

About incremental stability interest when analysing anti-windup control law performance*

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1 Introduction

Saturations are always present at the input of real plants. As a consequence, it was intensively investigated. When control law are designed without taking into account actuator saturations, closed loop performance can be dramatically degraded: even in the case of stable plant, pumping phenomena can be observed. When considering unstable plant, the closed loop system can be destabilized.

A classical approach to deal with is the anti windup design [11, 13]. A controller is first designed ignoring the input saturations. It is then modified to take into account the effects of the saturation nonlinearities [3]. The last step is the verification of the stability and the performance of the closed loop system. This approach was considered in a significant number of works [11, 13] and actually is widely applied in practise. Nevertheless, the main issue is to ensure that the anti-windup control actually ensures the expected performance level.

Recently in [12], many authors propose conditions guaranteeing the stability of the anti windup control law. In fact, most of the stability conditions were obtained by a direct or an implicit application of results strongly related to the small gain theorem or the passivity theorem [2]. Closed loop system stability is ensured in the sense that it guarantees that bounded \mathcal{L}_2 norm input signals lead to bounded \mathcal{L}_2 norm output signals. Note that closed loop systems with saturations are classically rewritten as the interconnection of a stationary linear system with a static nonlinearity. As a consequence, the global and exponential stability (in the Lyapunov sense) of the equilibrium point associated with the

null input is obtained.

It is crystal clear that such a result is significant and of interest. Nevertheless, automatic control praticiens are deeply interested by performance properties which are not ensured by stability. Due to the actuator saturation, the closed loop system is nonlinear. In this case, \mathcal{L}_2 stability does not guarantee the renewal of the properties attached to “linear systems” such as the unique steady state, or the stability of the equilibrium points associated to each constant input. These performance properties corresponds to usual practical specifications.

The main objective of this paper is to enlighten the interest of works based on the incremental norm in this specific context. As it has been proved by the first author [4, 6, 9, 7, 8, 5], incrementally stable systems have suitable steady-state properties, which are usually expected performance specifications. As a first point, a unique steady-state motion corresponds to a given input signal, independently of the initial condition and despite a vanishing perturbation on the input signal. As a second point, the steady state response to a constant (resp. periodic) input signal is also constant (resp. periodic).

Incremental stability is also suitable to study in quantitative way the closed loop system performance. In this context, the notion of (nonlinear) incremental performance is defined in the continuity of the (linear) H_∞ performance (i.e. through the use of a weighting function). When considering e.g. the (nonlinear) maps between the reference input and the tracking error, this (possibly linear) weighting function reflects our aim to keep the tracking error signal small with respect to the reference input signal.

The main contribution of this paper is to prove that it is always possible, for stable plants, to design an anti-windup scheme such the the closed loop system is incrementally stable. As a consequence, all the desirable qualitative properties are ensured.

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2 Incremental stability: some recall

The notations and terminology, which are recalled hereafter, are classical in the input-output context (see [17]). In the following, we use the \mathcal{L}_p spaces, i.e, the space of \mathcal{R}^n valued functions defined on \mathcal{R} , for which the p th power of the norm is integrable with $p \in [1, \infty)$, where the norm is defined by $\|f\|_p = (\int \|f(t)\|^p dt)^{1/p}$. The *causal truncation* at T of a function f , defined on \mathcal{R} , is denoted $P_T f$ and is defined by $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise. The *extended space* associated to \mathcal{L}_p is denoted by \mathcal{L}_p^e and corresponds to the space of \mathcal{R}^n valued functions defined on \mathcal{R} whose causal truncations belong to \mathcal{L}_p .

In the sequel, let us consider a nonlinear system, $y = \Sigma(u)$, which is described by this well-defined differential equation:

$$\Sigma \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)) \\ x(t_0) &= x_0 \\ y(t) &= h(t, x(t), u(t)) \end{cases} \quad (1)$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^p$. f (resp. h) is defined from $[t_0, \infty) \times \mathcal{R}^n \times \mathcal{R}^p$ into \mathcal{R}^n (resp. \mathcal{R}^m) and are assumed C^1 and such that $f(t, 0, 0) = 0$ and $h(t, 0, 0) = 0$ for all $t \geq t_0$.

Definition 2.1 Σ has a finite incremental gain on \mathcal{L}_p if there exists $\eta_p \geq 0$ such that

$$\|\Sigma(u_1) - \Sigma(u_2)\|_p \leq \eta_p \|u_1 - u_2\|_p$$

for all $u_1, u_2 \in \mathcal{L}_p$. The incremental gain of Σ coincides with the minimum value of η_p and is denoted $\|\Sigma\|_{\Delta, p}$. Σ is said to be *incrementally stable* (on \mathcal{L}_p) if it is stable, i.e. it maps \mathcal{L}_p to \mathcal{L}_p , and has a finite incremental gain.

Incremental norm interest

The previous definition may appear restrictive from an applicative point of view, since a limited class of possible inputs is considered for the system: as an example, a non-zero constant input does not belong to \mathcal{L}_2 . This restriction can be nevertheless bypassed using the link between the input-output stability properties on \mathcal{L}_2 and its extended space \mathcal{L}_2^e [17]. Indeed, if Σ_{x_0} has a finite incremental gain less or equal to η , then for all $T \geq 0$ and for all $u_1, u_2 \in \mathcal{L}_2^e$, the following relation is satisfied: $\|P_T(y_1 - y_2)\|_2 \leq \eta \|P_T(u_1 - u_2)\|_2$. This inequality clearly indicates that the input-output relation, which was already satisfied by the input signals inside \mathcal{L}_2 , remains valid inside \mathcal{L}_2^e . More generally, when studying the properties of the nonlinear system along a possible motion, the use of the extended space \mathcal{L}_2^e enables to consider a much larger class of possible inputs (e.g. non-zero constant inputs). As an illustration, introducing:

$$\mathcal{L}_\infty([0, T]) = \{f : \mathcal{R} \mapsto \mathcal{R}^n \mid \text{ess sup}_{t \in [0, T]} \|f(t)\| < \infty\}$$

it can be pointed out that the following inclusion $\mathcal{L}_\infty([0, T]) \subset \mathcal{L}_2([0, T])$ is true for each value of T . As a

consequence, the extended space, which is associated with \mathcal{L}_2 for a specific value of T , contains all the signals which have (almost everywhere) a finite amplitude on $[0, T]$. In conclusion, when characterizing the properties of the nonlinear system, the use of the extended space \mathcal{L}_2^e enables to take into account most of the possible input signals, which are generally considered in an application.

Quadratically incremental stable systems

As shown in [6], characterizing the incremental boundedness of a nonlinear system is a difficult problem, which typically involves solving a Hamilton-Jacobi type equation. As a matter of fact, in the same way as in the \mathcal{L}_2 gain context, a sufficient condition for the incremental boundedness of a nonlinear system has been obtained in [10], which now involves solving a Linear Matrix Inequality (LMI - the idea is to use a specific type of solution to the original Hamilton Jacobi type equation). Remember that LMIs correspond to convex constraints, for which efficient numerical algorithms have been proposed [1].

Let us assume that the output of system (1) is the full state i.e. $y = x$ then:

Theorem 2.1 [9, 10] *If there exist a positive and symmetric matrix P , and two strictly positive constants σ_{f_u} and ϵ such that*

$$(i) \quad P \frac{\partial f}{\partial x}(t, x, u) + \frac{\partial f}{\partial x}^T(t, x, u) P \leq -\epsilon I_n$$

$$(ii) \quad \left\| \frac{\partial f}{\partial u}(t, x, u) \right\| \leq \sigma_{f_u}$$

for all $t \geq t_0$, $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^p$, then Σ is (quadratically) incrementally stable on \mathcal{L}_p (for $p \in [1, \infty]$) for any initial condition $x_0 \in \mathcal{R}^n$.

Condition (i) can be weakened if this condition:

$$(i') \quad (x_1 - x_2)^T P (f(t, x_1, u) - f(t, x_2, u)) \leq -\epsilon \|x_1 - x_2\|^2$$

is satisfied for all $t \in \mathcal{R}$, $x_1, x_2 \in \mathcal{R}^n$ and $u \in \mathcal{R}^p$. Condition (i) implies condition (i') (they are equivalent if f is C^1).

Qualitative properties of quadratically incrementally stable systems

We first consider the Lyapunov property for the unperturbed motions of a quadratically incrementally stable system. More precisely, with reference to the motion which is associated with a specific input belonging to \mathcal{L}_2^e and with a specific initial condition, we characterize the behavior of the motion, which is associated with the same input but which is initialized with a different initial condition.

Theorem 2.2 [7, 5] *If Σ is a quadratically incrementally stable system, then all its unperturbed motions are uniformly globally exponentially stable, i.e. for any input $u_r \in \mathcal{L}_2^e$,*

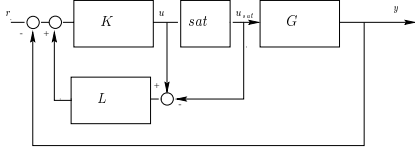


Figure 1: The closed-loop system

and for any initial condition $x_{0r} \in \mathcal{R}^n$, there exist two positive constants a and b satisfying for all $t_1 \geq t_0$ and for all $x_{0p} \in \mathcal{R}^n$:

$$\|\phi(t, t_1, x_r(t_1), u_r) - \phi(t, t_1, x_{0p}, u_r)\| \leq a \|x_r(t_1) - x_{0p}\| e^{-b(t-t_1)}$$

for all $t \geq t_1$ and where $x_r(t) = \phi(t, t_0, x_{0r}, u_r)$.

We now characterize the behavior of the system with respect to a perturbation on the system input. We first consider the effects of a vanishing perturbation.

Theorem 2.3 [5] *Let Σ , be a dynamical system which is quadratically incrementally stable. For any $\tilde{u}_r \in \mathcal{L}_2^e$ such that $u_r - \tilde{u}_r$ belongs to \mathcal{L}_2 , the following property is satisfied:*

$$\lim_{t \rightarrow \infty} \|\phi(t, t_0, x_0, u_r) - \phi(t, t_0, x_0, \tilde{u}_r)\| = 0$$

From Theorem 2.2 and Theorem 2.3, the nonlinear system has a unique steady state motion for a given input signal.

In the context of quadratically incrementally stable systems, the steady state response to a constant (resp. periodic) input signal is a constant (resp. periodic) output signal.

Theorem 2.4 [5] *Let Σ be a stationary and quadratically incrementally stable system. If the input of the system, namely u_r , is a T -periodic input, then the motion of the system is asymptotically T -periodic. Moreover there exists at least an initial condition for which the motion is T -periodic.*

Corollary 2.5 [5] *Let Σ be a stationary and quadratically incrementally stable system. The motion which is associated to constant input goes asymptotically to a constant and there exists at least, an equilibrium point for each possible constant input.*

3 Application to anti-windup compensator performance analysis

Without any loss of generality, the result is presented in the case of Single Input Single Output systems: the extension to Multi Input Multi Output systems is straightforward. We focus on the closed loop system defined as the interconnexion between a stable plant, referred to as G and its associated control law, referred to as K . Moreover, the plant actuator presents an input saturation which is defined by $\text{sat}(u) = u_{min}$ if $u < u_{min}$, $\text{sat}(u) = u$ if $u \in [u_{min}, u_{max}]$ and $\text{sat}(u) = u_{max}$ if $u > u_{max}$.

A classical idea to deal with saturations is to minimise the windup effect, by a suitable modification of K . A widespread modification is to augment the control law K , by a ‘‘feedback effect’’ L which only modifies the control law when a saturation appears (see figure 3).

In this specific context, we have this result:

Proposition 3.1 *There always exists a transfer function L such that the closed-loop is quadratically incrementally stable.*

Proof: Let us first note that the saturation can be rewritten as $\text{sat}(u) = u - \sigma(u)$ where $\sigma(u)$ is a dead-zone nonlinearity defined by: $\sigma(u) = u - u_{min}$ if $u < u_{min}$, $\sigma(u) = 0$ if $u \in [u_{min}, u_{max}]$ and $\sigma(u) = u - u_{max}$ if $u > u_{max}$. On this basis, we can consider that the closed-loop system is in fact the interconnexion between the dead zone nonlinearity and a linear system given by $H = -(I + KG)^{-1}K(G + L)$. Let us note that $\sigma(u)$ is an incrementally stable nonlinearity, with a slope restricted by 1. On \mathcal{R} (under a basic geometric type argument), we obviously have

$$|\sigma(u_1) - \sigma(u_2)| \leq |u_1 - u_2|$$

which ensures on \mathcal{L}_2 that

$$\|\sigma(u_1) - \sigma(u_2)\|_2 \leq \|u_1 - u_2\|_2.$$

Now if we choose $L = -G$ then the transfer function $H(s)$ is internally stable and $\|H(s)\|_\infty = 0$. Incremental stability of the closed loop system is then deduced from the incremental small gain theorem [2]. On this basis, it is possible to prove that the conditions (i)' and (ii) of theorem 2.2 are satisfied. As matter of fact, condition (i)' is a direct consequence of the Real Bounded Lemma and (ii) is obviously satisfied.

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