SOME RESULTS ON THE BEHAVIOR OF LIPSCHITZ CONTINUOUS SYSTEM*

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Abstract

This paper is devoted to the study of the behavior of Lipschitz continuous systems. We unify and improve through the reformulation of the Lipschitz continuity of a system in terms of the dissipativity of an associated fictitious system, our previous results concerning the Lyapunov stability of unperturbed motions associated with any inputs. Moreover, we show that the Lipschitz continuous systems have the steady-state property with respect to any inputs belonging to $\mathcal{L}_p^e$ with $p \in [1, \infty[$ (i.e., their asymptotic behavior is uniquely determined by the asymptotic behavior of the input). Finally, we characterize the behavior of stationary Lipschitz systems for periodic and almost periodic inputs. We end the paper by a example showing that these results allow us to explain the well-known properties attached to the behavior of a controlled missile.

1 Introduction

The general theorems, like the small gain theorem or the passive theorem, allow us to prove input-output stability of interconnected systems, and are, in many practical cases, usefull for proving the input-output stability of systems having nonlinearities ([12],[17][13], [3]). Unfortunately, the characterization of the stability in terms of input-output properties may not be sufficient with respect to the classical requirements attached to many applications e.g. airplane, missile,..., ship. The most significant example in this context is probably the response of the system with respect to step inputs: an application of the small gain (or passive) theorem in this context does not guarantee the existence of a constant steady-state for constant inputs[1]. This problem is particularly relevant in the context of linear systems perturbed by small non linear perturbations where, implicitly, we wish to recover some properties attached to stable linear systems such their behaviors with respect to specific inputs and/or with respect to modifications of the initial condition of the system. This remark points out the gap existing between the linear systems and the nonlinear systems which have a finite gain.

We have already pointed out these problems and propose to use the incremental norm (Lipschitz constant) in this context instead of the usual norm (see [4],[7] [8] and [9]). Indeed, in [8], we have shown that Lipschitz continuity of the system ensures asymptotic stability of the equilibria which is associated with any constant control. This first result is extended in [9], where we prove, considering nonlinear dynamical systems generated by differential equations, that the Lipschitz continuity of the system ensures the asymptotic stability of its unperturbed motions i.e., under perturbations of the initial state and for any prefixed input not necessarily constant, the state evolution asymptotically converges to the same motion.

All these results show that the use of the incremental norm (Lipschitz constant) and the use of incremental small gain or incremental passif theorem seems in some applications a better way to recover in the nonlinear context some properties of input-output stable linear systems.

In this paper, under the use of dissipativity concepts (see [15]), we firstly generalize and unify our previous results and obtain new results allowing to characterize the behavior of incrementally bounded systems in a more accurate way. Indeed, we prove that all Lipschitz systems, under an assumption concerning the observability of their state-space realization, have the steady-state property i.e., the asymptotic behavior of the system is uniquely determined by the asymptotic behavior of the input. Moreover, con-

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[1] A simple example is given by $\dot{x} = -f(x)$ where $f(x)$ is a C^1 function defined from $\mathcal{R}$ into $\mathcal{R}$, such that $< f(x)|x >> \epsilon\|x\|$ for all $x \in \mathcal{R}$ and such that there exists an interval where $\frac{\partial f}{\partial x} > 0$ for $x \in [a, b]$.

sidering time invariant nonlinear systems, we characterize their behavior with respect to periodic and almost periodic inputs: the motion associated to an almost periodic input (periodic) is asymptotically almost periodic (periodic) function. This result allows, for example, to prove that each constant value of the input has an associated equilibrium point as in the linear context.
All this results are obtained on the basis of the links existing between the incremental boundedness of a system and the dissipativity of an associated fictitious system. Finally, it seems important for us to explain in this context, the role of each assumption leading to the proof of these differents results. To this purpose, we have separated, on the one hand, the properties due to the incremental boundedness of the system and on the other hand, the role of the different assumptions concerning the state-space realization of the system on the announced results. Due to the space limitation, only the proof concerning the steady-state property of Lipschitz systems is given. The complete proofs of the present results are given in [5].
This paper is organized as follows. In section 2, we introduce the notations and recall some definitions. Section 3 is devoted to restating incremental boundedness in terms of the dissipativity of a fictitious system and to establish the properties associated with the available storage function of this fictitious system. The results concerning the Lyapunov stability of the unperturbed motion are set in Section 4. Section 5 contains the main result of this paper concerning the steady-state property of incrementally stable systems. In Section 6, we characterize the behavior of incrementally bounded systems for periodic and almost periodic inputs. Finally, in Section 7, a simple example, extracted from [6], concerning the incremental stability of a controlled missile, is presented.

2 Notations and definitions

The notations and terminology, which are recalled hereafter, are classical in the input-output context (see [13]). In the following, we use the $\mathcal{L}_p$ spaces i.e the space of $\mathcal{R}^n$ valued functions defined on $\mathcal{R}$, for which the *pth* power of the norm is integrable when $1 \leq p < \infty$, with the norm defined by $\|f\|_p = \sqrt[p]{\int \|f(t)\|^p dt}$. The *causal truncation* at T of a function f, defined on $\mathcal{R}$, is denoted $P_T f$ and is defined by $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise. The *anticausal truncation* of f at T is defined as $Q_T f = f - P_T f$. The *extended space* associed to $\mathcal{L}_p$ is denoted by $\mathcal{L}_p^e$ and corresponds to space of $\mathcal{R}^n$ valued functions defined on $\mathcal{R}$ whose causal truncations belong to $\mathcal{L}_p$.
A dynamical system, Σ, is defined as a causal operator from $\mathcal{L}_p^e$ into $\mathcal{L}_p^e$. Σ is *stationary* if for all $u_1, u_2 \in \mathcal{L}_p^e$ related by $u_1(t) = u_2(t+T)$ the ouputs $y_1 = \Sigma(u_1)$ and $y_2 = \Sigma(u_2)$ are similary related i.e. $y_1(t) = y_2(t+T)$. Σ is said to be a *finite gain stable* system if there exists $\gamma > 0$ such that $\|y\|_p \leq \gamma \|u\|_p$ for all $u \in \mathcal{L}_p$. Σ has a *finite incremental gain* if there exists $\eta > 0$ such that $\|y_1 - y_2\|_p \leq \eta \|u_1 - u_2\|_p$ for all $u_1, u_2 \in \mathcal{L}_p$. Σ is said to be *incrementally stable* if it is stable, i.e., it maps $\mathcal{L}_p$ to $\mathcal{L}_p$, and has a finite incremental gain.
The *state space representation* of $y = \Sigma(u)$, is defined through the introduction of an abstract set, X, called the *state-space* and the definition of a *state transition function*, ϕ, defined from[2] $\mathcal{R}_2^+ \times X \times \mathcal{L}_p^e$ into X and a *read-out function*, r, defined from $X \times \mathcal{R}^m$ to $\mathcal{R}^q$. The state and the output of Σ at time t are respectively equal to $x(t) = \phi(t, t_0, x_0, u)$ and $y(t) = r(t, \phi(t, t_0, x_0, u), u(t))$, where, by consistency of the state, the *initial state* is such that $x_0 = \phi(t_0, t_0, x_0, u)$. Noting that the state-space representation of dynamical systems always exists (Theorem 1 in [14]). X is assumed to be a normed and complete space. With reference to the state-space representation of Σ, we introduce this notation $\Sigma = (\mathcal{R}^m, \mathcal{L}_p^e, \mathcal{R}^q, \mathcal{L}_p^e, X, r, \phi)$.
In the sequel, $x_r(t)$ denotes an *unperturbed motion* of Σ associated with the input, $u_r \in \mathcal{L}_p^e$, and with a specific initial condition, $x_{0r} \in X$ i.e. $x_r(t) = \phi(t, t_0, x_{0r}, u_r)$.
We finally introduce some definitions characterizing the links between the input and the output of the system and its state. Same kind of definition already appears in previous works (see especially [14] and also [11]). Let us define

$$\Delta y_{(t_i, x_1, x_2, u)}(t) = r(t, \phi(t, t_i, x_1, u), u(t)) \\ - r(t, \phi(t, t_0, x_2, u), u(t))$$

Definition 2.1 *A state of Σ, x_o, is said to be observable if for all $x \in X$ and all $(t_i, t) \in \mathcal{R}_2^+$, there exists an input $u_o \in \mathcal{L}_p^e$ such that if $\Delta y_{(t_i, x_o, x, u_o)} = 0$ a.e. then $x = x_o$. Σ is said to be irreductible if all states belonging to X are observable. The state-space of Σ is said to be uniformly irreductible if it is irreductible and there exist β_i, a function of class $\mathcal{K}_\infty$, and a constant $T_i > 0$ such that $\|P_{t+T_i} Q_t \Delta y_{(t, x_1, x_2, u)}\|_p^p \geq \beta_i(\|x_1 - x_2\|)$ for all $x_1, x_2 \in X$, $t \geq t_0$ and some $u_o \in \mathcal{L}_p^e$.*

Definition 2.2 *The unperturbed motion, $x_r(t)$, is said to be uniformly observable if there exist β, a function of class $\mathcal{K}$, and a constant $T_o > 0$ such that $\|P_{t+T_0} Q_t \Delta y_{(t, x_r(t), x, u_r)}\|_p^p \geq \beta(\|x_r(t) - x\|)$ for all $x \in X$ and $t \geq t_0$.*

Definition 2.3 *The state space of Σ is said to be reachable from x_0 if given any $x \in X$ and $t \in \mathcal{R}$, there exist $u \in \mathcal{L}_p$ and $T_r \geq 0$ such that $x = \phi(t, t - T_r, x_0, u)$*
The state space is said to be uniformly and isotropically reachable from x_0 if there exist α_r, a function of class $\mathcal{K}$, and $T_r > 0$ such $\|P_t Q_{t-T_r}(u_1 - u_2)\|_p^p < \alpha_r(\|x_1 - x_2\|)$ for all $x_1, x_2 \in X$ and $t \geq t_0$ where $u_1, u_2 \in \mathcal{L}_p$ and $x_i = \phi(t, t - T_r, x_0, u_i)$ with $i \in \{1, 2\}$.

3 Dissipativity approach for incremental systems

In the following, we introduce a fictitious system which allows to restate incremental boundedness of the initial

[2] $\mathcal{R}_2^+ \triangleq \{(t_1, t_0) \in \mathcal{R}^2 | t_1 \geq t_0\}$

system in terms of the dissipativity of this fictitious system with respect a specific supply rate function. Before setting specific results concerning the incremental boundedness of Σ, we recall some definitions and notations which have been introduced by Willems in [15].
Let us associated to a system, $y = \Sigma(u)$, defined from $\mathcal{L}_p^e$ into $\mathcal{L}_p^e$, a real value function, $w(t) = w(u(t), y(t))$, defined on $\mathcal{L}_p^e \times \mathcal{L}_p^e$ which is called the supply rate. This function is assumed to be integrable on a finite support.

Definition 3.1 [15] *The available storage, S_a, of a dynamical system Σ with supply rate $w(t)$ is the function from $X \times \mathcal{R}$ into $\mathcal{R}^e$ defined by*

$$S_a(x, t_0) = \sup_{x\rightarrow} - \int_{t_0}^{T} w(\tau) d\tau \quad (1)$$

where the notation $\sup_{x\rightarrow}$ denotes the supremum over all motions starting in state x at time t_0 and where the supremum is take over all $u \in \mathcal{L}_p^e$.

Let us associated with the dynamical system, $\Sigma = (\mathcal{R}^m, \mathcal{L}_p^e, \mathcal{R}^q, \mathcal{L}_p^e, X, r, \phi)$, a fictitious dynamical system, $\Sigma_f = (\mathcal{R}^m \times \mathcal{R}^m, \mathcal{L}_p^e \times \mathcal{L}_p^e, \mathcal{R}^q, \mathcal{L}_p^e, X \times X, r_f, \phi_f)$, defines by $y_f = \Sigma_f(u_1, u_2) = \Sigma(u_1) - \Sigma(u_2)$ and its associated supply rate function defines by $w_f(t) = \eta^p \|u_1(t) - u_2(t)\|^p - \|y_f(t)\|^p$.

Lemma 3.1 *The dynamical system, Σ, has a incremental gain less or equal η if and only if $S_a(x_0, x_0, t_0) = 0$ where S_a is the available storage function of Σ_f, defined from $X \times X \times \mathcal{R}$ into $\mathcal{R}^e$, with supply rate $w_f(t)$.*

We give in the sequel two quite elementary Lemmas which are in fact the basis of all results linking input-output stability and Lyapunov stability.

Lemma 3.2 *Let Σ a system with a finite incremental gain less or equal η. For any input $u_1, u_2 \in \mathcal{L}_p^e$ and any time $t_2 \geq t_1 \geq t_0$, one has*

$$(i)\ \ S_a(x_1(t_1), x_2(t_1), t_1) \leq \int_{t_0}^{t_1} \eta^p \|u_1(\tau) - u_2(\tau)\|^p d\tau$$

$$(ii)\ \ S_a(x_1(t_1), x_1(t_1), t_1) = 0$$

$$(iii)\ \ S_a(x_1(t_1), x_2(t_1), t_1) + \int_{t_1}^{t_2} w_f(\tau) d\tau \geq S_a(x_1(t_2), x_2(t_2), t_2)$$

where $x_i(t) = \phi(t, t_0, x_0, u_i)$ with $i \in \{1, 2\}$.

Lemma 3.3 *Under the assumption of Lemma 3.2 and for any $x_1, x_2 \in X$, $u \in \mathcal{L}_p^e$ and any time $t_2 \geq t_1 \geq t_0$ one has*

$$S_a(x_1, x_2, t_1) \geq \int_{t_1}^{t_2} \|y_1(\tau) - y_2(\tau)\|^p d\tau$$

where $y_i(t) = r(t, \phi(t, t_1, x_i, u), u(t))$ with $i \in \{1, 2\}$.

4 Internal stability

We can prove the following theorem:

Theorem 4.1 *Let $\Sigma = (\mathcal{R}^m, \mathcal{L}_p^e, \mathcal{R}^q, \mathcal{L}_p^e, X, r, \phi)$ a dynamical system with a finite incremental gain.*

(i) If X is uniformly irreductible and uniformly isotropically reachable from x_0 then all unperturbed motions are uniformly stable.

(ii) If X is reachable from x_0 and the unperturbed motion is uniformly observable then the unperturbed motion is uniformly globally attractive.

(iii) If X is uniformly isotropically reachable from x_0 and the unperturbed motion is uniformly observable then the unperturbed motion is uniformly globally assymptotically stable.

This result can be prove under the definition of a function, V, from $\mathcal{R} \times X$ into $\mathcal{R}$ and related to S_a in this way:

$$V(t, x) = S_a(x_r(t), x_r(t) + x, t) \quad (2)$$

where $x_r(t)$ is the unperturbed motion and $x(t)$ the difference between the unperturbed motion and the perturbed one. Now, on the basis of Lemmas 3.2 and 3.3 and definitions concerning the property of the state-space realization of Σ, V has the following properties:

Lemma 4.2 *Let $\Sigma = (\mathcal{R}^m, \mathcal{L}_p^e, \mathcal{R}^q, \mathcal{L}_p^e, X, r, \phi)$ a dynamical system with a finite incremental gain and V the function defines by (2).*
(i) If X is uniformly irreductible then

$$V(t, x) \geq \beta_i(\|x\|) \quad \text{for all } x \in X$$

(ii) If X is uniformly isotropically reachable from x_0 then

$$V(t, x) \leq \eta^p \alpha_r(\|x\|) \quad \text{for all } x \in X$$

(iii) If the unperturbed motion is uniformly observable then for any $t_1 \geq t_0$ and $T \geq T_o$ one has:

$$V(t_1 + T, x(t_1 + T)) - V(t_1, x(t_1)) \leq -\beta(\|x(t_1)\|)$$

where $x(t) = \phi(t, t_0, x_0, u_1) - \phi(t, t_0, x_0, u_2)$, $u_1(t) = u_r(t)$ for all $t \geq t_0$ and $u_2 \in \mathcal{L}_p^e$ and $u_2(t) = u_r(t)$ for all $t \geq t_1$.

Remarks:

(*i*) It is important to note that V is not a Lyapunov function for the system, since it is not necessary a continuous function of x. Despiste this point, the Lyapunov stability can be proved using the upper bound provided by point (*ii*) of Lemma 4.2.

(*ii*) It is clear that other results can be inferred, when modifying some of the assumptions. The assumption concerning the reachability of the state space can be removed if V is assumed to be a continuous function of x for fixed t (see [10] [1]). It seems difficult in our context to remove the assumption concerning a uniformly irreductible

X, Indeed, even when assuming $V(t,.)$ continuous and X irreductible, (so as to guarantee that $V(t,x) \geq 0$ for all $x \neq 0$), this does not ensure that V is positive definite. Finally, using the notion introduced in [11], it is possible to consider the asymptotic stability with respect to a tube around the unperturbed motion.

5 Steady-state property of incrementally stable systems

We set the main theorem of this article.

Theorem 5.1 *Let* $\Sigma = (\mathcal{R}^m, \mathcal{L}_p^e, \mathcal{R}^q, \mathcal{L}_p^e, X, r, \phi)$ *a dynamical system with a finite incremental gain. If the unperturbed motion associated with* u_r *is uniformly observable then for any* $\widetilde{u}_r \in \mathcal{L}_p^e$ *such* $u_r - \widetilde{u}_r$ *belongs to* $\mathcal{L}_p$*, one has*

$$\lim_{t\to\infty} \|\phi(t,t_0,x_0,u_r) - \phi(t,t_0,x_0,\widetilde{u}_r)\| = 0$$

Proof: Let us define a function V, from $\mathcal{R} \times X$ into $\mathcal{R}$ by $V(t,x) = S_a(x_r(t), x_r(t)+x, t)$ where $x_r(t) = \phi(t,t_0,x_0,u_r)$ and $x(t) = \phi(t,t_0,x_0,u_r) - \phi(t,t_0,x_0,\widetilde{u}_r)$. On this basis, the theorem is proved if $\lim_{t\to\infty}\|x(t)\| = 0$.
To this purpose, using (iii) of Lemma 3.2, one has

$$V(t_1+T, x(t_1+T)) - V(t_1, x(t_1)) \leq$$
$$\int_{t_1}^{t_1+T} (-\|y_f(\tau)\|^p + \eta^p \|u_r(\tau) - \widetilde{u}_r(\tau)\|^p) d\tau$$

with $y_f(t) = r(t, \phi(t,t_1,\widetilde{x}_t,\widetilde{u}_r), \widetilde{u}_r(t)) - r(t, \phi(t,t_1,x_t, u_r), u_r(t))$ where $x_t = x_r(t_1)$ and $\widetilde{x}_t = x_r(t_1) + x(t_1)$.
Let us rewrite y_f as the difference of two terms y_{f_1} and y_{f_2} respectively define by[3]

$$y_{f_1} = r(t, \phi(t,t_1,\widetilde{x}_t,\widetilde{u}_r), \widetilde{u}_r) - r(t, \phi(t,t_1,\widetilde{x}_t,u_r), u_r)$$
$$y_{f_2} = r(t, \phi(t,t_1,x_t,u_r), u_r) - r(t, \phi(t,t_1,\widetilde{x}_t,u_r), u_r).$$

y_{f_1} clearly corresponds to the output of the fictitious system, Σ_f, which has been submitted to the input $u_1 = u_2 = \widetilde{u}_r$ from $t = t_0$ to $t = t_1$ and which is now submitted to the inputs u_r and $\widetilde{u}_r$. Thus, on the basis of (ii) and (iii) of Lemma 3.2, and the fact that $S_a \geq 0$ (by uniform observability), one has

$$\int_{t_1}^{t_1+T} \|y_{f_1}(\tau)\|^p d\tau \leq \int_{t_1}^{t_1+T} \eta^p \|u_r(\tau) - \widetilde{u}_r(\tau)\|^p d\tau$$

The second term, i.e. y_{f_2}, corresponds to the output of the fictitious system which is submitted to the same input (under the initial condition $x_f(t_1) = (x_t, \widetilde{x}_t)^T$). Then using the uniform observability, one has

$$\int_{t_1}^{t_1+T} \|y_{f_2}(\tau)\|^p d\tau \geq \beta(\|x_t - \widetilde{x}_t\|)$$

for all $T \geq T_o$ where T_o is the time constant associated to the uniform observability property. Through a simple manipulation on the norm,[4]we deduce the following inequality
$V(t_1+T, x(t_1+T)) - V(t_1, x(t_1))$

$$\leq -\beta(\|x(t_1)\|) + 2\int_{t_1}^{t_1+T} \eta^p \|u_r(\tau) - \widetilde{u}_r(\tau)\|^p d\tau \tag{3}$$

Let us now assume that $x(t)$ does not go to zero, then for all $T_{\epsilon_1} \geq t_0$, there exists $\epsilon > 0$ and a sequence of times t_i such that $\|x(t_i)\| > \epsilon$ where the sequence of time is such that $t_0 \geq T_{\epsilon_1}$ and $\lim_{i\to\infty} t_i = \infty$.
Let us define then $\lambda = \frac{1}{2}\inf_{\|x\|=\epsilon}\beta(\|x\|)$. The difference between the two inputs belongs to $\mathcal{L}_p$, and this guarantees the existence of a time T_{ϵ_2}, such that
$\int_t^\infty \eta^p \|u_r(\tau) - \widetilde{u}_r(\tau)\|^p d\tau < \lambda$ for $t \geq T_{\epsilon_2}$. Then, for $t \geq \max(T_{\epsilon_1}, T_{\epsilon_2})$ and by inequality (3), the sequence defined by $K_i = V(t_i, x(t_i))$ is strictly decreasing. V is however lower bounded and it has finite values along all the possible motions of Σ_f (see point (i) of Lemma 3.2). The contradiction is consequently proven. □

An immediate consequence of Theorem 5.1 is the following corollary:

Corollary 5.2 *Under the assumptions of Theorem 5.1 and the reachability from* x_0 *of the state-space of* Σ*, one has for any* $x_{0r}, x_{0p} \in X$

$$\lim_{t\to\infty} \|\phi(t,t_0,x_{0p},u_r) - \phi(t,t_0,x_{0r},\widetilde{u}_r)\| = 0$$

6 Periodic and almost periodic inputs

In this section, we restrict our attention to the case of stationary dynamical systems and we characterize the behavior of this specific class of systems for different types of input.
In the sequel, a system possesses an equilibrium point, namely x_e, if there exists an input, namely u_e, belonging to $\mathcal{L}_p^e$ with a finite amplitude almost everywhere and such that for all $t \geq t_0$ one has $x_e = \phi(t,t_0,x_e,u_e)$.

Definition 6.1 [2] *A continuous function,* f*, defined from* $\mathcal{R}$ *into* $\mathcal{R}$ *is said to be almost periodic if for any* $\epsilon > 0$*, there exists a positive number* $l(\epsilon)$ *such that any interval of length* $l(\epsilon)$ *contains a* τ *for which* $\|f(t+\tau) - f(t)\| \leq \epsilon$ *for all* $t \in \mathcal{R}$.

Definition 6.2 *A motion,* $x(t)$*, defined from* $[t_0, \infty)$ *into* X *is said to be weakly* [5] *asymptotically almost periodic if for any* $\epsilon > 0$*, there exists a positive number* $l(\epsilon)$ *and* $T_\epsilon > 0$ *such that any interval of length* $l(\epsilon)$ *contains a* τ *for which* $\|x(t+\tau) - x(t)\| \leq \epsilon$ *for all* $t > T_\epsilon$.

[3] The notations are slightly modified for a lake of space

[4] $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$ and thus $\|x\| - \|y\| \leq \|x-y\|$

[5] We introduce that because the motion is not necessary a continuous function of time at the opposite of definition given in [16].

Theorem 6.1 *Let Σ, a stationary dynamical system, with a finite incremental gain. Its state-space is assumed uniformly istropically reachable and the motion associated with u_r is assumed uniformly observable then*

(*i*) *If u_r is an uniformly continuous function of time then there exist, for any $\epsilon > 0$, two finite constants $\sigma_u > 0$ and T_ϵ such that for any $|\sigma| \leq \sigma_u$ and all $t \geq T_\epsilon$, one has*

$$\|x_r(t) - x_r(t-\sigma)\| < \epsilon$$

Moreover, there exists at least an initial condition for which the motion is an uniformly continuous function of time for all $t \geq t_0$.

(*ii*) *If u_r is a periodic input and the system possesses an equilibrium point then the unperturbed motion is asymptotically periodic. Moreover, there exists at least an initial condition such that the output associated with this input is a perodic function almost everywhere i.e. $\|Q_t P_{t+T}(y_{(t+T)} - y_t)\|_p = 0$ for all t where y_t is the output associated with u_r and $y_{(t+T)}$ is defined by $y_{(t+T)}(t) = y_t(t+T)$.*

(*iii*) *If u_r is an almost periodic input and the system possesses an equilibrium point then the unperturbed motion is weakly asymptotically almost periodic. Moreover, there exists at least an initial condition such that the motion associated with this input is an almost periodic function.*

Remarks:

(*i*) In the context of constant inputs, the point (*ii*) of Theorem 6.1 implies that it is possible to associate with each constant input an equilibrium point which is globally asymptotically stable by Theorem 4.1.

(*ii*) In [12], this result is proved for incremental systems which are the interconnectionxion between memoryless nonlinearities and a linear time invariant system with a finite gain. In this context all the assumptions concerning the realization of the nonlinear system can be removed.

7 A simple example

The previous results are used in [6] so as to analyze the behavior of a PI controlled missile. Missiles are indeed a perfect example of nonlinear systems, whose behavior is closely related to the behavior of a linear system (e.g. steady-state with respect to any constant inputs, the steady-state unique property and periodic motions for periodic inputs, ...).

Consider the closed-loop system, obtained by applying a PI controller to a missile pitch channel. We give in this section a basic result, which explains in an analytic way why this closed-loop system is incrementally stable. To this purpose, we just consider the principal nonlinearity, namely the aerodynamic moment. This first step has been successfully extended in [6] to a more realistic model of missile controlled by a classical PI[6] under the computation of the H_∞ norm of a specific transfer matrix. The nonlinear dynamics of the pitch or the yaw channels of a missile can be described by the following equations:

$$\begin{aligned} \dot{q} &= \frac{M_q}{I_y} q + \frac{M(\alpha)}{I_y} + \frac{M_\eta}{I_y}\eta \\ \dot{\alpha} &= q + \frac{Z(\alpha)}{mV} \end{aligned}$$

where q is the pitch rate and α the angle of attack.
Let us consider the following controller defined by:

$$u(t) = k_3\left(-\frac{\Gamma_{co}(t)}{V} - q(t)\right)$$

Under this simple feedback, the closed-loop system is described by (introducing suitable notations):

$$\begin{aligned} \dot{q} &= aq + R(\alpha) + B_1\Gamma_{co} \\ \dot{\alpha} &= q + P(\alpha) \end{aligned}$$

and its linearisation for a specific constant value of $\alpha = \alpha_0$ is given by

$$\begin{aligned} \dot{\bar{q}} &= a\bar{q} + R'(\alpha_0)\bar{\alpha} + B_1\delta_{\Gamma_{co}} \\ \dot{\bar{\alpha}} &= \bar{q} + P'(\alpha_0)\bar{\alpha} \end{aligned}$$

Let us study the nonlinear closed-loop system with respect to the nonlinearity, which is associated to the aerodynamic moment $R(\alpha)$. To this purpose, we consider the transfer function between a fictitious input, u, acting on $\dot{\bar{q}}$ and $\bar{\alpha}$ which is given by:

$$\frac{\bar{\alpha}}{u} = G(s) = \frac{1}{s^2 + (a + P'(\alpha_0))s - R'(\alpha_0) + a * P'(\alpha_0)}$$

Assume there exist a specific value α_0 of α and a gain k_3, such that the damping of this transfer function is greater than 0.9. The H_∞ norm of $G(s)$ is then given by

$$\|G(s)\|_\infty \approx |-R'(\alpha_0) + a * P'(\alpha_0)|^{-1}$$

On this basis, recalling that the incremental gain of $R(\alpha)$ is linked to the value of its derivatives under this relation $\|R\|_\Delta \leq \max_x \bar{\sigma}\frac{\partial R(x)}{\partial x}$ and invoking the small incremental gain Theorem [17, 3], we deduce that the nonlinear system is incrementally stable if the derivative of $R(\alpha)$ verifies this constraint:

$$|R'(\alpha) - R'(\alpha_0)| < |-R'(\alpha_0) + a * P'(\alpha_0)| \quad (4)$$

Finally, it is interesting to note that the condition given by (4) is a necessary condition for the stability of the system. Indeed, let us consider the interconnection of $G(s)$ with a linear feedback, i.e, $u = -k\bar{\alpha}$ and calculate the denominator of this closed-loop system:

$$P_{bf} = s^2 + (a + P'(\alpha_0))s - R'(\alpha_0) + a * P'(\alpha_0) + k$$

[6] The two nonlinearities associated to the aerodynamic effects are taken into account and the actuator dynamics are modelized by a second order transfer function.

Typically the coefficients of the unperturbed denominator satisfy these inequalities:

$$a + P'(\alpha_0)) > 0 \quad -R'(\alpha_0) + a * P'(\alpha_0) > 0$$

We can then deduce that the interconnection is marginally stable for $k = R'(\alpha_0) - a * P'(\alpha_0)$ and unstable for $k < R'(\alpha_0) - a * P'(\alpha_0)$. It is clear that condition (4) is a necessary and sufficient condition for the incremental stability of the missile.
This simple fact explain probably why the classically used heuristic method (which consists in deducing the stability of the nonlinear system from the stability of its linearisations) works well in this context.
This study can be extended to a realistic model of the closed-loop system associated to a controlled missile [6].

8 Conclusion

In this paper, through the use of simple arguments concerning the dissipativity property of a fictitious system, we have characterized the behavior of Lipschitz continuous systems toward specific inputs and modifications of the initial condition. The main result of this paper is the proof of the steady-state property of this kind of system. On this basis, we have characterized the behavior of stationary Lipschitz continuous systems for periodic and almost periodic inputs. All these results show, once of more, that the input-output approach allows us to bypass, in some cases, the associated difficulties in search of Lyapunov function for the nonlinear systems.

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