Lipschitz continuous neural networks on $\mathcal{L}_p$[1]

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Abstract

This paper presents simple conditions ensuring that dynamical neural networks are incrementally stable, that is Lipschitz continuous, on $\mathcal{L}_p$. A first interest of this result is that it ensures obviously the continuity of the system as an operator from a signal space to another signal space. This property may be interpreted in this context as the ability for dynamical neural networks to interpolate. In some sense, it is an extension of a well-known property of static neural networks.
A second interest of this result is linked to the fact that the behaviors of Lipschitz continuous systems with respect to specific inputs or initial condition problems can be completely analyzed. Indeed, Lipschitz continuous systems have the steady-state property with respect to any inputs belonging to $\mathcal{L}_p^e$ with $p \in [1, \infty]$, *i.e.*, their asymptotic behavior is uniquely determined by the asymptotic behavior of the input. Moreover, the Lipschitz continuity guarantees the existence of globally asymptotic stable (in sense of Lyapunov) equilibrium points for all constant inputs.

1 Introduction

Recently, the study of dynamical properties of artificial neural networks has received a great deal of interest [13, 12, 14, 17, 4]. Most of the work focus on the "classification capabilities" of this class of systems. The main idea is to use the domain of attraction of each equilibrium point to classify specific information. The nonlinearities of neural networks allow to "shape" complex domains of attraction which can be useful for classifying. In this framework, the authors propose conditions ensuring asymptotic stability of equilibrium points and estimate the domain of attraction for each stable equilibrium point. The aim of this paper is to propose a framework allowing to analyze the "interpolation capabilities" of the dynamical neural networks. This property was well studied in the case of static neural networks but was never considered, to the best of our knowledge, in the case of dynamical neural networks.

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As a matter of fact, we propose in this paper simple (structural) conditions concerning the weight matrices which ensure the Lipschitz continuity of the systems on $\mathcal{L}_p$.
It thus guarantees the continuity of the system as an operator from a signal space to another signal space, *i.e.*, a small variation of inputs makes a small variation of the output. Moreover, Lipschitz continuity can be clearly interpreted as a "continuous interpolation" ability of the dynamical neural networks.
As it was proved in recent work, ([8, 9, 10, 6, 7]), Lipschitz continuous system owns many interesting properties. More precisely:

- The system has an equilibrium point for each constant input;
- Each equilibrium point is globally asymptotically stable;
- If a constant input is "close" to another constant input then the two equilibrium points are "close".

Many another properties are provided by Lipschitz continuity (for details see, *e.g.*, [6, 7]). For example, Lipschitz continuous systems have the steady state properties, *i.e.*, the asymptotic behavior (the output of neural network) is uniquely determined by the asymptotic behavior of the input. This property correspond to the idea that the information related to the neural network behavior is contained only in the interconnection and in the different values of the associated weights but not in the "state" of the dynamical system.
Most of the dynamical neural networks considered in the literature has the following structure ([17]):

$$\dot{x}_i = -x_i + \sigma\left(\sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{1_{ij}}u_j\right) + \sum_{j=1}^{n} b_{2_{ij}}u_j \quad (1)$$

where each $x_i(t)$, $i = 1, \cdots, n$, is a real value function of time which corresponds to the internal state of the ith artificial neuron and each $u_i(t)$, $i = 1, \cdots, m$, is also a real value function of time which corresponds to an external input signal to the neural network. The coefficients a_{ij},$b_{1_{ij}}$ and $b_{2_{ij}}$ denote the weights of the various network connections. The function $\sigma : \mathcal{R} \to \mathcal{R}$ is the activation function. Typically, the function σ is "sigmoidal", *e.g.*, $\sigma(x) = 1/(1 + e^{-x})$.

In the following, we consider that the dynamical neural networks system are nonlinear causal operators defined from a signal space into another signal space and described by:

$$\Sigma(u)\begin{cases}\dot{x}(t) &= -x(t)+\sigma^{(n)}(Ax(t)+B_1u(t))+B_2u(t)\\ y(t) &= Cx(t)\\ x(t_0) &= x_0\end{cases} \tag{2}$$

where $A=\{a_{ij}\}$ belongs to $\mathcal{R}^{n\times n}$, $B_1=\{b_{1_{ij}}\}, B_2=\{b_{2_{ij}}\}$ belong to $\mathcal{R}^{n\times m}$, C belongs to $\mathcal{R}^{p\times n}$ and $\sigma^{(n)}:\mathcal{R}^n\to\mathcal{R}^n$ is a diagonal map.

$$\sigma^{(n)}:\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\mapsto\begin{pmatrix}\sigma(x_1)\\ \vdots\\ \sigma(x_n)\end{pmatrix} \tag{3}$$

where σ is a sigmoidal type function, *i.e.*, non decreasing with the properties that both $\lim_{x\to-\infty}\sigma(x)$ and $\lim_{x\to\infty}\sigma(x)$ exist. We moreover assume in the sequel that σ is C^1 and Lipschitz.
Finally, we assume that input-output signal spaces are two $\mathcal{L}_p^e$ spaces *i.e.* u and y belong to $\mathcal{L}_p^e$ spaces.
The aim of this paper is to provide simple conditions on the weight matrices which ensure the Lipschitz continuity (the incremental boundedness) of the dynamical neural networks on $\mathcal{L}_p$, *i.e.*, there exists a finite constant η such that $\|\Sigma(u_1)-\Sigma(u_2)\|_p\le\eta\|u_1-u_2\|_p$ for any $u_1,u_2\in\mathcal{L}_p$.
The paper is organized as follows. In section 2.1, basic notions about input-output stability are firstly recall. The notion of quadratic incremental stability is then introduced in section 2.2 while section 2.3 presents qualitative properties of incrementally stable systems. Section 3, which is the main section of this paper, presents simple conditions ensuring that the dynamical neural networks are incrementally stable on $\mathcal{L}_p$. Finally, section 4 presents a result concerning the adaptation of weight matrices.

2 Lipschitz continuity: some recalls

2.1 Notations, definitions

The notations and terminology, which are recalled hereafter, are classical in the input-output context (see [3, 19]). The $\mathcal{L}_p$ spaces for $p\in[1,\infty)$ are introduced as the spaces of $\mathcal{R}^n$ valued functions defined on $[0,+\infty)$, for which the *pth* norm is integrable. The $\mathcal{L}_p$ norm is defined as: $\|f\|_p=\left(\int_0^{+\infty}\|f(t)\|^p dt\right)^{\frac{1}{p}}$ where $\|.\|$ denotes the Euclidean norm. $\mathcal{L}_\infty$ is the spaces of $\mathcal{R}^n$ valued functions defined on $[0,+\infty)$, for which $ess\sup_{t\ge0}|f(t)|$ is finite. The $\mathcal{L}_\infty$ norm is defined as: $\|f\|_\infty=ess\sup_{t\ge0}|f(t)|$. The *causal truncation* at T of a function f, defined on $[0,+\infty)$, is denoted P_Tf and is defined by $P_Tf(t)=f(t)$ for $0\le t\le T$ and 0 otherwise. The *extended space* associated with $\mathcal{L}_p$, which is denoted as $\mathcal{L}_p^e$, corresponds to the space of $\mathcal{R}^n$ valued functions defined on $[0,+\infty)$, whose causal truncations belong to $\mathcal{L}_p$. Let A a real matrix, $\bar{\sigma}(A)$ denotes its maximal singular value.

Let us consider the following nonlinear system denoted by $y=\Sigma_{x_0}(u)$ and described by this differential equation:

$$\Sigma_{x_0}\begin{cases}\dot{x}(t) &= f(x(t),u(t))\\ y(t) &= h(x(t),u(t))\\ x(t_0) &= x_0\end{cases} \tag{4}$$

where $x(t)\in\mathcal{R}^n, u(t)\in\mathcal{R}^m$, and $y(t)\in\mathcal{R}^p$. f, resp. h, are defined from $\mathcal{R}^n\times\mathcal{R}^m$ into $\mathcal{R}^n$ (resp. into $\mathcal{R}^p$) and are assumed C^1 and uniformly Lipschitz. The system is then well-defined, *i.e.*, the solution, $x(t)=\phi(t,t_0,x_0,u)$, exists, is unique and belongs to $\mathcal{L}_p^e$ for all $x_0\in\mathcal{R}^n$ and u belonging to $\mathcal{L}_p^e$.

We can now introduce the notion of (incremental) $\mathcal{L}_p$ gain.

Definition 2.1 *Σ_{x_0} is said to be a weakly finite ($\mathcal{L}_p$) gain stable system if there exists $\gamma\ge0$, $\beta\ge0$ such that $\|\Sigma_{x_0}(u)\|_p\le\gamma\|u\|_p+\beta$ for all $u\in\mathcal{L}_p$. Σ_{x_0} is said to be finite ($\mathcal{L}_p$) gain stable when $\beta=0$. The ($\mathcal{L}_p$) gain of Σ_{x_0}, denoted by $\|\Sigma_{x_0}\|_{i,p}$, is defined as the minimal value of γ.*

Definition 2.2 *Σ_{x_0} has a finite incremental gain on $\mathcal{L}_p$ if there exists $\eta\ge0$ such that $\|\Sigma_{x_0}(u_1)-\Sigma_{x_0}(u_2)\|_p\le\eta\|u_1-u_2\|_p$ for all $u_1,u_2\in\mathcal{L}_p$. The incremental gain of Σ_{x_0} on $\mathcal{L}_p$, denoted $\|\Sigma_{x_0}\|_{\Delta,p}$, is defined as the minimal value of η. Σ_{x_0} is said to be incrementally stable if it is stable, i.e., it maps $\mathcal{L}_p$ to $\mathcal{L}_p$, and if it has a finite incremental gain on $\mathcal{L}_p$.*

The above definitions may appear restrictive from an applicative point of view, since a limited class of possible input signals is considered for the system: as an example, a non-zero constant input does not belong to $\mathcal{L}_p$ for $p\in[0,\infty)$. This restriction can be nevertheless bypassed using the link between the input-output stability properties on $\mathcal{L}_p$ and its extended space $\mathcal{L}_p^e$. Indeed, if Σ_{x_0} has a finite incremental gain less or equal to η, then for all $T\ge0$ and for all $u_1,u_2\in\mathcal{L}_p^e$, the following relation is satisfied:

$$\|P_T(y_1-y_2)\|_p\le\eta\|P_T(u_1-u_2)\|_p$$

This inequality clearly indicates that the input-output relation, which was already satisfied by the input signals inside $\mathcal{L}_p$, remains valid inside $\mathcal{L}_p^e$.
More generally, when studying the properties of the nonlinear system along a possible motion, the use of the extended space $\mathcal{L}_p^e$ enables to consider a much larger class of possible inputs, *e.g.*, non-zero constant inputs. In conclusion, when characterizing the properties of the

nonlinear system, the use of the extended space $\mathcal{L}_p^e$ enables to take into account most of the possible input signals, which are generally considered in an application.

2.2 Quadratic incremental boundedness

Since testing incremental boundedness of a nonlinear system is a difficult problem [8], in [11], we introduced a stronger notion, named *quadratic* incremental boundedness, which is easier to handle. This notion is now recalled:

Theorem 2.1 *If there exist a positive and symmetric matrix P, and finite constants ϵ and $\sigma_{f_u}, \sigma_{h_x}$ and σ_{h_u} satisfying the two following conditions for all $t \geq t_0$, $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$:*

$$(i) \quad P\frac{\partial f}{\partial x}(x,u) + \frac{\partial f}{\partial x}^T(x,u)P \leq -\epsilon I_n$$

$$(ii) \quad \bar{\sigma}\left(\frac{\partial f}{\partial u}\right) < \sigma_{f_u} \quad \bar{\sigma}\left(\frac{\partial h}{\partial x}\right) < \sigma_{h_x} \quad \bar{\sigma}\left(\frac{\partial h}{\partial u}\right) < \sigma_{h_u}$$

then the system (4) is incrementally bounded on $\mathcal{L}_p$ for $p \in [1, \infty]$ for any initial condition $x_0 \in \mathcal{R}^n$.

Definition 2.3 *A system which satisfies the conditions of theorem 2.1 is said to be quadratically incrementally bounded.*

2.3 Qualitative properties of quadratically incrementally bounded systems

The following section presents the qualitative properties, which are associated to this class of nonlinear systems (see [9, 10, 7] for the proofs of the theorems, which are presented in this section).
The first subsection is devoted to the characterization of the variation of the system trajectory, with respect to a variation of the input signal. The second subsection considers the case of a fixed input signal with a perturbation of the initial condition.

Input-output behavior of incrementally bounded systems

We would like to characterize the behavior of the system with respect to a perturbation on the system input. We first consider the effects of a vanishing perturbation.

Theorem 2.2 *Let Σ, be a dynamical system which is quadratically incrementally stable. For any $\widetilde{u}_r \in \mathcal{L}_p^e$ such that $u_r - \widetilde{u}_r$ belongs to $\mathcal{L}_p$, the following property is satisfied:*

$$\lim_{t\to\infty} \|\phi(t, t_0, x_0, u_r) - \phi(t, t_0, x_0, \widetilde{u}_r)\| = 0$$

Behavior with respect to the initial condition

We study in this section the Lyapunov property for the unperturbed motions of a quadratically incrementally bounded system. More precisely, with reference to the motion which is associated with a specific input belonging to $\mathcal{L}_p^e$ and with a specific initial condition, we characterize the behavior of the motion, which is associated with the same input but which is initialized with a different initial condition.

Theorem 2.3 *If Σ is a quadratically incrementally bounded system, then all its unperturbed motions are uniformly globally exponentially stable, i.e., for any input $u_r \in \mathcal{L}_p^e$ and for any initial condition $x_{0r} \in \mathcal{R}^n$, there exist two positive constants a and b satisfying for all $t_1 \geq t_0$ and for all $x_{0p} \in \mathcal{R}^n$:*

$$\|\phi(t, \tilde{t}, x_r(\tilde{t}), u_r) - \phi(t, \tilde{t}, x_{0p}, u_r)\| \leq a\|x_r(\tilde{t}) - x_{0p}\|e^{-b(t-\tilde{t})}$$

for all $t \geq \tilde{t}$ and where $x_r(t) = \phi(t, t_0, x_{0r}, u_r)$.

As a consequence of the two above theorems, the nonlinear system has a unique steady state motion for a given input signal. We now consider the effect of a persistent perturbation with a finite amplitude.

Constant and periodic inputs

In the context of quadratically incrementally stable systems, the steady state response to a constant (resp. periodic) input signal is a constant (resp. periodic) output signal.

Theorem 2.4 *Let Σ quadratically incrementally bounded system. Let us assume that there exists $u_e \in \mathcal{L}_\infty$ and $x_e \in \mathcal{R}^n$ such that $x_e = \phi(t, t_0, x_e, u_e)$ for any $t \geq t_0$ then*
(i) if the input of the system is a T-periodic input, then the motion of the system is asymptotically T-periodic. Moreover, there exists an initial condition for which the motion is T-periodic.
(ii) if the input of the system is a constant input, then the motion of the system goes asymptotically to a constant. Moreover, there exists an equilibrium point for each possible constant input.

Remarks:
(i) The proofs of all previous results are available in [7].
(ii) The assumption concerning the existence of equilibrium point x_e, ensures the boundedness of trajectories associated to inputs which are a finite amplitute almost everywhere.
(iii) Theorem 2.4 can be extended to the larger class of almost periodic input signals [6, 7].

3 Condition for incrementally boundedness of neural networks

3.1 Feed-forward type neural networks

Proposition 3.1 *If A is an upper triangular matrix such that $a_{ii} \leq 0$ and $a_{i,j} = 0$ for $i > j$, then system (2), i.e, $y = \Sigma(u)$, is a quadratically incrementally bounded system.*

Proof: We shows in the sequel that the conditions of theorem 2.1 are satisfied taking a suitable diagonal and strictly positive definite matrix.
To this purpose, let us write the linearization (the Gateaux derivative), namely $\bar{y} = D\Sigma_G[u_r](\bar{u})$ of Σ at $u_r \in \mathcal{L}_p^e$

$$\left\{ \begin{array}{rcl} \dot{\bar{x}}(t) & = & -\bar{x}(t) + M(t)(A\bar{x}(t) + B_1\bar{u}(t)) + B_2\bar{u}(t) \\ \bar{x}(t_0) & = & 0 \\ \bar{y}(t) & = & C\bar{x}(t) \end{array} \right. \tag{5}$$

where $M(t) = \dfrac{\partial \sigma^{(n)}}{\partial x}[Ax_r(t) + B_1u_r(t)]$ with $x_r(t) = \phi(t, t_0, x_0, u_r)$.

By definition of $\sigma^{(n)}$, $M(t)$ is a diagonal matrix, *i.e.*, $M(t) = diag(M_1(t), \cdots, M_n(t))$. We prove in the sequel that there exists a diagonal and strictly positive definite matrix, namely $D = diag(d_1, \cdots, d_n)$, such the following quantity:

$$\Pi = \bar{x}(t)^T D(-I + M(t)A)\bar{x}(t)$$

is strictly negative.
One has

$$\begin{array}{rcl} \Pi & = & \sum_{i=1}^n \bar{x}_i d_i(-1 + M_i a_{ii})\bar{x}_i + ... \\ & & ... + \sum_{i=1}^{n-1} \sum_{k=i+1}^n \bar{x}_i d_i M_i a_{ik} \bar{x}_k \end{array} \tag{6}$$

The first term of the right side of (6) has the following upper bound:

$$\sum_{i=1}^n \bar{x}_i d_i(-1 + M_i a_{ii})\bar{x}_i \leq -\sum_{i=1}^n |\bar{x}_i| d_i |\bar{x}_i| \leq 0$$

since $d_i > 0$.
σ is assumed non decreasing, C^1 and Lipschitz continuous on $\mathcal{R}$, then there exits a finite constant, namely L, such that for any $i \in [1, n]$, one has:

$$0 \leq M_i \leq L$$

We then deduce that the second term of (6) satisfies the following inequality:

$$\sum_{k=i+1}^n \bar{x}_i d_i M_i a_{ik} \bar{x}_k \leq |\bar{x}_i| \sum_{k=i+1}^n L d_i |a_{ik}| |\bar{x}_k|$$

and thus

$$\Pi \leq (|\bar{x}_1|, \cdots, |\bar{x}_n|) D\mathcal{A}(|\bar{x}_1|, \cdots, |\bar{x}_n|)^T$$

where

$$\mathcal{A} = \begin{bmatrix} -1 & L|a_{12}| & \cdots & L|a_{1n}| \\ 0 & -1 & \cdots & L|a_{2n}| \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

In fact, $-\mathcal{A}$ is a M-Matrix[1] which allows to ensure that there exists a strictly positive definite and diagonal matrix D such that

$$\bar{x}^T(D\mathcal{A} + \mathcal{A}^T D)\bar{x} < 0$$

which proves that condition (i) of theorem 2.1 is satisfied. Since condition (ii) of theorem 2.1 is also satisfied, we conclude that $y = \Sigma(u)$ is a quadratically incrementally bounded system. ■

3.2 Positive type neural networks

Proposition 3.2 *If $A + A^T$ is a strictly negative definite matrix then system (2), i.e, $y = \Sigma(u)$, is a quadratically incrementally bounded system.*

Proof: We show in the sequel that the conditions of theorem 2.1 are also satisfied. To this purpose, let us pick, $P = -(A + A^T)$ which is, by definition of A, positive definite. Following the definition introduced in the proof of the previous proposition, we have:

$$\begin{array}{rcl} \Pi & = & -2\bar{x}^T(A + A^T)(-I + MA)\bar{x} \\ & = & -2\bar{x}^T(A + A^T)\bar{x} - 2\bar{x}^T(A + A^T)M(A + A^T)\bar{x} \end{array}$$

which is strictly negative since $M(t)$ is a positive matrix and $A + A^T$ is a strictly negative definite matrix. We conclude that $y = \Sigma(u)$ is a quadratically incrementally bounded system. ■

3.3 Row and column diagonally dominant type neural networks

The conditions of theorem 2.1 are just a sufficient condition for incremental stability. As matter of fact, we present in the sequel, two others types of dynamical neural networks, which are incrementally bounded (but not necessary quadratically incrementally bounded). Nevertheless, those systems possess all the qualitative properties attached to quadratically incrementally bounded systems (described in subsection 2.2).

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[1] Let us recall the following lemma (see [1])
Lemma Let W be a real and square matrix with non positive off diagonal elements. then the following condition are equivalent:

- (i) W is a *M-Matrix*.
- (ii) The real parts of the eigenvalues of W is positive.
- (iii) There exits a diagonal matrix $D = diag(d_1, \cdots, d_n)$ with $d_i > 0$ such that $DW + W^T D$ is positive definite.

Proposition 3.3 *If the elements of the matrix A is such that*

$$a_{ii} < 0 \text{ and } |a_{ii}| > \sum_{i \neq j} |a_{ij}|$$

then system (2), i.e, $y = \Sigma(u)$, is an incrementally bounded system on $\mathcal{L}_p$ for any $p \in [1\infty]$.

Proof: In the sequel, we use a slide modification of the condition given by theorem 2.1. As matter of fact, we replace the quadratic type Lyapunov function by:

$$V(\bar{x}) = \max_i |\bar{x}_i|$$

So, in this case, if $|\bar{x}_i| \geq |\bar{x}_j|$ for all $j \neq i$, the derivative[2] of V is given by:

$$\dot{V} = sign(\bar{x}_i)[-\bar{x}_i + M_i a_{ii}\bar{x}_i + \sum_{j \neq i} M_i a_{ij}\bar{x}_j]$$

The property of A ensures that there exists $\epsilon > 0$ such that:

$$\begin{array}{rcl} \dot{V} & \leq & sign(\bar{x}_i)[-\bar{x}_i + \bar{x}_i[Mi(t)(-|a_{ii}| + \sum_{i \neq j} |a_{ij}|)] \\ & \leq & -\max_i |\bar{x}_i| \end{array}$$

which allows to prove the exponential stability of all the linearizations of the system. On this basis, using well-known result, we claim that all the linearizations of the system are $\mathcal{L}_p$ stable. The means value theorem (in norm) on $\mathcal{L}_p$ allows to conclude. ■

Proposition 3.4 *If the elements of the matrix A is such that*

$$a_{ii} < 0 \text{ and } |a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

then system (2), i.e, $y = \Sigma(u)$, is an incrementally stable system on $\mathcal{L}_p$ for any $p \in [1\infty]$.

Proof: We firstly note that A is a stable matrix and thus A^{-1} exists. On this basis, let us consider the following change of coordinates: $z = Ax$ and let us rewritten the system as:

$$\left\{ \begin{array}{rcl} \dot{z}(t) & = & -z(t) + A\sigma^{(n)}(z(t) + B_1u(t)) + AB_2u(t) \\ y(t) & = & CA^{-1}z(t) \end{array} \right.$$

The drift of the linearization is thus given by:

$$\dot{\bar{z}}(t) = (-I + AM(t))\bar{z}(t)$$

As in the previous proof, we introduce a specific (non quadratic) Lyapunov function:

$$V(\bar{z}) = \sum_i |\bar{z}_i|$$

which, as in the previous proof, allows to conclude the exponential stability of all linearizations of the system. ■

[2] We just provide in a sequel a sketch of proof, see [5] to handle the problem linked to dicountinuous derivatives.

4 A first idea for the adaptation of the matrix of weights

In this section, we propose a first way allowing to estimate the matrix of weights.
For the sake of simplicity, we just consider the feed-forward type neural networks and assume that A is an invertible matrix and that $B_2 = 0$.
We introduce new coordinates, namely z, which are related to x by this relation $z = Ax$. On this basis, system (2) can be rewritten as

$$\dot{z}(t) = -z(t) + A\sigma^{(n)}(z(t) + B_1u(t)) \tag{7}$$

Let us introduce this new system:

$$\dot{z}(t) = -z(t) + (A_0 + \sum_{i=1}^{m} \theta_i \mathcal{F}_i)\sigma^{(n)}(z(t) + B_1u(t)) \tag{8}$$

where $\theta_i, i = 1, \cdots p$, is a parameter and $\mathcal{F}_i, i = 1, \cdots p$, is a constant matrix belonging to $\mathcal{R}^{n \times n}$. We finally assume that there exists a vector of parameter, namely $\theta_v = (\theta_{v1}, \cdots, \theta_{vp})^T$, such that

$$A = A_0 + \sum_{i=1}^{m} \theta_{vi}\mathcal{F}_i$$

Let us associated with system (8), the following "estimator" (the mechanism of adjustment is clearly linked the classical MRAS scheme, see [2])):

$$\left\{ \begin{array}{rcl} \dot{\hat{z}} & = & -z + A_0\sigma^{(n)}(\hat{z} + B_1u) + \sum_i \hat{\theta}_i \mathcal{F}_i \sigma^{(n)}(z + B_1u) \\ \dot{\hat{\theta}}_1 & = & \sigma^{(n)}(z + B_1u)^T \mathcal{F}_1^T P(\hat{z} - z) \\ \vdots & & \vdots \\ \dot{\hat{\theta}}_m & = & \sigma^{(n)}(z + B_1u)^T \mathcal{F}_m^T P(\hat{z} - z) \end{array} \right.$$

where P is a suitable symmetric and strictly positive matrix.

Proposition 4.1 *If A_0 is an upper triangular matrix such that $a_{ii} < 0$ and $a_{i,j} = 0$ for $i > j$ then there exists a symmetric and positive definite matrix, $P \in \mathcal{R}^{n \times n}$, such that*

$$\lim_{t \to \infty} \hat{z}(t) - z(t) = 0$$

Moreover, $\hat{\theta} - \theta_v$ is bounded for any $t \geq t_0$.

Proof: The dynamic associated to the error vector, i.e., $e_z = \hat{z} - z$ and $e_\theta = \hat{\theta} - \theta$, is given by the following differential equation:

$$\left\{ \begin{array}{rcl} \dot{e_z} & = & -e_z + f(\hat{z}, z, u) + \Sigma_i e_{\theta_i} \mathcal{F}_i \sigma^{(n)}(z + B_1u) \\ \dot{e_{\theta_1}} & = & \sigma^{(n)}(z + B_1u)^T \mathcal{F}_1^T Pe_z \\ \vdots & & \vdots \\ \dot{e_{\theta_m}} & = & \sigma^{(n)}(z + B_1u)^T \mathcal{F}_m^T Pe_z \end{array} \right. \tag{9}$$

where $f(\hat{z}, z, u) = A_0\sigma^{(n)}(\hat{z} + B_1u) - A_0\sigma^{(n)}(z + B_1u)$

Let us define the Lyapunov function:

$$V = e_z^T P e_z + e_\theta^T e_\theta$$

and its associated derivative along a particular trajectory of system (4):

$$\dot{V} \;\; = \;\; -e_z^T P e_z + e_z^T P f(\hat{z}, z, u)$$

since the crossing term between e_z and e_θ is removed. Following the proof of proposition 3.1, we can prove that there exists a diagonal matrix, $P > 0$ such that:

$$P(-I + A_0 M(t)) + (-I + A_0 M(t))^T P < 0$$

with $M(t) = \dfrac{\partial \sigma^{(n)}}{\partial x}[z_r + B_1 u_r])$. Under the use of use of mean value type arguments, the previous inequality implies that

$$e_z^T P(-e_z + A_0(\sigma^{(n)}(\hat{z} + B_1 u) - \sigma^{(n)}(z + B_1 u)) < 0$$

This inequality allows to ensure that there exists $\epsilon > 0$ such that $\dot{V} \leq -\epsilon \|e_z\|^2$. On this basis, we deduce that the error are bounded and that e_z goes to zero asymptotically (see [2]). ■

As matter of fact, under some conditions on the persistent excitation [2], following the result provides in [15], we can easily prove that the estimated parameters approach their correct values, *i.e.*,

$$\lim_{t \to \infty} \hat{\theta}(t) = \theta_v$$

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