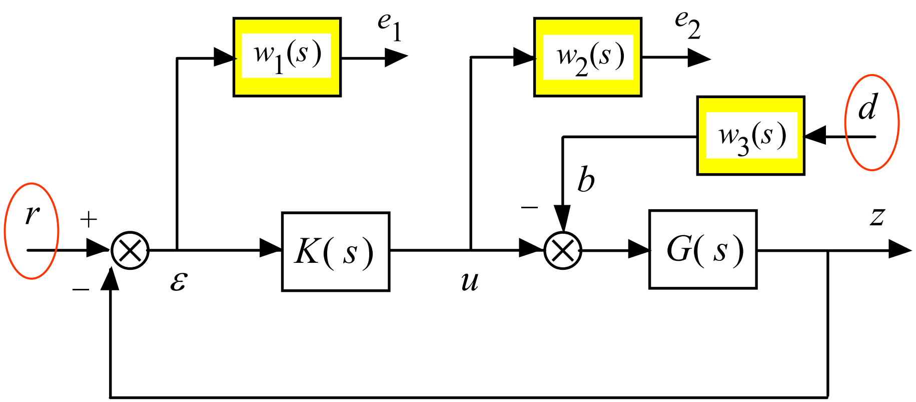
H_∞ – Typical 4 block problem

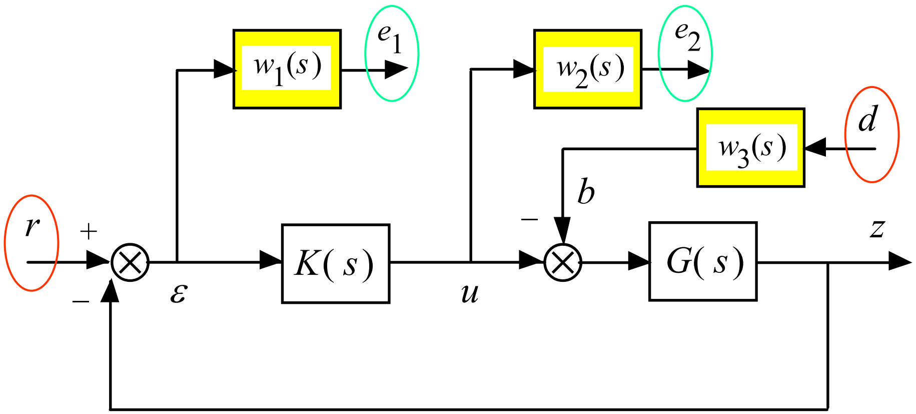


- 4 block criterion: two input / two output criterion

$$\begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix} = \begin{pmatrix} w_1(s)\,S(s) & w_1(s)\,S(s)\,G(s)\,w_3(s) \\ w_2(s)\,K(s)\,S(s) & w_2(s)\,K(s)\,S(s)\,G(s)\,w_3(s) \end{pmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix}$$

$$S = (I + GK)^{-1}$$

H_∞ – Typical 4 block problem



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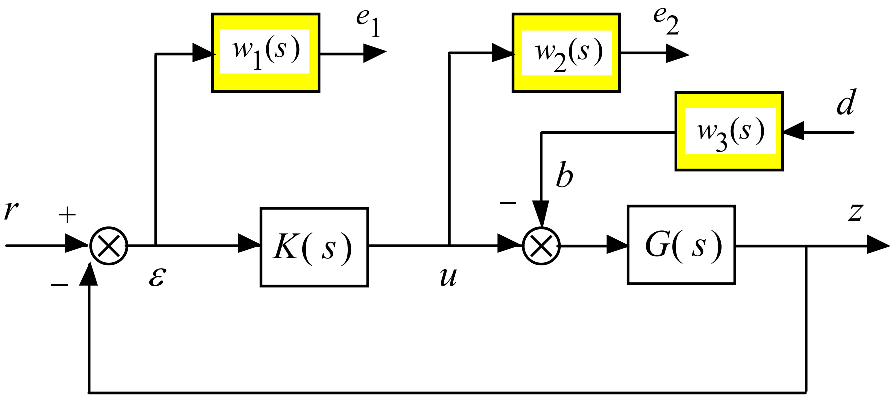

- 4 block problem: two input / two output criterion
- W_i : degrees of freedom

$$\begin{pmatrix} E_1(s) \\ E_2(s) \end{pmatrix} = \begin{pmatrix} w_1(s)S(s) & w_1(s)S(s)G(s)w_3(s) \\ w_2(s)K(s)S(s) & w_2(s)K(s)S(s)G(s)w_3(s) \end{pmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix}$$

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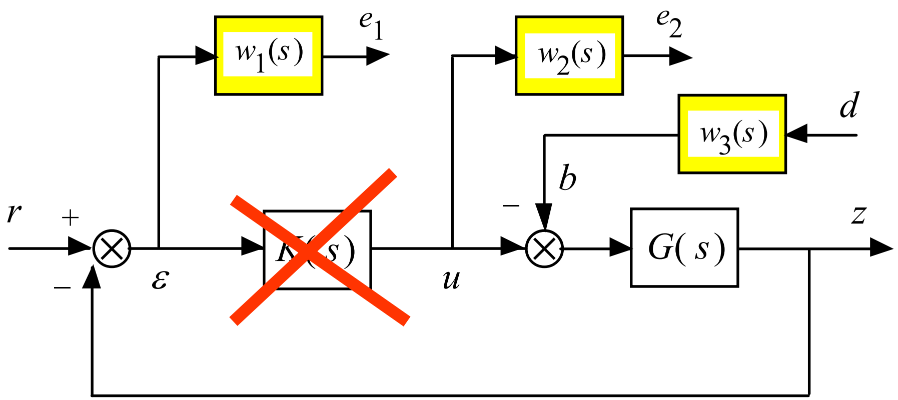
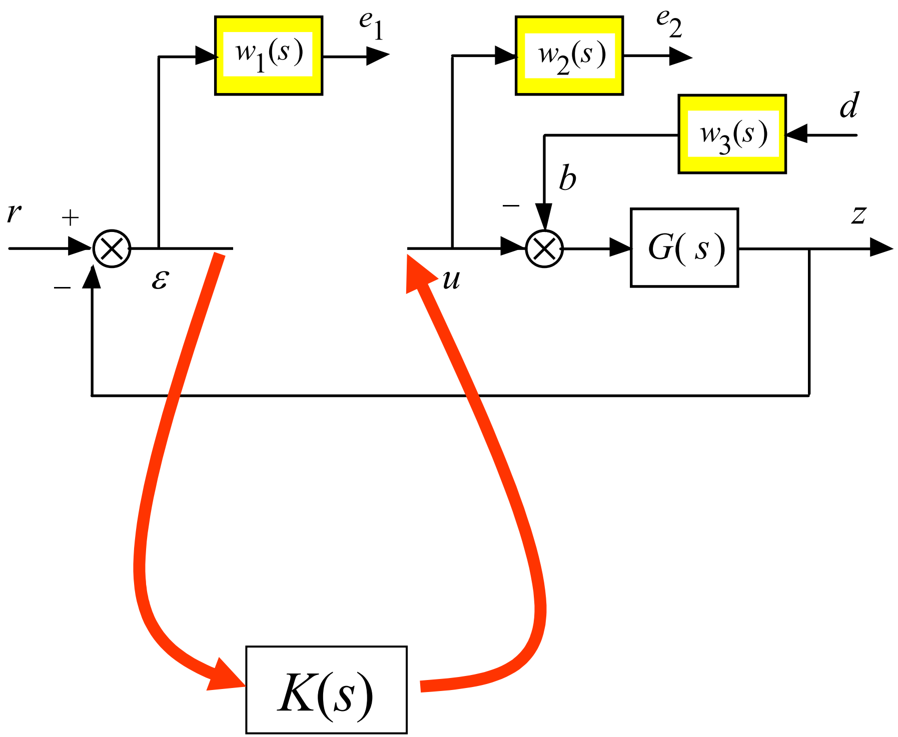


- Standard form
  - Isolate $K(s)$

H_∞ – Typical 4 block problem



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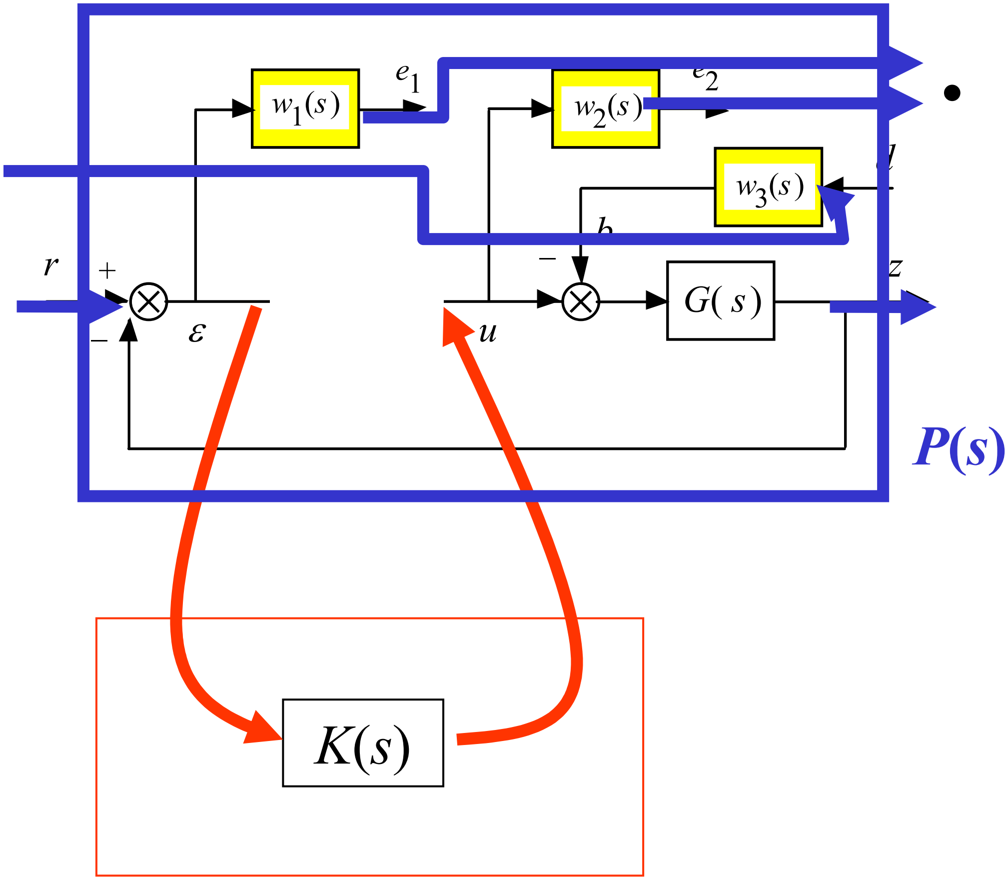
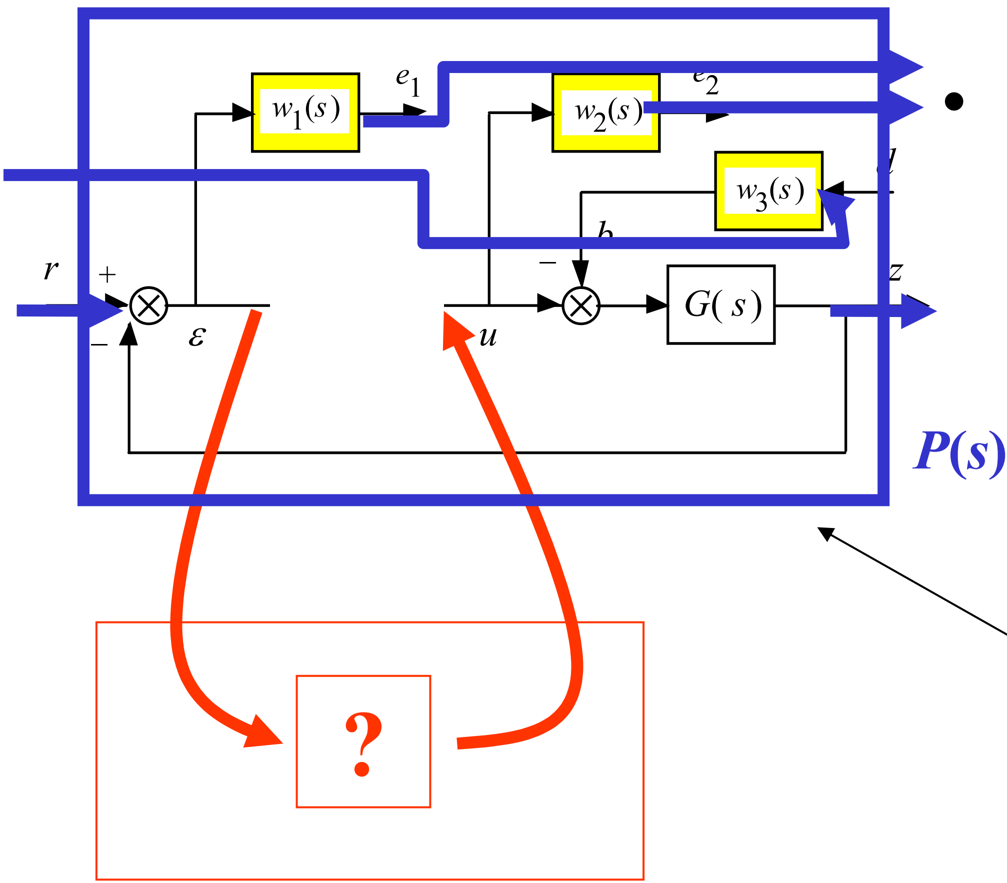


- Standard form
  - Isolate $K(s)$
  - split the graph (or the equations) in two parts

H_∞ – Typical 4 block problem



- Standard form
  - isolate $K(s)$
  - split the graphe (or the equations) in two parts
  - obtain the augmented plan $P(s)$

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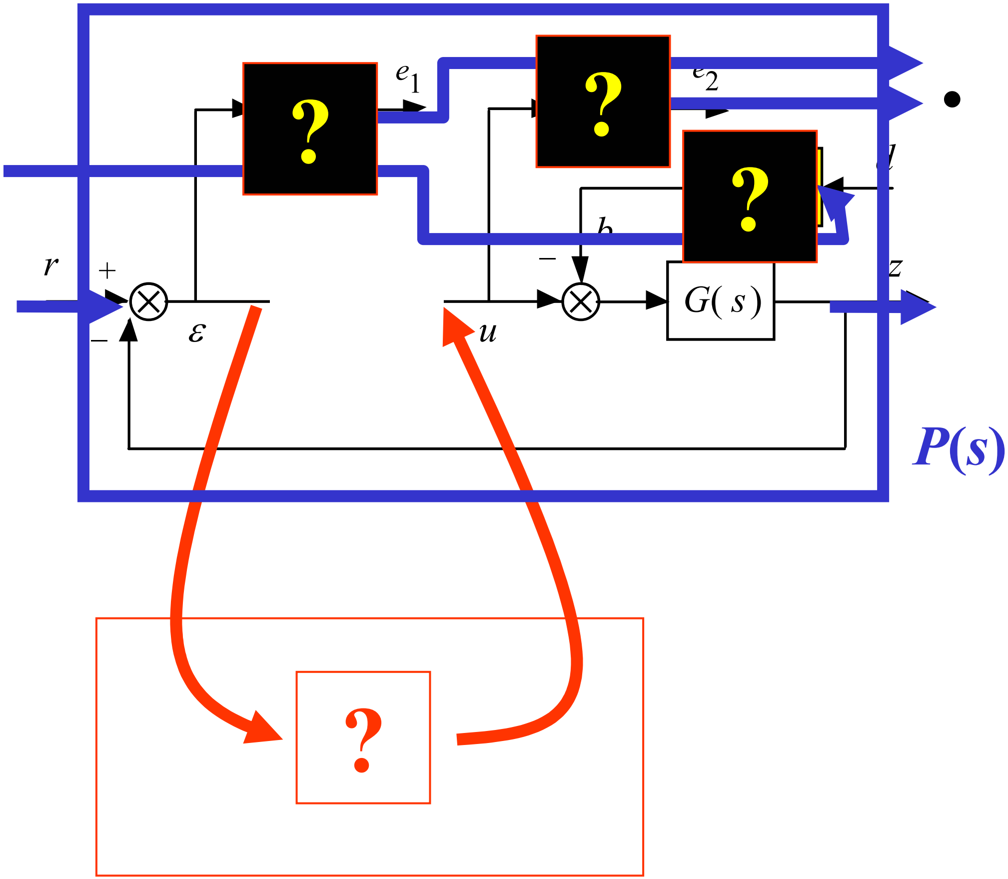
H_∞ – Typical 4 block problem



- Standard form
  - isolate $K(s)$
  - split the graphe (or the equations) in two parts
  - obtain the augmented plan $P(s)$
  - **How to choose the weights W_i**

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H_∞ – Methodology

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Considered problem

What can we do with the **NP engineer questions**?

Rewrite the specs

Convex criterion

Immersion
Multiplier
Relaxation

Convex optimisation

The solution

Global minima

re not mutually exclusive

Supélec

H_∞ – Methodology

How to choose the weighting functions?

Block properties of the induced norm

$$\left\| \begin{pmatrix} L & M \\ N & O \end{pmatrix} \right\|_\infty < \gamma \Rightarrow \begin{cases} \|L\|_\infty < \gamma \\ \|M\|_\infty < \gamma \\ \|N\|_\infty < \gamma \\ \|O\|_\infty < \gamma \\ \left\| \begin{pmatrix} L \\ N \end{pmatrix} \right\|_\infty < \gamma \\ \left\| \begin{pmatrix} M \\ O \end{pmatrix} \right\|_\infty < \gamma \\ \left\| \begin{pmatrix} L & M \end{pmatrix} \right\|_\infty < \gamma \\ \left\| \begin{pmatrix} N & O \end{pmatrix} \right\|_\infty < \gamma \end{cases}$$

H_∞ – Methodology

Application of this property to the 4 block criterion $\left\| \begin{pmatrix} w_1 S & w_1 S\,G\,w_3 \\ w_2 K\,S & w_2 K\,S\,G\,w_3 \end{pmatrix} \right\|_\infty < \gamma$

gives:

$$\|w_1 S\|_\infty < \gamma$$

$$\|w_2 K S\|_\infty < \gamma$$

$$\|w_1 w_3 S G\|_\infty < \gamma$$

$$\|w_2 w_3 K S G\|_\infty < \gamma$$

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H_∞ – Methodology

Application of this property to the 4 block criterion $\left\| \begin{pmatrix} w_1 S & w_1 S\, G\, w_3 \\ w_2 K\, S & w_2 K\, S\, G\, w_3 \end{pmatrix} \right\|_\infty < \gamma$

gives:

H_∞ norm property

$$\|w_1 S\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R}\ |S(j\omega)| < \frac{\gamma}{|w_1(j\omega)|}$$

$$\|w_2 K S\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R}\ |K(j\omega)\, S(j\omega)| < \frac{\gamma}{|w_2(j\omega)|}$$

$$\|w_1 w_3 S G\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R}\ |S(j\omega)\, G(j\omega)| < \frac{\gamma}{|w_1(j\omega)\, w_3(j\omega)|}$$

$$\|w_2 w_3 K S G\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R}\ |K(j\omega)\, S(j\omega)\, G(j\omega)| < \frac{\gamma}{|w_2(j\omega)\, w_3(j\omega)|}$$

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H_∞ – Methodology

Application of this property to the 4 block criterion

$$\left\| \begin{pmatrix} w_1 S & w_1 S\, G\, w_3 \\ w_2 K\, S & w_2 K\, S\, G\, w_3 \end{pmatrix} \right\|_\infty < \gamma$$

gives:

Closed loop function

$$\|w_1 S\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R} \; |S(j\omega)| < \frac{\gamma}{|w_1(j\omega)|}$$

$$\|w_2 K S\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R} \; |K(j\omega)\, S(j\omega)| < \frac{\gamma}{|w_2(j\omega)|}$$

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H_∞ – Methodology

Application of this property to the 4 blocks criterion $\left\| \begin{pmatrix} w_1 S & w_1 S\,G\,w_3 \\ w_2 K\,S & w_2 K\,S\,G\,w_3 \end{pmatrix} \right\|_\infty < \gamma$

gives:

Closed loop function

Frequency constraints

$$\|w_1 S\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R} \ |S(j\omega)| < \frac{\gamma}{|w_1(j\omega)|}$$

$$\|w_2 K S\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R} \ |K(j\omega)\,S(j\omega)| < \frac{\gamma}{|w_2(j\omega)|}$$

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H_∞ – Methodology

Shaping properties

Closed loop function

Frequency constraints

$$\|w_1 S\|_\infty < \gamma \Leftrightarrow \forall \omega \in \mathbf{R}\ |S(j\omega)| < \frac{\gamma}{|w_1(j\omega)|}$$

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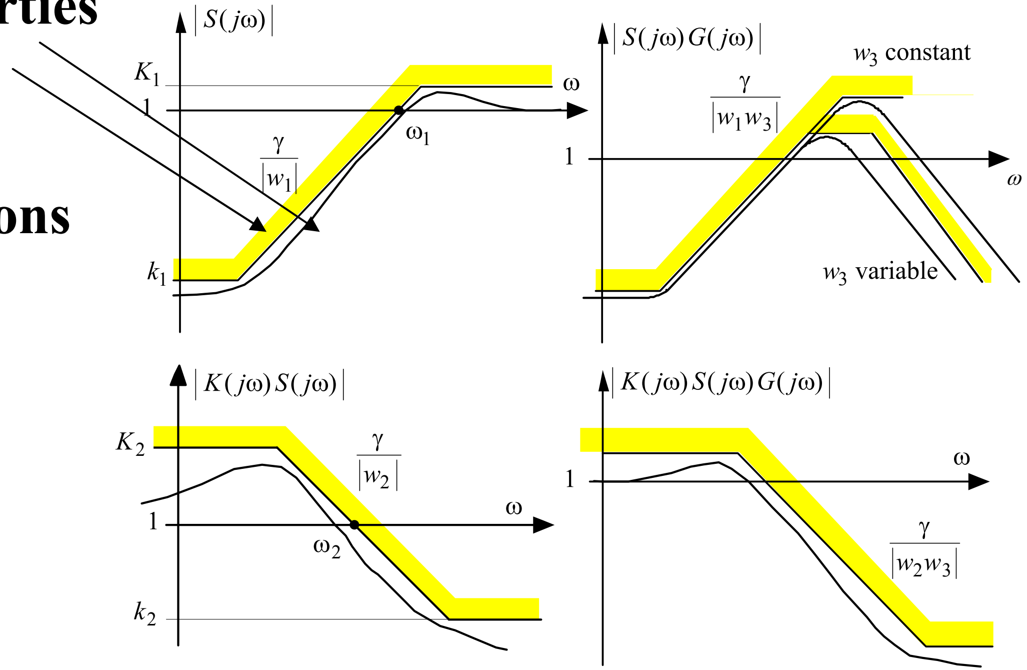
H_∞ – Methodology

Shaping properties

on closed loop

transfer functions



H_∞ – Methodology

- How to choose the weighting functions?
  - Use the shaping property
  - A game theory interpretation can be used too (worst case noise)

  - More generally, properties of the induced norm offer interesting possibilities for methodology and applications.
  - Many properties of induced norm are meaningfull both from physical and mathematical point of view.

H_∞ – Flexibility of standard form

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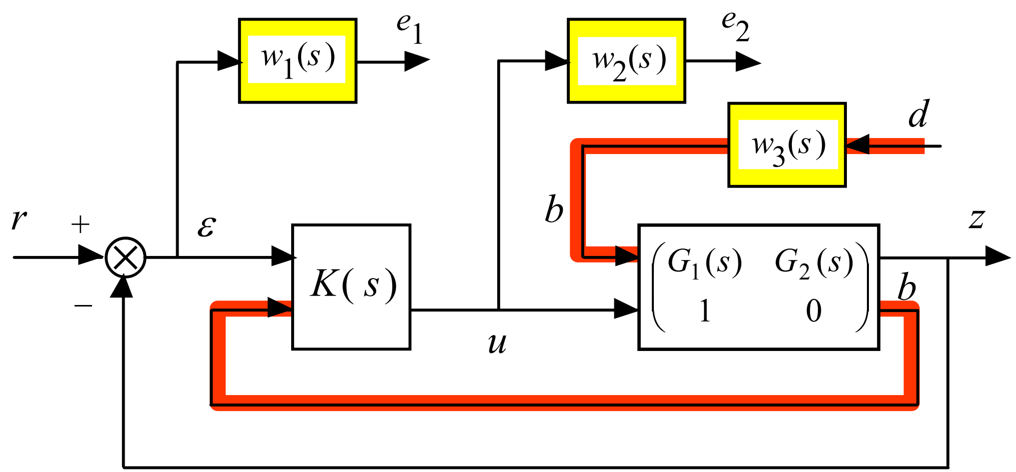
H_∞ – Flexibility of standard form

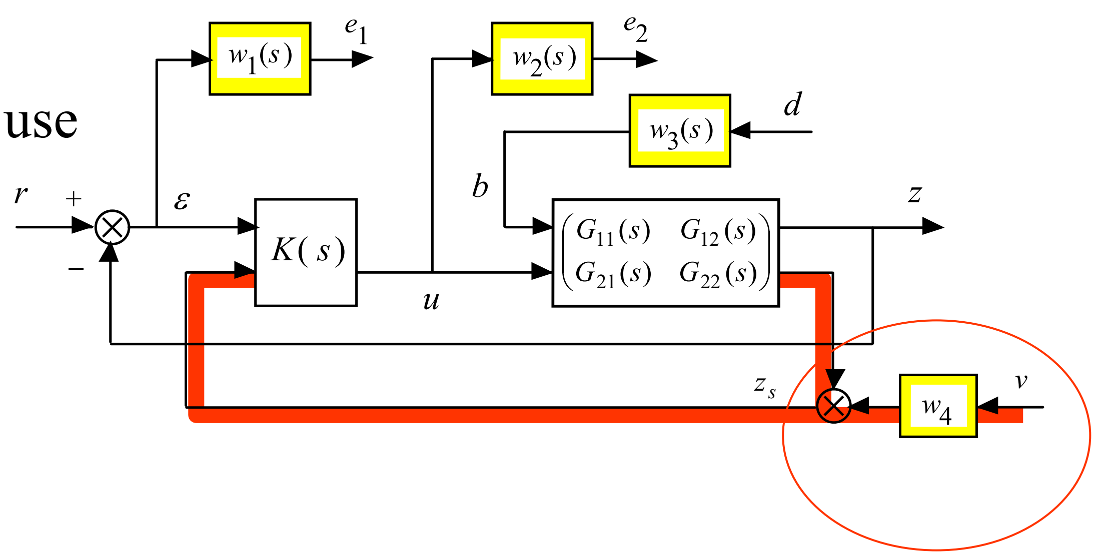
- The standard form of the problem allows to deal with many different problem in a unique framework

- Measurable perturbation rejection



H_∞ – Flexibility of standard form

- The standard form of the problem allows to deal with many different problem in the unique framework
- Extra measurement use



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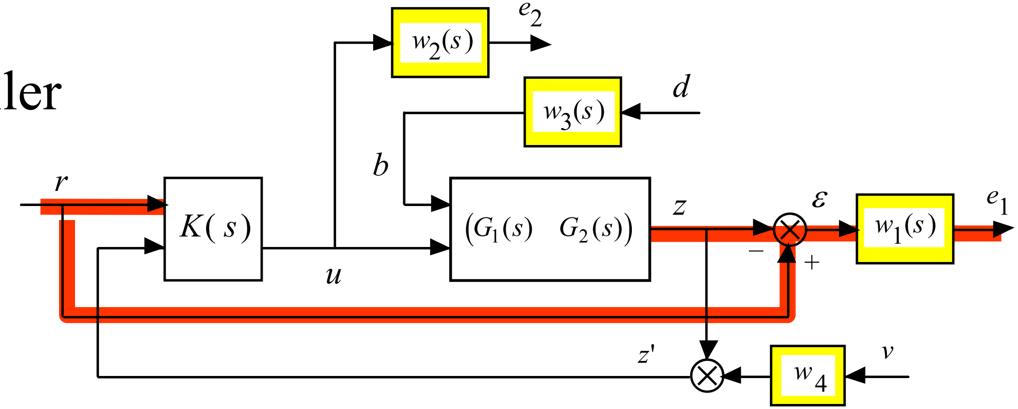
H_∞ – Flexibility of standard form

- The standard form of the problem allows to deal with many different problem in the unique framework

- Two degrees of freedom controller



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H_∞ – Applications

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H_∞ – Applications

Since 1990, the methodology had been largely developed and this approach is now currently used

This method is applied in many domains, to cite a few

- Dam regulation (Supélec-EDF,INRA-Cémagref)
- Electromechanical drive (Supélec-CRAN)
- Irrigation canal control (INRA-Cémagref-LAP)
- Nuclear power plant (EDF > 2 Phd)
- Hydraulic power plant (Supélec-EDF)
- Missile control (Supélec-Aerospatiale > 4 Phd)
- Launch vehicle (Supelec-CNES > 2 Phd)
- Freezing control process (INRA-Cémagref)
- Magnetic Bearing (Supélec-S2M)
- …

and many places (research departments and industries)

H_∞ – Summary

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H_∞ – Summary

- **Standard form** is very powerful (idea of graph separation)
- Properties of the **induced norm** offer many interesting possibilities
- Induced norm properties are closely related to the idea of energy, amplification. Other approach using game theory offer other methodology.
- With an adapted methodology, this method can deal with performance and robustness specs in the same time with an **input/output point of view** (it allows to guarantee multivariable stability margin in the output feedback case!).

μ analysis

μ – Definition

Supélec

μ – Definition

- Def. Structured set of perturbations:

$$\underline{\Delta} := \left\{ \begin{array}{c} \Delta = \operatorname{diag}\left\{\Delta_1, \ldots, \Delta_q, \delta_1 I_{r_1}, \ldots, \delta_r I_{r_r}, \varepsilon_1 I_{c_1}, \ldots, \varepsilon_c I_{c_c}\right\} \in \mathbf{C}^{k\times k} \\ \Delta_i \in \mathbf{C}^{k_i\times k_i} \;\; ; \;\; \delta_i \in \mathbf{R} \;\; ; \;\; \varepsilon_i \in \mathbf{C} \end{array} \right\}$$

- Def. Structured singular value of P for the structure $\underline{\mathbf{\Delta}}$:

$$\mu_{\underline{\Delta}}(P) := \left(\inf_{\Delta \in \underline{\Delta}} \left(\overline{\sigma}(\Delta) : \det\left(I - \Delta P\right) = 0\right) \right)^{-1}$$

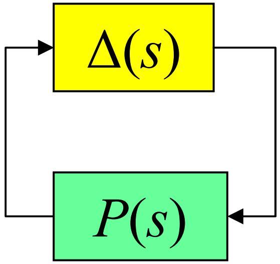
$$\mu_{\underline{\Delta}}(P) := 0 \text{ si } \forall \Delta \in \underline{\Delta} \;\; \det\left(I - \Delta P\right) \neq 0$$

- If $\mu \neq 0$, μ^{-1} is the size of the smallest Δ that makes I-ΔP rank deficient.

μ – Definition – Property

- *Def. Structured operator Δ(s)*

$$\forall \omega \in \Re \quad \Delta(j\omega) \in \underline{\Delta}$$



- *Fundamental property*
  - $P(s)$ *stable*
  - Connexion $P(s)$*$\Delta(s)$ *is internally stable* for all structured $\Delta(s)$ *stable and* $\|\Delta(s)\|_\infty < \alpha$ *if and only if*

$$\forall \omega \quad \mu_{\underline{\Delta}}\left(P(j\omega)\right) \leq \alpha^{-1}$$

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μ – Definition

- J.C. Doyle. "Structured Uncertainty in Control Systems Design". 24th Conf. on Decision and Control, pp. 260-265, Ft Lauderdale, Floride, 1985.
- K. Zhou, J.C. Doyle, K. Glover, Robust and Optimal Control. Prentice-Hall, 1996.

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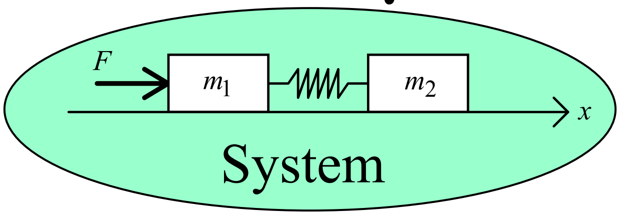
μ – basic example

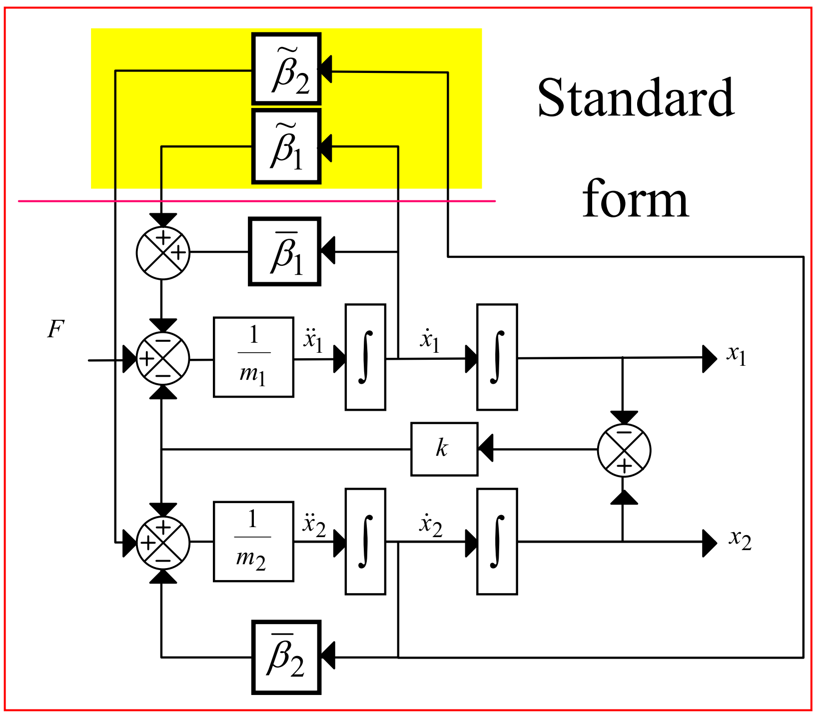
μ – basic example

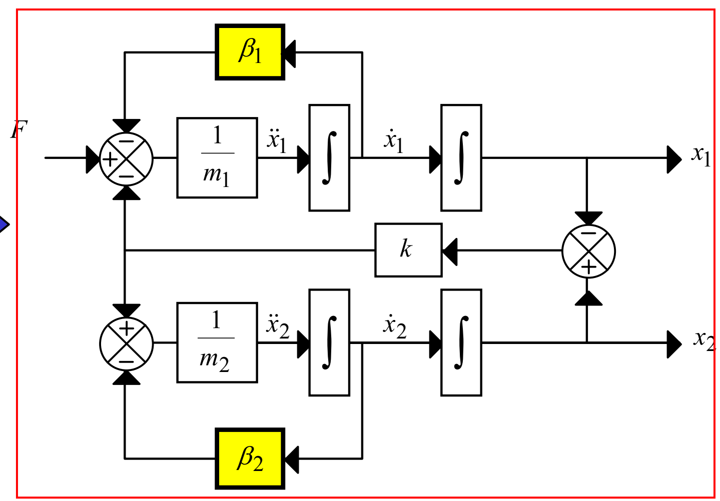


Model

$$\begin{cases} m_1 \ddot{x}_1 = F - k(x_2 - x_1) - \beta_1 \dot{x}_1 \\ m_2 \ddot{x}_2 = +k(x_2 - x_1) - \beta_2 \dot{x}_2 \end{cases}$$

+ uncertainties on β_1 and β_2



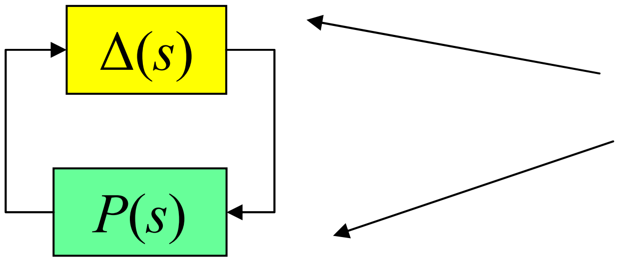


Supélec

μ –uncertainty interpretation

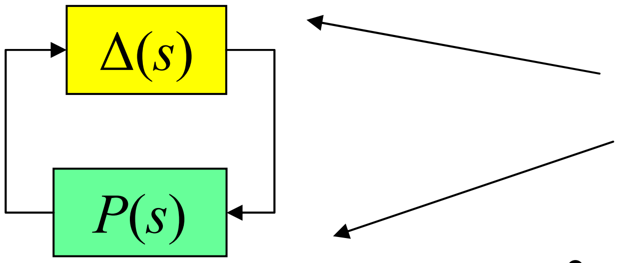
- In general, uncertainties stand for
  - Unknown bounded parameters
  - Imperfect measurements (high frequencies …)
  - Simplification of equations (intentional … or not)
  - ...

μ – standard form



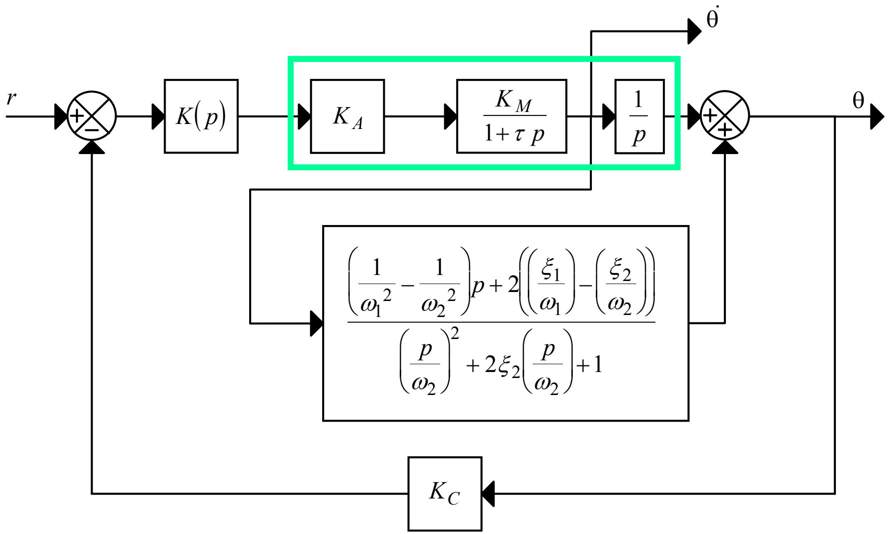
- The standard form allows to separate
  - the uncertainties
  - known part of the model

μ – standard form



- The standard form allows to separate
  - the uncertainties
  - known part of the model
- Is this approach universal?
  - Can we compute the standard form in more complex situations ?
  - Technical and long answer …
  - Let see an example

μ – standard form



- Electrical motor with flexible mode

$K_M \in [10 \times 0.75, 10 \times 1.25]$

$\tau \in [0.015 \times 0.75, 0.015 \times 1.25]$

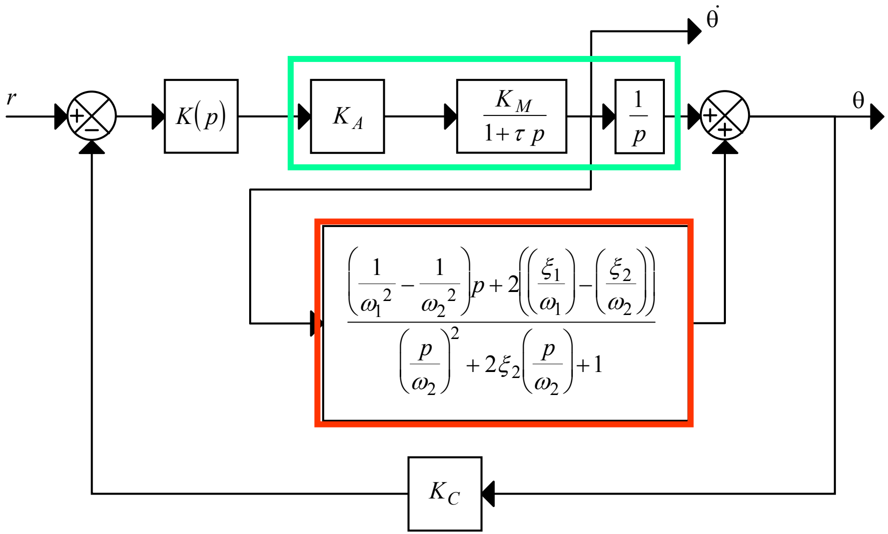
$\xi_1 \in [0.02 \times 0.75, 0.02 \times 1.25]$

$\xi_2 \in [0.01 \times 0.75, 0.01 \times 1.25]$

$\omega_1 \in [200 \times (1-\beta), 200 \times (1+\beta)]$

$\omega_2 \in [300 \times (1-\beta), 300 \times (1+\beta)]$ $\quad \beta = (1-0.88456) \times 0.5$

μ – standard form



- Electrical motor with flexible modes

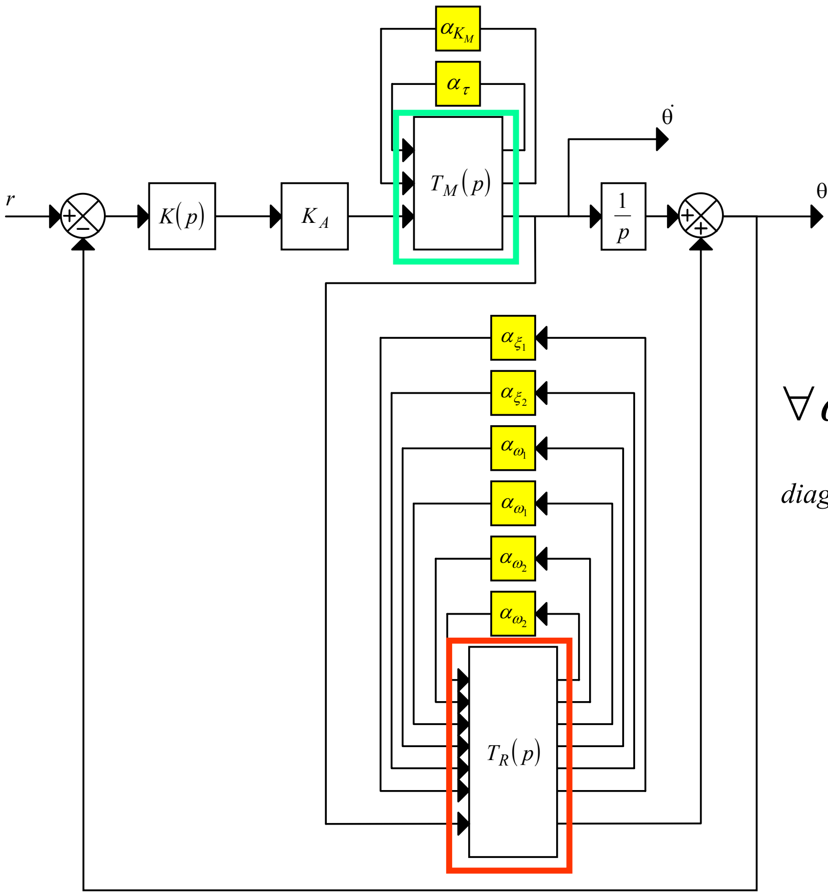
$$K_M \in [10\times 0.75, 10\times 1.25]$$
$$\tau \in [0.015\times 0.75, 0.015\times 1.25]$$
$$\xi_1 \in [0.02\times 0.75, 0.02\times 1.25]$$
$$\xi_2 \in [0.01\times 0.75, 0.01\times 1.25]$$
$$\omega_1 \in [200\times(1-\beta), 200\times(1+\beta)]$$
$$\omega_2 \in [300\times(1-\beta), 300\times(1+\beta)] \quad \beta=(1-0.88456)\times 0.5$$

Flexible-mode block in the diagram:

$$\frac{\left(\frac{1}{\omega_1^2}-\frac{1}{\omega_2^2}\right)p+2\left(\left(\frac{\xi_1}{\omega_1}\right)-\left(\frac{\xi_2}{\omega_2}\right)\right)}{\left(\frac{p}{\omega_2}\right)^2+2\xi_2\left(\frac{p}{\omega_2}\right)+1}$$

Supélec

μ – standard form



- We want to separate unknown parameters to check the fundamental property:

$$\forall\omega \;\; \mu_{diag\left(\mathbf{R},\mathbf{R},\mathbf{R},\mathbf{R},\mathbf{R}I_2,\mathbf{R}I_2\right)}\left(T_\mu(i\omega)\right) < 1$$

$$diag\left(\alpha_{K_M},\alpha_\tau,\alpha_{\xi_1},\alpha_{\xi_2},\alpha_{\omega_1},\alpha_{\omega_1},\alpha_{\omega_2},\alpha_{\omega_2}\right) \in diag\left(\mathbf{R},\mathbf{R},\mathbf{R},\mathbf{R},\mathbf{R}I_2,\mathbf{R}I_2\right)$$

- We have to compute the state space form of T_M and T_R

μ – standard form

- For the central part of the motor model, we obtain the following result (symbolic computation approach)

$$T_M(p) = \left(\begin{array}{c|c} -\dfrac{1}{\tau_0} & \begin{bmatrix} -K_{M1} & -\dfrac{\tau_1}{\tau_0} & -K_{M0} \end{bmatrix} \\ \hline \begin{bmatrix} 0 \\ -\dfrac{1}{\tau_0} \\ -\dfrac{1}{\tau_0} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\dfrac{\tau_1}{\tau_0} & 0 \\ 0 & -\dfrac{\tau_1}{\tau_0} & 0 \end{bmatrix} \end{array}\right)$$

$$\begin{cases} K_{M_0} = 42 \\ K_{M_1} = 42 \times 0.25 \\ \tau_0 = 0.015 \\ \tau_1 = 0.015 \times 0.25 \end{cases}$$

Stephane.font@supelec.fr

Supélec

μ – standard form

- And for the flexible mode we obtain

$$
T_R(p)=\left(\begin{array}{c|c}
\begin{bmatrix}
0 & -2\xi_{1_0}\omega_{2_0}{}^2 & 0 & 0 & -2\xi_{1_0}\omega_{2_0}{}^2 \\
0 & -2\xi_{2_0}\omega_{2_0}{}^2 & 0 & 0 & -2\xi_{2_0}\omega_{2_0}{}^2 \\
\dfrac{2}{\omega_{1_0}{}^2} & \dfrac{-2\omega_{2_0}{}^2}{\omega_{1_0}{}^2} & 0 & 0 & \dfrac{\left(\omega_{1_0}{}^2-\omega_{2_0}{}^2\right)}{\omega_{1_0}{}^2} \\
0 & -2\omega_{2_0}{}^2 & 0 & 0 & -2\omega_{2_0}{}^2 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
&
\begin{bmatrix}
\xi_{1_1} & 0 & 0 & 0 & \xi_{1_0}\omega_{2_1} & \xi_{1_0}\omega_{2_0}\omega_{2_1} & \xi_{1_0}\omega_{2_0}{}^2 \\
0 & \xi_{2_1} & 0 & 0 & 0 & \xi_{2_0}\omega_{2_1} & \xi_{2_0}\omega_{2_0} \\
0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & -\dfrac{\omega_{1_1}}{\omega_{1_0}{}^2} & -\dfrac{\omega_{2_1}}{\omega_{1_0}{}^2} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}{}^2} & \dfrac{-\omega_{1_0}{}^2+\omega_{2_0}{}^2}{\omega_{1_0}{}^2} \\
0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0}\omega_{2_1} & \omega_{2_0}{}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\\ \hline
\begin{bmatrix}
0 & -2\omega_{2_0}{}^2 & 0 & 0 & -\omega_{2_0}{}^2 \\
0 & -2\omega_{2_0} & 0 & 0 & -\omega_{2_0} \\
\dfrac{2}{\omega_{1_0}} & -2\dfrac{\omega_{2_0}{}^2}{\omega_{1_0}{}^2} & 0 & 0 & -\dfrac{\omega_{2_0}{}^2}{\omega_{1_0}{}^2} \\
0 & -2\dfrac{\omega_{2_0}{}^2}{\omega_{1_0}} & 0 & 0 & -\dfrac{\omega_{2_0}{}^2}{\omega_{1_0}}\;0 \\
0 & -2\omega_{2_0} & 0 & 0 & -\omega_{2_0} \\
0 & -2 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
&
\begin{bmatrix}
0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0}\omega_{2_1} & \omega_{2_0}{}^2 \\
0 & 0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0} \\
0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & -\dfrac{\omega_{1_1}}{\omega_{1_0}{}^2} & \dfrac{\omega_{2_1}}{\omega_{1_0}{}^2} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}{}^2} & \dfrac{\omega_{2_0}{}^2}{\omega_{1_0}{}^2} \\
0 & 0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & \dfrac{\omega_{2_1}}{\omega_{1_0}} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}} & \dfrac{\omega_{2_0}{}^2}{\omega_{1_0}} \\
0 & 0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}\right)
$$

$$
\begin{cases}
\xi_{1_0} = 0.02 \\
\xi_{1_1} = 0.02 \times 0.25 \\
\omega_{1_0} = 200 \\
\omega_{1_1} = 200 \times (1 - 0.88456) \times 0.5
\end{cases}
\qquad
\begin{cases}
\xi_{2_0} = 0.01 \\
\xi_{2_1} = 0.01 \times 0.25 \\
\omega_{2_0} = 300 \\
\omega_{2_1} = 300 \times (1 - 0.88456) \times 0.5
\end{cases}
$$

μ – standard form

- And for the flexible mode we obtain
- Don't make it manually!

$$
T_R(p)=\left(\begin{array}{c|c}
\begin{bmatrix}
0 & -2\xi_{1_0}\omega_{2_0}^{\ 2} & 0 & 0 & -2\xi_{1_0}\omega_{2_0}^{\ 2}\\
0 & -2\xi_{2_0}\omega_{2_0}^{\ 2} & 0 & 0 & -2\xi_{2_0}\omega_{2_0}^{\ 2}\\
\dfrac{2}{\omega_{1_0}^{\ 2}} & \dfrac{-2\omega_{2_0}^{\ 2}}{\omega_{1_0}^{\ 2}} & 0 & 0 & \dfrac{\left(\omega_{1_0}^{\ 2}-\omega_{2_0}^{\ 2}\right)}{\omega_{1_0}^{\ 2}}\\
0 & -2\omega_{2_0}^{\ 2} & 0 & 0 & -2\omega_{2_0}^{\ 2}\\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
&
\begin{bmatrix}
\xi_{1_1} & 0 & 0 & 0 & \xi_{1_0}\omega_{2_1} & \xi_{1_0}\omega_{2_0}\omega_{2_1} & \xi_{1_0}\omega_{2_0}^{\ 2}\\
0 & \xi_{2_1} & 0 & 0 & 0 & \xi_{2_0}\omega_{2_1} & \xi_{2_0}\omega_{2_0}\\
0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & -\dfrac{\omega_{1_1}}{\omega_{1_0}^{\ 2}} & -\dfrac{\omega_{2_1}}{\omega_{1_0}^{\ 2}} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}^{\ 2}} & \dfrac{-\omega_{1_0}^{\ 2}+\omega_{2_0}^{\ 2}}{\omega_{1_0}^{\ 2}}\\
0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0}\omega_{2_1} & \omega_{2_0}^{\ 2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\\ \hline
\begin{bmatrix}
0 & -2\omega_{2_0}^{\ 2} & 0 & 0 & -\omega_{2_0}^{\ 2}\\
0 & -2\omega_{2_0} & 0 & 0 & -\omega_{2_0}\\
\dfrac{2}{\omega_{1_0}} & -2\dfrac{\omega_{2_0}^{\ 2}}{\omega_{1_0}^{\ 2}} & 0 & 0 & -\dfrac{\omega_{2_0}^{\ 2}}{\omega_{1_0}^{\ 2}}\\
0 & -2\dfrac{\omega_{2_0}^{\ 2}}{\omega_{1_0}} & 0 & 0 & -\dfrac{\omega_{2_0}^{\ 2}}{\omega_{1_0}} & 0\\
0 & -2\omega_{2_0} & 0 & 0 & -\omega_{2_0}\\
0 & -2 & 0 & 0 & -1\\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
&
\begin{bmatrix}
0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0}\omega_{2_1} & \omega_{2_0}^{\ 2}\\
0 & 0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0}\\
0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & -\dfrac{\omega_{1_1}}{\omega_{1_0}^{\ 2}} & \dfrac{\omega_{2_1}}{\omega_{1_0}^{\ 2}} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}^{\ 2}} & \dfrac{\omega_{2_0}^{\ 2}}{\omega_{1_0}^{\ 2}}\\
0 & 0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & \dfrac{\omega_{2_1}}{\omega_{1_0}} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}} & \dfrac{\omega_{2_0}^{\ 2}}{\omega_{1_0}}\\
0 & 0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0}\\
0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}\right)
$$

$$
\begin{cases}
\xi_{1_0}=0.02\\
\xi_{1_1}=0.02\times 0.25\\
\omega_{1_0}=200\\
\omega_{1_1}=200\times(1-0.88456)\times 0.5
\end{cases}
\qquad
\begin{cases}
\xi_{2_0}=0.01\\
\xi_{2_1}=0.01\times 0.25\\
\omega_{2_0}=300\\
\omega_{2_1}=300\times(1-0.88456)\times 0.5
\end{cases}
$$

Supélec

μ – standard form

- And for the flexible mode we obtain
- Don't make it by hand !
- *A free toolbox is available (numerical approach using Matlab):*
  - *J. F. Magny, CERT, Toulouse.*

$$
T_R(p)=\left(\begin{array}{c|c}
\begin{bmatrix}
0 & -2\xi_{1_0}\omega_{2_0}{}^2 & 0 & 0 & -2\xi_{1_0}\omega_{2_0}{}^2 \\
0 & -2\xi_{2_0}\omega_{2_0}{}^2 & 0 & 0 & -2\xi_{2_0}\omega_{2_0}{}^2 \\
\dfrac{2}{\omega_{1_0}{}^2} & \dfrac{-2\omega_{2_0}{}^2}{\omega_{1_0}{}^2} & 0 & 0 & \dfrac{\left(\omega_{1_0}{}^2-\omega_{2_0}{}^2\right)}{\omega_{1_0}{}^2} \\
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0 & 0 & 0 & 1 & 0
\end{bmatrix}
&
\begin{bmatrix}
\xi_{1_1} & 0 & 0 & 0 & \xi_{1_0}\omega_{2_1} & \xi_{1_0}\omega_{2_0}\omega_{2_1} & \xi_{1_0}\omega_{2_0}{}^2 \\
0 & \xi_{2_1} & 0 & 0 & 0 & \xi_{2_0}\omega_{2_1} & \xi_{2_0}\omega_{2_0} \\
0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & -\dfrac{\omega_{1_1}}{\omega_{1_0}{}^2} & -\dfrac{\omega_{2_1}}{\omega_{1_0}{}^2} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}{}^2} & \dfrac{-\omega_{1_0}{}^2+\omega_{2_0}{}^2}{\omega_{1_0}{}^2} \\
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0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\\ \hline
\begin{bmatrix}
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\dfrac{2}{\omega_{1_0}} & -2\dfrac{\omega_{2_0}{}^2}{\omega_{1_0}{}^2} & 0 & 0 & -\dfrac{\omega_{2_0}{}^2}{\omega_{1_0}{}^2} \\
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0 & -2\omega_{2_0} & 0 & 0 & -\omega_{2_0} \\
0 & -2 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
&
\begin{bmatrix}
0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0}\omega_{2_1} & \omega_{2_0}{}^2 \\
0 & 0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0} \\
0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & -\dfrac{\omega_{1_1}}{\omega_{1_0}{}^2} & \dfrac{\omega_{2_1}}{\omega_{1_0}{}^2} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}{}^2} & \dfrac{\omega_{2_0}{}^2}{\omega_{1_0}{}^2} \\
0 & 0 & 0 & -\dfrac{\omega_{1_1}}{\omega_{1_0}} & \dfrac{\omega_{2_1}}{\omega_{1_0}} & \dfrac{\omega_{2_0}\omega_{2_1}}{\omega_{1_0}} & \dfrac{\omega_{2_0}{}^2}{\omega_{1_0}} \\
0 & 0 & 0 & 0 & 0 & \omega_{2_1} & \omega_{2_0} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
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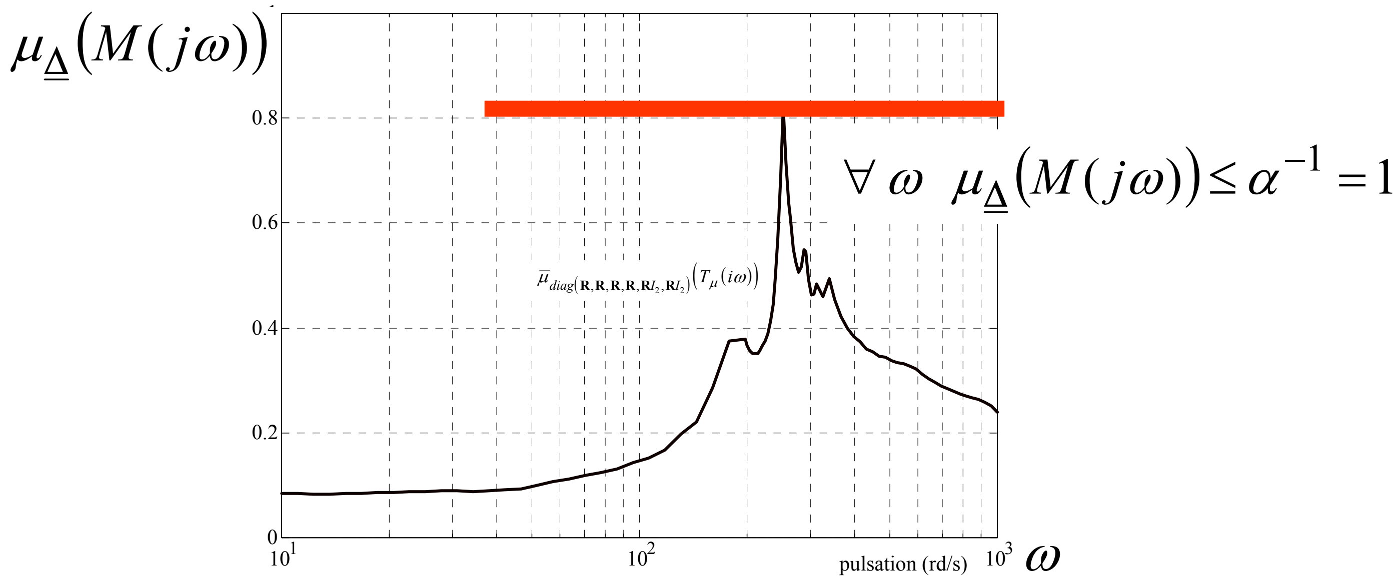
\qquad
\begin{cases}
\xi_{2_0} = 0.01 \\
\xi_{2_1} = 0.01\times 0.25 \\
\omega_{2_0} = 300 \\
\omega_{2_1} = 300\times(1\text{ - }0.88456)\times 0.5
\end{cases}
$$

Supélec

μ – standard form

- So we can check the property :



- Stability is obtained for all models in the defined set

μ – standard form

- The standard form allows to separate
  - deterministic and know part of the model
  - from the uncertainties
- Is this approach universal?
  - **Yes** (in fact *almost* all the time)
  - Using adapted tools, standard form can be easily computed

μ – computation

Supélec

μ – computation

- Direct computation
  - The value of μ can be approximated by gridding method for small dimension of the problem

Supélec

μ – computation

- Direct computation
  - The value of μ can be approximated by gridding method for small dimension of the problem
  - Exact computation is possible for some degenerated situations (for some *very small problem*, upper bound of μ is equal to μ, and so μ can be easily computed).

μ – computation

- Direct computation
  - The value of μ can be approximated by gridding method for small dimension of the problem
  - Exact computation is possible for some degenerated situations (for some small problem, upper bound of μ is equal to μ, and so μ can be easily computed).
  - In general, the exact computation is a non polynomial problem considering the number of uncertainties.

- Braatz, R.D., P.M. Young, J.C. Doyle and M. Morari. "Computational Complexity of mu Calculation, IEEE Trans. *Autom. Control*, AC 39 n° 5, pp. 1000-1002, 1994.

Supélec

μ – computation

- Indirect computation using properties of μ

P1 $\forall\, \underline{\Delta},\ \forall\, \alpha \in \mathbf{C},\ \forall\, P \in \mathbf{C}^{k\times k},\ \mu_{\underline{\Delta}}(\alpha P) = |\alpha|\, \mu_{\underline{\Delta}}(P)$

P2 $\underline{\Delta}_1 = \mathbf{C}^{k\times k} \quad \mu_{\underline{\Delta}_1}(P) = \overline{\sigma}(P)$

P3 $\underline{\Delta}_2 = \{\delta I_k,\ \delta \in \mathbf{R}\} \quad \mu_{\underline{\Delta}_2}(P) = \rho_R(P) := \sup_{\lambda \in \mathbf{R}} \left(|\lambda| : \det(\lambda I_k - P) = 0\right)$

P4 $\underline{\Delta}_3 = \{\delta I_k,\ \delta \in \mathbf{C}\} \quad \mu_{\underline{\Delta}_3}(P) = \rho(P) := \sup_{\lambda \in \mathbf{C}} \left(|\lambda| : \det(\lambda I_k - P) = 0\right)$

P5 $\forall\, P \in \mathbf{C}^{k\times k}\ \underline{\Delta}_a \subset \underline{\Delta}_b\ \rho_R(P) \le \mu_{\underline{\Delta}_a}(P) \le \mu_{\underline{\Delta}_b}(P) \le \overline{\sigma}(P)$

P6 $\forall\, P \in \mathbf{C}^{k\times k}, \forall D \in \mathbf{D},\ \mu_{\underline{\Delta}}(P) \le \overline{\sigma}\left(D P D^{-1}\right)$

P7 $\forall\, P \in \mathbf{C}^{k\times k}, \forall U \in \mathbf{U},\ \rho_R(U P) \le \mu_{\underline{\Delta}}(P)$

P8 $\forall\, P \in \mathbf{C}^{k\times k}, Q \in \mathbf{Q},\ \rho_R(Q P) \le \mu_{\underline{\Delta}}(P)$

Supélec

μ – computation

- Indirect computation using properties of μ

P1 $\forall\, \underline{\Delta},\ \forall\, \alpha \in \mathbf{C},\ \forall\, P \in \mathbf{C}^{k\times k},\ \mu_{\underline{\Delta}}(\alpha P) = |\alpha|\, \mu_{\underline{\Delta}}(P)$

P2 $\underline{\Delta}_1 = \mathbf{C}^{k\times k} \quad \mu_{\underline{\Delta}_1}(P) = \overline{\sigma}(P)$

P3 $\underline{\Delta}_2 = \{\delta I_k,\ \delta \in \mathbf{R}\} \quad \mu_{\underline{\Delta}_2}(P) = \rho_R(P) := \sup_{\lambda \in \mathbf{R}} \left(|\lambda| : \det(\lambda I_k - P) = 0 \right)$

P4 $\underline{\Delta}_3 = \{\delta I_k,\ \delta \in \mathbf{C}\} \quad \mu_{\underline{\Delta}_3}(P) = \rho(P) := \sup_{\lambda \in \mathbf{C}} \left(|\lambda| : \det(\lambda I_k - P) = 0 \right)$

P5 $\forall\, P \in \mathbf{C}^{k\times k} \quad \underline{\Delta}_a \subset \underline{\Delta}_b \quad \rho_R(P) \le \mu_{\underline{\Delta}_a}(P) \le \mu_{\underline{\Delta}_b}(P) \le \overline{\sigma}(P)$

P6 $\forall\, P \in \mathbf{C}^{k\times k},\ \forall D \in \mathbf{D},\ \mu_{\underline{\Delta}}(P) \le \overline{\sigma}\left(D P D^{-1}\right)$

P7 $\forall\, P \in \mathbf{C}^{k\times k}, \forall U \in \mathbf{U},\ \rho_R(U P) \le \mu_{\underline{\Delta}}(P)$

P8 $\forall\, P \in \mathbf{C}^{k\times k}, Q \in \mathbf{Q},\ \rho_R(Q P) \le \mu_{\underline{\Delta}}(P)$

Supélec

μ – computation

- *Upper bound*

$$P6 \Rightarrow \forall\, P \in \mathbf{C}^{k\times k},\ \ \mu_{\underline{\Delta}}(P) \le \min_{D\in\mathbf{D}} \overline{\sigma}\left(D\,P\,D^{-1}\right)$$

convex problem D hermitian, $D \ge 0,\ D\Delta = \Delta D$

- *LMI formulation*

$$\exists\, D \in \mathbf{D}:\ \overline{\sigma}\left(D\,P\,D^{-1}\right) \le \alpha$$

$$\exists\, D \in \mathbf{D}:\ \overline{\lambda}\left(D^{-1}P^{*}D\,D\,P\,D^{-1}\right) \le \alpha^{2}$$

$$\exists\, D \in \mathbf{D}:\ D^{-1}P^{*}D^{2}P\,D^{-1} - \alpha^{2} I \le 0$$

$$\exists\, D \in \mathbf{D}:\ P^{*}D^{2}P - \alpha^{2} D^{2} \le 0$$

$$\exists\, D' \in \mathbf{D}:\ P^{*}D'P - \alpha^{2} D' \le 0$$

$$\alpha^{*} = \min_{D\in\mathbf{D}} \alpha$$

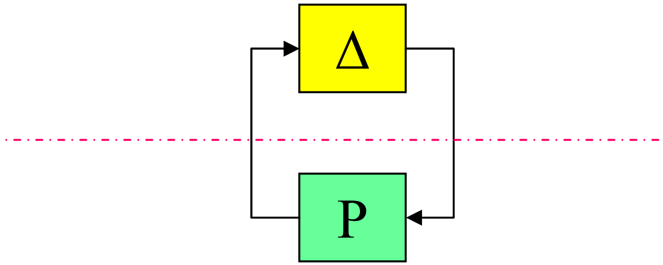
$$\alpha \ge 0$$

$$P^{*}D\,P - \alpha^{2} D \le 0$$

μ – graph separation approach

Supélec

μ – graph separation approach

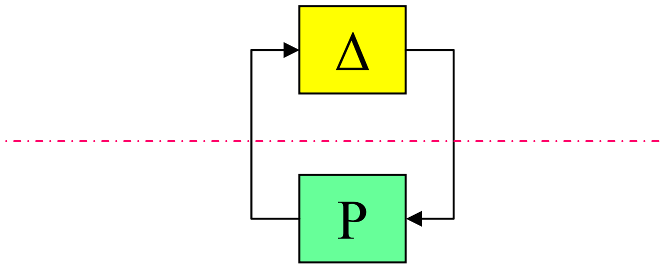


Does the closed loop have a good property *Pcl* ?

⇓

Use properties of each operator (P_a, P_b) that guarantee a global property of the closed loop P_{cl}

μ – graph separation approach



Do the closed loop have a good property *Pcl* ?

$\Downarrow$

Use properties of each operator (P_a, P_b) that guarantee a global property of the close loop P_{cl}

- If P_{cl} is "closed loop stability property"
  - Sufficient condition (small gain theorem):

$$P_a : \|\Delta\| < 1 \text{ et } P_b : \|P\| \leq 1$$

  - Sufficient condition (passivity condition):

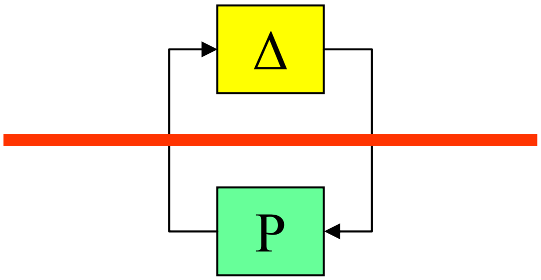
$$\exists \varepsilon, \quad \forall t > 0, \quad \forall x, \|x\| < 1$$

$$P_a : \int_0^t x^T H_1 x > \varepsilon > 0 \text{ et } P_b : \int_0^t x^T H_2 x < -\varepsilon < 0$$

  - others …

μ – graph separation approach

- Application to the structured singular value
  - The property *Pcl* is: rank deficiency of *I*-Δ*P*



$P_a : \overline{\sigma}(\Delta) < \alpha^{-1}$ and $P_b : \overline{\sigma}(P) = \alpha$

$$\Downarrow$$

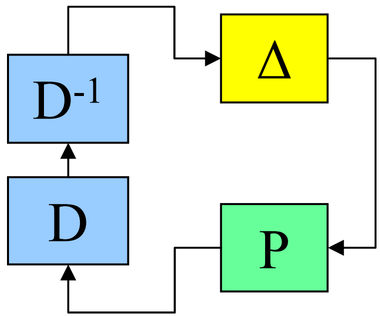
$P_{cl} : I - \Delta P$ not rank deficient

- With the mu formalism : $\mu_{\underline{\Delta}}(P) \leq \alpha = \overline{\sigma}(P)$
- (There is no rank deficiency in the ball whose diameter is **α**)
- *Note: we can obtain this result using properties*

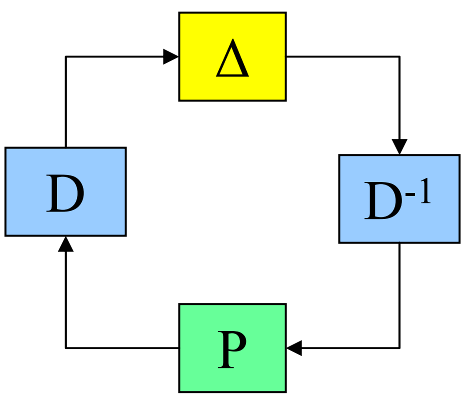
$$\underbrace{\Delta \subset \mathbf{C}^{k\times k} \Rightarrow \mu_{\underline{\Delta}}(P) \leq \underbrace{\mu_{\mathbf{C}^{k\times k}}(P) = \overline{\sigma}(P)}_{P2}}_{P5}$$

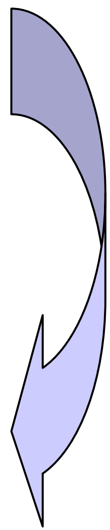
μ – graph separation approach

- We can use a multiplier D



- And reorganized the operators

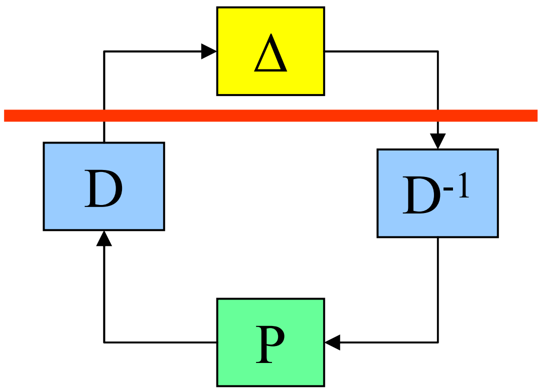




Conservation of the property P_{cl} must be checked

μ – graph separation approach

- Graph separation



$P_a : \bar{\sigma}(\Delta) < \alpha^{-1}$ and $P_b : \bar{\sigma}(DPD^{-1}) = \alpha$

$\Downarrow$

P_{cl}

- With the mu formalism : $\mu_{\underline{\Delta}}(P) \leq \alpha = \bar{\sigma}(DPD^{-1})$
- We get a less conservatism result

Note: we find out the property P6.

μ – graph separation approach

- This graph separation approach can be **generalized to many classes of operators** (nonlinear memoryless functions, time varying systems, nonlinear systems)

- Safonov, M. G., Stability and Robustness of multivariable Feedback Systems, MIT Press, Cambridge, 1980.

Conclusions

Supélec

Conclusion

- What is a good method in automatic control ?

- For:
  - Identification
  - Synthesis
  - analysis

- Looking for methods who are:
  - general
  - easy to use
  - CAD (computer-aided design)

Supélec

Conclusion

- Since 1980, robustness theory had been developed in this state of mind (impressive development after 1992)
  - *General* criterion formulation
  - Specs are *formalised*
  - Parametric uncertainties are *considered in the design*
  - Neglected dynamics are *considered in the design*
  - Formulation as an optimisation problem is very general and allows a *CAD approach*

Conclusions

μ analysis and H_∞ share some fundamental concepts and objects :

- **standard form**
- **input/ouput approach**
- **induced norms and associated properties**
- **multiplier approach**
- **possibility to reach tractable problems**

Those concept are important and gather many idea of post modern robust control

Our aim is to enlarge the field of robust methods to the nonlinear systems, while preserving:

- **global properties for stability**
- **global performance specs**
- **direct consideration of robust demands (margin…)**