Nonlinear performance of a PI controlled missile: an explanation*

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Abstract. The primary aim of this paper is to investigate the practical interest of the incremental norm approach for analyzing (realistic) nonlinear dynamical systems. In this framework indeed, incremental stability, a stronger notion than $\mathcal{L}_2$-gain stability, ensures suitable qualitative and quantitative properties. On the one hand, the qualitative properties essentially correspond to (steady-state) input/output properties, which are *not* necessarily obtained when ensuring only $\mathcal{L}_2$-gain stability. On the other hand, it is possible to analyze quantitative robustness performance properties using

the notion of (nonlinear) incremental performance, the latter being defined in the continuity of the (linear) H_∞ performance (i.e. through the use of a weighting function). As testing incremental properties is a difficult problem, stronger, but computationally more attractive, notions are introduced, namely quadratic incremental stability and ! performance. Testing these properties reduces indeed to solving convex optimization problems over Linear Matrix Inequalities (LMIs).

As an illustration, we consider a classical missile problem, which was already treated using several (linear and nonlinear) approaches. We focus here on the analysis of the *nonlinear* behavior of this PI controlled missile: using the notions of quadratic incremental stability and performance, the closed loop nonlinear missile is proved to meet desirable control specifications.

Keywords. Incremental norm, Missile control, Nonlinear performance analysis, Linear Matrix Inequalities.

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1 Introduction

1.1 Presentation of the general problem

Even if classical engineering design methods often take into account design specifications indirectly or in an heuristic way, suitable control laws can be nevertheless obtained using these techniques. This is especially true in the non-linear context. As an alternative to the validation of these controllers using intensive time domain simulations, it would be interesting to develop an *analysis* method, which enables to guarantee that the design specifications are met by the controller.

Concerning these design specifications, which are to be satisfied by the non-linear closed loop system, we first note that this desirable qualitative properties is often defined in industrial problems in the continuity of the linear context.
As a first requirement, we try to guarantee steaty state behavior of the system, such a constant (resp. periodic) output should be asymptotically obtained as the response to a constant (resp. periodic) input.

As a second requirement, it is often necessary to take into account the variable and unpredictable nature of the reference input signals, which act on the system. Using the fact that the state vector of a system entirely summarizes the past of this system at a given instant, it suffices to analyze the behavior of the system for a given set of initial states, so as to understand the effect of the past values of the reference input on the future behavior of the closed loop system.

As a matter of fact, in the linear context, the asymptotic stability of the system automatically ensures these two requirements. This is *no longer true* in the *nonlinear* context (a simple counterexample is presented in section 5 which illustrates this classical fact).

Furthermore, some quantitative properties are desirable, which guarantee for example that the closed loop system exhibits nice properties when considering a particular type of input signal. In the linear context, this can be done through the use of linear H_∞ criteria. The main goal in this context is to define weighting functions which reflect typical specifications such as keeping the tracking error signal small with respect to the reference input signal [59].

Moreover, all above requirements have to be *robustly* satisfied despite the discrepancy between the physical system and its associated model(s). A possible way to take into account these uncertainties is also provided, in the linear context, by the use of the H_∞ norm.
In a nonlinear context, a solution to the above problems is provided by the incremental norm framework. As a matter of fact, incrementally stable systems were proved to exhibit many desirable qualitative [22, 23, 19, 18] and quantitative [21, 25] nonlinear properties.
Remember that a nonlinear system is said to be incrementally bounded (i.e. Liptchitz continuous) on $\mathcal{L}_2$ if there exists a positive finite constant η such that $\|H(u_1) - H(u_2)\|_2 \leq \eta \|u_1 - u_2\|_2$ for all u_1 and u_2 in $\mathcal{L}_2$.

Since these properties, which correspond to typical control specifications, are not necessarily obtained when only guaranteeing the $\mathcal{L}_2$-gain stability of the closed loop system (a counterexample will be presented in incremental norm framework appears especially interesting in an engineering context.

The specific aim of this paper is to show the practical interest of the incremental norm approach, by applying this framework to a realistic missile problem. This is done in two steps.

- *A realistic PI controlled missile is first proved to be incrementally stable.* Note that the missile model and specifications were extracted from [38] and that this particular control problem was intensively treated using several (linear and nonlinear) approaches (see e.g. [35, 52, 2] and included references). Note that the properties of the nonlinear closed loop missile system are often not explored (e.g. in [35, 52]).

- This incremental stability property enables to analyze many qualitative and quantitative properties of the nonlinear missile control system. Typical design specifications are especially proved to be met by the nonlinear closed loop missile.

Anticipating the following subsections, which briefly present the properties of incrementally stable systems, we would like to already emphasize two of these properties. Firstly, a constant (resp. periodic) output is proved to be asymptotically obtained as the response to a constant (resp. periodic) input. As a second point, this nonlinear closed loop system is proved to exhibit a suitable incremental nonlinear performance property: using a weighting function as in H_∞ control, it is possible to guarantee a certain behavior of the output as the response to a given set of reference input signals.

1.2 About the incremental approach to nonlinear system analysis

The incremental norm framework is useful for analyzing the properties of nonlinear closed loop systems from both qualitative and quantitative points of view. Even if we focus in this paper on some of these properties, other interesting properties could be nevertheless considered, and especially attractive sensitivity properties (see [25] for details).

In a qualitative way, incrementally stable systems possess suitable steady-state properties, and the effect of a non zero initial condition is guaranteed to decay asymptotically to zero. As a first point, a unique steady-state motion corresponds to a given input signal, independently of the initial condition and despite a vanishing perturbation on the input signal (when analyzing the response of the nonlinear closed loop system to a reference input signal, remember that introducing an unknown initial condition can be interesting, since it allows to consider the past unknown values of the reference input signal - see the first subsection). As a second point, the steady state response to a constant (resp. periodic) input signal is also constant (resp. periodic).

In a quantitative way, it is possible to analyze the robustness and performance properties of a nonlinear closed loop system. The weighted incremental norm approach was indeed originally introduced as a means to extend classical

H_∞ control concepts into a nonlinear context [17, 21, 25]: in a linear context, the original idea of [59] was to recast the initial design problem into a well defined optimization problem, involving the minimization of a *weighted* H_∞ norm. In the same way, in the incremental norm approach, the idea is to define the robustness and performance properties of a (nonlinear) system by adding a suitable weighting function, which reflects the desired properties for the closed loop.

As a matter of fact, the original motivation for developing the incremental norm framework can be more precisely found in two different classical problems, namely the extension of linear H_∞ control to the nonlinear case (see above) and the analysis and design of gain-scheduled control systems. Gain scheduled control is indeed a widespread engineering method, which was only recently addressed from a theoretical point of view (see e.g. [49, 39, 51, 50, 29]). [20] especially emphasizes that the incremental norm framework is suitable for analyzing gain scheduled controlled systems.

In most classical gain-scheduling design techniques[1], the underlying idea is to derive in an heuristic way global properties for the closed loop (nonlinear) system from local ones. As a great advantage of the incremental approach, it is possible to link in a rigorous way local and global properties of nonlinear systems.

Note finally that some basic properties of incrementally stable systems will be

recalled in the following sections. The reader is referred to [17, 21, 25] for a more complete presentation.

1.3 Practical analysis of incremental properties

Generally speaking, testing the incremental properties of a nonlinear operator is a difficult problem, which typically involves solving Hamilton-Jacobi type equations [21]. In the present paper, the main idea is rather to achieve a compromise between the conservatism and the complexity of the tests, which are associated to (incremental) stability criteria (the interest of such an idea was already emphasized by Safonov in [41, 40]).

To this purpose, we focus in this paper on the notions of *quadratic* incremental stability and performance (a similar approach was adopted, for instance, for the $\mathcal{L}_2$ gain estimation in [10]). In this new context, sufficient conditions for the incremental boundedness of a nonlinear system can be obtained, which now involve solving a Linear Matrix Inequality (LMI) problem. The underlying idea is to use a specific type of solution to the original Hamilton Jacobi type equation. Remember that LMIs correspond to convex constraints, for which efficient numerical algorithms have been proposed [4].

As a matter of fact, the analysis of the PI controlled missile will be performed in this paper using a large range of "LMI-based" criteria, from the

[1]The idea is to consider various linear time invariant (LTI) plant models, which correspond to the time invariant linearizations of the plant model at various trim points. An LTI controller is then synthesized for each of these LTI models and the nonlinear gain-scheduled controller is finally obtained as an interpolation of the LTI controllers between the trim points.

simplest one (namely, the circle criterion [57]) to the most sophisticated ones (derived from the multiplier approach, which is applied, either to the analysis of LTI uncertain systems [31] or to the analysis of input-output properties [54, 6]).

1.4 The missile problem

As a matter of fact, when designing a missile control system, the problem is threefold.

- The missile aerodynamic model is highly nonlinear.
- This aerodynamic model, which is derived from flight-mechanics equations and then identified using

  experimental data, is moreover uncertain. It is consequently necessary to take into account uncertainties in both linear and nonlinear parts of the model.
- The performance requirement is usually (very) strong. We would like indeed to ensure, firstly the fastest tracking and regulation dynamics, secondly a correct transient behavior and thirdly satisfactory robustness in the face of uncertain dynamics (actuators, bending modes, delays) and large (linear and nonlinear) aerodynamic uncertainties [13].

This control design problem typically remains unsolved from a theoretical point of view. As a matter of fact, the nonlinear nature of the problem is rarely considered even in analysis problems.

Nevertheless, a properly synthesized PI controller, whose design essentially uses the frozen-time linearizations of the missile model, often works well in practice. In the same way, it is possible to analyze the robustness properties of an LTI (Linear Time invariant) missile model in the presence of LTI model uncertainties [56, 13, 16]: the results appear here again rather satisfactory from a practical point of view. *To some extent, the contribution of the present paper is also to provide a theoretical framework and associated practical computational tools, in order to prove the (usually observed) good quality of well-known LTI designs, such as PI control.*

1.5 Content of this paper

The paper is organized as follows. The missile model and design objectives are presented in sections 2.1 and 2.2. The PI controller is presented in section 2.3, using a classical engineering method (see e.g. [27, 3]) [2].

Basic notions about input/output stability are recalled in section 3, such as the use of (extended) $\mathcal{L}_2$ spaces and the notion of (incremental) $\mathcal{L}_2$ gain. The notion of quadratic incremental stability is defined in section 4.1, while section

[2] In an obvious way, the purpose of this paper is not to propose the best PI controller, which could be synthesized for this particular problem. Our aim is rather to synthesize a PI controller in a classical way, so as to be able to then study its non linear properties in our framework.

4.2 proposes (computationally feasible) sufficient conditions for quadratic incremental stability. Qualitative properties of incrementally stable systems are presented in section 5. The closed loop missile is proved to be quadratically incrementally stable in section 6, using either the circle criterion or an LMI condition. Section 7 further investigates the quantitative robustness and performance properties of a generic nonlinear closed loop in the context of the incremental norm approach. We then focus in section 8 on a practical way to investigate the robust incremental performance properties, and apply this technique in section 9, in order to prove the incremental performance property of the missile, despite nonlinearities and neglected dynamics at the missile input (noting here again that the performance is defined through the use of a weighting function). Concluding remarks end the paper.

2 Missile model and design objectives

The missile model is described in the first subsection. The second one contains the design objectives, while the third one is devoted to the design of the PI controller.

2.1 Missile model

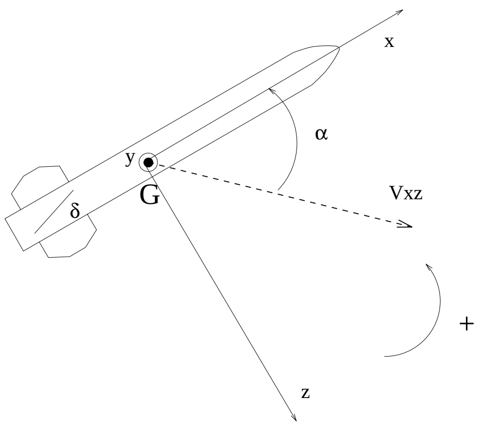


Figure 1: Definition of the missile variables.

We extract from [38] the pitch axis model of a missile, which flies at Mach 3 and at an altitude of 20,000 ft. Because of its realistic nature, the associated control problem was intensively considered: see e.g. [35, 52, 2].

The idea is to use the tail deflection δ so as to track an acceleration maneuver. The missile is modeled as a rigid body (see figure 1), with control input δ and measured outputs η (acceleration output) and q (rate output). The state of the missile involves the angle of attack α and the pitch rate q:

$$\begin{aligned}\dot{\alpha} &= cos(\alpha)K_\alpha M C_n(\alpha,\delta,M) + q \\ \dot{q} &= K_q M^2 C_m(\alpha,\delta,M)\end{aligned} \tag{1}$$

whereas the normal acceleration output η is given by:

$$\eta = \frac{K_z}{g} M^2 C_n(\alpha,\delta,M)$$

The stability derivatives C_n and C_m are nonlinear functions of α and δ:

$$\begin{aligned}C_n(\alpha,\delta,M) &= a_n\alpha^3 + b_n|\alpha|\alpha + c_n(2 - M/3)\alpha + d_n\delta \\ C_m(\alpha,\delta,M) &= a_m\alpha^3 + b_m|\alpha|\alpha + c_m(-7 + 8M/3)\alpha + d_m\delta.\end{aligned}$$

For this specific missile model, the two nice polynomial functions above are thus given. Note nevertheless that these coefficients are generally only poorly known in practice, or even that these functions are not determinated. It is interesting to emphasize that our approach could be nevertheless applied even in these cases.

See Table 1 for the description of various missile variables and Appendix A for the associated numerical data.

α	angle of attack (radians)
q	pitch rotational rate in the plane (Gx, Gz) (rad/s)
M	Mach number
δ	actual tail deflection angle (radians)
δ_c	commanded tail deflection angle (radians)
η_c	commanded acceleration in g's
η	actual acceleration in g's

Table 1: Missile variables.

The actuator is modeled as a second order linear transfer function:

$$\ddot{\delta} = -\omega_a^2\delta - 2\xi_a\omega_a\dot{\delta} + \omega_a^2\delta_c$$

where δ is the actual tail deflection and δ_c the commanded tail deflection.

An open loop analysis of this missile model is first performed for a given Mach M and for a given tail deflection δ, namely $M = 3$ and $\delta = 0$. A highly nonlinear behavior is exhibited (see [47] for details). In fact, one can note:

1. the existence of 2 stable and 1 unstable (physically possible) equilibrium points (α, q), respectively $(4.69\times10^{-2}, 3.24\times10^{-2})$, $(-4.69\times10^{-2}, -3.24\times10^{-2})$ and $(0, 0)$;

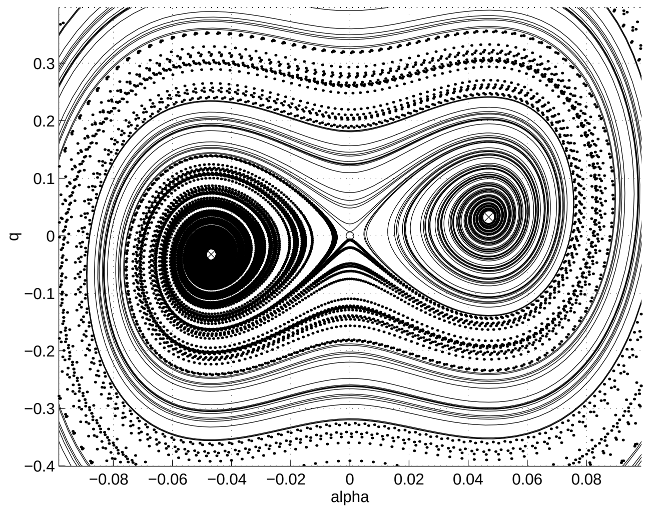


Figure 2: Trajectories associated to the two equilibrium points

2. a complicated stability domain in the space of the state vector, see figure 2: the dotted lines correspond to trajectories converging to the first stable equilibrium point $(-4.69 \times 10^{-2},\ -3.24 \times 10^{-2})$, and the full lines to the trajectories converging to the second stable equilibrium point $(4.69 \times 10^{-2},\ 3.24 \times 10^{-2})$;

3. a strong discrepancy between the behavior of the nonlinear system and the behavior of its time invariant linearizations. Depending on the considered equilibrium point, these linearizations are indeed stable or unstable.

2.2 Design objectives

When applying a step input signal to the reference input $\eta_c(t)$, we would like to obtain a time constant which is less than $0.35\,s$, a maximal overshoot no greater than 10% or 20% and a steady state error which is less than 5%. This performance must be obtained despite the coarse modeling of the stability derivatives over the operating range $-20° \leq \alpha(t) \leq +20°$.

The controller must moreover track a sinusoidal command with low or middle frequency, i.e. a frequency less than 2 rad/s, in $\eta_c(t)$ with a sufficient accuracy (less than 10 % of tracking error). This last requirement indeed corresponds to a classical maneuver specification.

2.3 PI controller design

A PI controller is designed on the time invariant linearization of the nonlinear system at $\alpha_0 = 0.15$ radians (about 8.6 degrees), by applying a classical engineering method (see e.g. [27]). The controller has the following structure (see

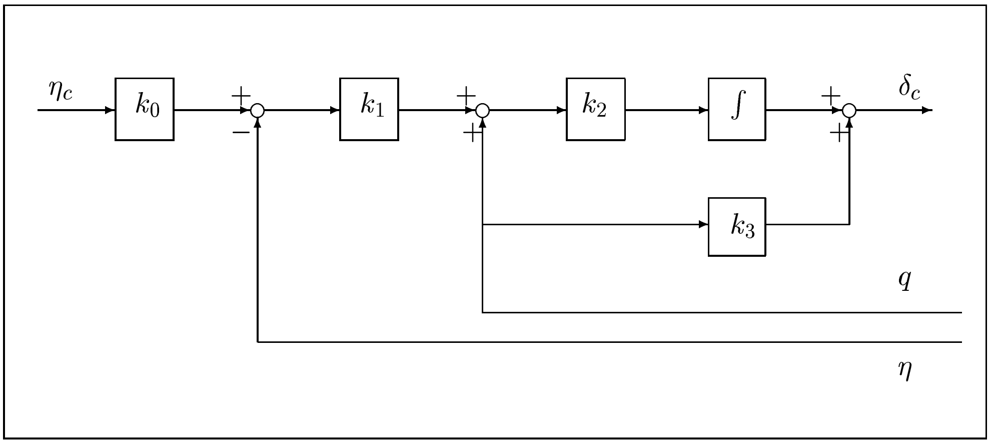


Figure 3: PI controller structure.

figure 3):

$$u(s) = \frac{k_3 s + k_2}{s} q(s) + \frac{k_1 k_2}{s}(k_0 \eta_c(s) - \eta(s)).$$

with

$$k_0 = 1.12, \quad k_1 = 0.0867, \quad k_2 = 2.5 \quad \text{and} \quad k_3 = 0.5.$$

For a reference signal η_c, k_0 is obtained by pointing out that, at the equilibrium point (q_e, η_e), the input of the integrator in the PI controller must tend towards 0. As a consequence, we have:

$$k_1(k_0\eta_c - \eta_e) + q_e = 0.$$

On the other hand, from the missile equations (1):

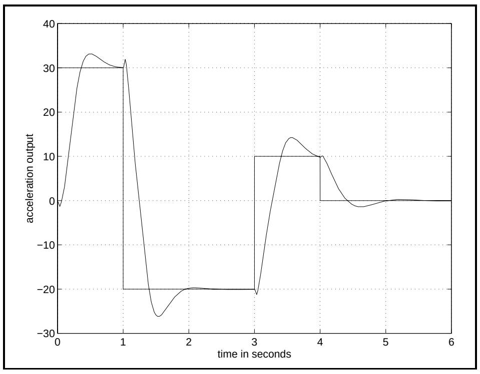
$$q_e \approx -g\frac{K_\alpha}{K_z M}\eta_e.$$

As a consequence, $\eta_e = \eta_c$ if

$$k_0 = g\frac{K_\alpha}{K_z M k_1} + 1 = 1.12.$$

Nonlinear simulations emphasize the good results obtained with this simple PI controller (see figure 4). Note that despite the highly nonlinear behavior of the open loop missile, a satisfactory closed-loop system is achieved with only one set of controller parameters, i.e. it is not necessary to use a gain scheduled PI controller. As a matter of fact, it can be observed that when the "worst-case" linearization is correctly controlled, then the nonlinear system appears also correctly controlled, using the same PI controller.

Up to now, performance was estimated using intensive simulations (such as the ones above) or by analyzing time invariant linearizations of the closed loop [56, 13, 16]. The purpose of this paper is to propose performance and robustness criteria (and associated computationally efficient tests), which enable to *guarantee* the performance of a PI controlled *nonlinear* missile.



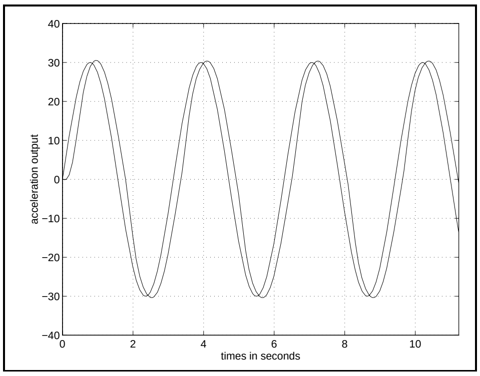


Figure 4: Acceleration output as the response to a sequence of step commands (left) and to a sinusoidal command (right).

3 Input/output stability: some recalls

3.1 Notations and definitions

The notations and terminology, which are recalled hereafter, are classical in the input-output context (see [6, 54]). The $\mathcal{L}_2$ spaces are introduced as the spaces of $\mathcal{R}^n$ valued functions defined on $[0, +\infty)$, for which the quadratic norm is integrable. $< .,. >$ denotes the scalar product, defined on $\mathcal{L}_2$: $< f, g >= \int_0^{+\infty} f(t)^T g(t) dt$. The $\mathcal{L}_2$ norm is defined as: $\|f\|_2 = \sqrt{\int_0^{+\infty} \|f(t)\|^2 dt}$ where $\|.\|$ denotes the Euclidean norm. The *causal truncation* at T of a function f, defined on $[0, +\infty)$, is denoted $P_T f$ and is defined by $P_T f(t) = f(t)$ for $0 \leq t \leq T$ and 0 otherwise. The *extended space* associated with $\mathcal{L}_2$, which is denoted as $\mathcal{L}_2^e$, corresponds to the space of $\mathcal{R}^n$ valued functions defined on $[0, +\infty)$, whose causal truncations belong to $\mathcal{L}_2$.

We consider a nonlinear system which is described by the following differential equation:

$$\Sigma_{x_0} \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)) \\ y(t) &= h(t, x(t), u(t)) \\ x(0) &= x_0 \end{cases} \tag{2}$$

where $x(t) \in \mathcal{R}^n$, $y(t) \in \mathcal{R}^m$, and $u(t) \in \mathcal{R}^p$. f and h, which are defined from $\mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^p$ into $\mathcal{R}^n$ and $\mathcal{R}^p$ respectively,

are assumed to be C^1 and to satisfy $f(t, 0, 0) = 0$ and $h(t, 0, 0) = 0$. The system is moreover assumed to be well-defined, i.e. the solution, $x(t) = \phi(t, 0, x_0, u)$, exists, is unique and belongs to $\mathcal{L}_2^e$ for all $x_0 \in \mathcal{R}^n$ and for any $u \in \mathcal{L}_2^e$ (where ϕ is the flow of equation (2) and is defined from

$\mathcal{R}^+ \times \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{L}_2^e$ into $\mathcal{R}^n$).

We can now introduce the notion of (incremental) $\mathcal{L}_2$ gain and (incremental) passivity.

Definition 3.1 *Σ_{x_0} is said to be a weakly finite ($\mathcal{L}_2$) gain stable system if there exist $\gamma \geq 0$ and $\beta \geq 0$ such that $\|\Sigma_{x_0}(u)\|_2 \leq \gamma\|u\|_2 + \beta$ for all $u \in \mathcal{L}_2$. Σ_{x_0} is said to be finite ($\mathcal{L}_2$) gain stable when $\beta = 0$. The ($\mathcal{L}_2$) gain of Σ_{x_0}, denoted by $\|\Sigma_{x_0}\|_i$, is defined as the minimal value of γ.*

Definition 3.2 *Σ_{x_0} has a finite incremental gain if there exists $\eta \geq 0$ such that $\|\Sigma_{x_0}(u_1) - \Sigma_{x_0}(u_2)\|_2 \leq \eta\|u_1 - u_2\|_2$ for all $u_1, u_2 \in \mathcal{L}_2$. The incremental gain of Σ_{x_0}, denoted $\|\Sigma_{x_0}\|_\Delta$, is defined as the minimal value of η. Σ_{x_0} is said to be incrementally stable if it is stable, i.e. it maps $\mathcal{L}_2$ to $\mathcal{L}_2$, and if it has a finite incremental gain.*

Definition 3.3 *Σ_{x_0} is incrementally passive if the scalar product $< \Sigma_{x_0}(u_1) - \Sigma_{x_0}(u_2)|u_1 - u_2 > \geq 0$ for all $u_1, u_2 \in \mathcal{L}_2$. If there exists $\eta \geq 0$ such that $< \Sigma_{x_0}(u_1) - \Sigma_{x_0}(u_2)|u_1 - u_2 > \geq \eta\|u_1 - u_2\|_2^2$ for all $u_1, u_2 \in \mathcal{L}_2$, then Σ_{x_0} is strictly incrementally passive.*

Remark: An $\mathcal{L}_2$ gain stable (respectively a passive), linear time invariant system is also incrementally stable (respectively incrementally passive).

3.2 Discussion

The above definitions may appear restrictive from an applicative point of view, since a limited class of possible input signals is considered for the system: as an example, a non-zero constant input does not belong to $\mathcal{L}_2$. This restriction can be nevertheless bypassed using the link between the input-output stability properties on $\mathcal{L}_2$ and its extended space $\mathcal{L}_2^e$. This link was investigated in details by Willems (see the book [54]). The extended space concept was introduced in the stability analysis framework by Sandberg (i.e. [45]).

Indeed, if Σ_{x_0} has a finite incremental gain less or equal to η, then for all $T \geq 0$ and for all $u_1, u_2 \in \mathcal{L}_2^e$, the following relation is satisfied:

$$\|P_T(y_1 - y_2)\|_2 \leq \eta\|P_T(u_1 - u_2)\|_2$$

where $y_i = \Sigma_{x_0}u_i$. This inequality clearly indicates that the input-output relation, which was already satisfied by the input signals inside $\mathcal{L}_2$, remains valid inside $\mathcal{L}_2^e$.

More generally, when studying the properties of the nonlinear system along a possible motion, the use of the extended space $\mathcal{L}_2^e$ enables to consider a much larger class of possible inputs (e.g. non-zero constant inputs). As an illustration, introducing:

$$\begin{aligned}
\mathcal{L}_\infty([0,T]) &= \{f : [0,\, T] \mapsto \mathcal{R}^n \mid \operatorname{ess\,sup}_{t\in[0,T]} \|f(t)\| < \infty\} \\
\mathcal{L}_2([0,T]) &= \{f : [0,\, T] \mapsto \mathcal{R}^n \mid \textstyle\int_0^T \|f(t)\|^2 dt < \infty\}
\end{aligned}$$

the inclusion $\mathcal{L}_\infty([0,T]) \subset \mathcal{L}_2([0,T])$ is true for each value of T. As a consequence, the extended space, which is associated with $\mathcal{L}_2$ for a specific value of T, contains all the signals which have (almost everywhere) a finite amplitude on $[0,T]$.

In conclusion, when characterizing the properties of the nonlinear system, the use of the extended space $\mathcal{L}_2^e$ enables to take into account most of the possible input signals, which are generally considered in an application.

4 Quadratic incremental stability

Since testing incremental stability of a nonlinear system is a difficult problem, we introduce a stronger notion, named *quadratic* incremental stability, which is easier to handle. This notion is defined in the first subsection, while an associated test is introduced in the second one. This (LMI-based) test is computationally efficient.

4.1 Definition and property

In this section, we focus on the input to state properties of the system: a sufficient condition of incremental stability is proposed in the following theorem for the case where $y = x$ (i.e. $h(t,x,u) = x$ in equation (2)). This theorem is an extension of a result, which was previously published in [24]. It is straightforward to extend it to the case of input to output properties (i.e. the case where $h(t,x,u) \neq x$). Discussing these properties would be nevertheless beyond the scope of this paper.

Theorem 4.1 *If there exist a symmetric, positive definite matrix P and two positive constants ϵ and σ_{f_u} such that the two following conditions:*

$$(i) \quad P\,\frac{\partial f}{\partial x}(t,x,u) + \frac{\partial f}{\partial x}^T(t,x,u)\,P \leq -\epsilon I_n$$

$$(ii) \qquad \left(\frac{\partial f}{\partial u}\right)^T \left(\frac{\partial f}{\partial u}\right) \leq \sigma_{f_u} I$$

are satisfied for all $t \in \mathcal{R}$, $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^p$, then system (2) is incrementally stable for any initial condition $x_0 \in \mathcal{R}^n$.

With reference to the quadratic stability concept, we introduce then the following definition.

Definition 4.1 *A system which satisfies the conditions of theorem 4.1 is said to be quadratically incrementally stable.*

First note that quadratic incremental stability is a stronger concept than incremental stability. A quadratically incrementally stable system is incrementally stable and, in addition, it satisfies conditions (i) and (ii) of Theorem 4.1. But quadratic incremental stability can be easier to test than incremental stability, through conditions (i) and (ii) of Theorem 4.1.

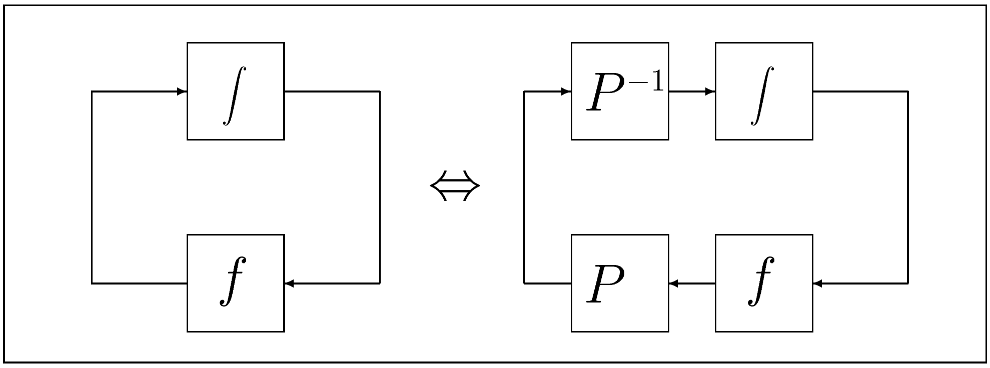


Figure 5: Incremental stability using a passivity condition

Note finally that the above theorem could be considered as reminiscent of the incremental stability theorem of [58], in the context of the connection of two (incremental) passive operators. The system $\dot{x} = f(x)$ can be indeed modeled as the connection of integrators with the (memoryless) nonlinearity $f(.)$ (see figure 5). We remark then that the incremental stability property of this system is not modified by the introduction of any (invertible) matrices P and P^{-1}. If P is positive definite, the operator $P^{-1}\int$ becomes then (incrementally) passive. Matrix P can be referred as a *multiplier* [58, 54]. Condition (i) is finally obtained by applying the (incremental) passivity theorem [58, 6], which is given below.

Theorem 4.2 *[6] Consider the well posed feedback system $u_1 = e_1 + H_2 e_2$ and $u_2 = e_2 - H_1 e_1$, where H_1 and H_2 map $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$. Let $H_1 0 = 0$ and $H_2 0 = 0$. If H_1 has finite incremental gain and is strictly incrementally passive and if H_2 is incrementally passive, then the feedback system is incrementally stable.*

A similar theorem can be obtained when considering incremental gains of the sub-systems H_1 and H_2.

Theorem 4.3 *[6] Consider the well posed feedback system $u_1 = e_1 + H_2 e_2$ and $u_2 = e_2 - H_1 e_1$, where H_1 and H_2 map $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$. Assume that H_1 (resp. H_2) is incrementally stable with the incremental gain η_1 (resp. η_2). If $\eta_1 \eta_2 < 1$, then the feedback system is incrementally stable.*

4.2 Sufficient conditions and their computation

Checking condition (i) of theorem 4.1 may reveal uneasy in practice, so that we propose in this subsection an associated sufficient condition.

• To this aim, we first assume that the set of all matrices, which are obtained for every value of x and u:

$$\frac{\partial f}{\partial x}(t, x, u)$$

is embedded in the set of matrices $F(\theta(t))$ defined as (polytopic model - see [4]):

$$F(\theta(t)) = \sum_{i=1}^{r} \theta_i(t) A_i, \quad \sum_{i=1}^{r} \theta_i(t) = 1 \quad \text{and} \quad \theta_i(t) \geq 0$$

where each $\theta_i(t)$ is a function of x and u. On the one hand, this embedding can lead to conservative results since the set of matrices $F(\theta(t))$ can be bigger than the set of matrices $\frac{\partial f}{\partial x}(t, x, u)$. But on the other hand, numerically attractive conditions, ensuring condition (i), are obtained using the polytopic model.

Condition (i) of theorem 4.1 can then be interpreted in the following way. The problem is now to find a quadratic Lyapunov function $V = \delta x^T P \delta x$, which proves the stability of the time-varying system:

$$\dot{\delta x} = F(\theta(t)) \delta x \tag{3}$$

and thus the stability of all the time varying linearizations of the nonlinear system (2).

In this context, condition (i) of theorem 4.1 holds if there exists a positive definite matrix P satisfying (see *e.g.* [40, 4]):

$$A_i^T P + P A_i < 0 \quad i = 1, \ldots, r. \tag{4}$$

Condition (ii) of theorem 4.1 can then be interpreted as testing whether the maximal singular value of the matrix $\frac{\partial f}{\partial u}(t, x, u)$ is uniformly bounded.

• A more general LFT (Linear Fractional Transformation) model can be used for $F(\theta(t))$, namely (see *e.g.* [4, 10, 48]):

$$F(\theta(t)) = A + B_p \Delta(\theta(t)) \left(I - D_{qp} \Delta(\theta(t)) \right) C_q$$

where A, B_p, C_q and D_{qp} are real matrices and where:

$$\Delta(\theta(t)) = \mathbf{diag} \left(\theta_1(t) I_{n_1}, \ldots, \theta_r(t) I_{n_r} \right).$$

Each $\theta_i(t)$ is a function of x and u, which is known to lie inside an interval $[a_i,\ b_i]$: without loss of generality, we assume in the sequel that $b_i = 0$.

Condition (i) of theorem 4.1 holds now if there exist a definite positive matrix P and r positive definite matrices S_i, with dimension $n_i \times n_i$, satisfying (see *e.g.* [48]):

$$\begin{bmatrix} A^T P + PA & PB_p + C_q^T \Lambda S \\ B_p^T P + C_q \Lambda S & D_{qp}^T \Lambda S + \Lambda S D_{qp} - S \end{bmatrix} < 0 \tag{5}$$

with $\Lambda = \mathbf{diag} \left(a_1 I_{n_1}, \ldots, a_r I_{n_r} \right)$ and $S = \mathbf{diag} \left(S_1, \ldots, S_r \right)$.

Remarks:
(i) Conditions (4) and (5) correspond to a feasibility problem over Linear Matrix Inequalities [4], which can be efficiently solved using *e.g.* the technique of

[53].
(*ii*) More accurate models and less conservative conditions could be obtained (see e.g. [47] and included references).
(*iii*) If $r = 1$ and $n_1 = 1$, condition (5) is satisfied if and only if the (graphical) circle criterion holds [55].

5 Qualitative properties of quadratically incrementally stable systems

We will prove in the following section that the missile control system is quadratically incrementally stable. The present section presents the qualitative properties, which are associated to this class of nonlinear systems (see [22, 23, 19] for the proofs of the theorems, which are presented in this section).

The first subsection considers the case of a fixed input signal with a perturbation of the initial condition. The second subsection is devoted to the characterization of the system trajectory variation, with respect to a variation of the input signal. As a consequence of these results, there exists a unique steady-state motion, which corresponds to a given input signal, independently of the initial condition and despite a vanishing perturbation on the input signal. More specifically, in the final subsection, we present a suitable steady-state property for quadratically incrementally stable systems, namely the steady state response to a constant (resp. periodic) input signal is a constant (resp. periodic) output signal.

We first motivate our interest for the incremental norm approach with the following example.

5.1 Motivating example

We consider the following nonlinear system:

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & -70 \\ 70 & -14 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} p \\
q &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + w \\
p &= -f(q) \\
z &= p
\end{aligned}
$$

with $f(q) = 90q^3 - 200q|q| + 120q$. $\mathcal{L}_2$ gain stability between input w and output z can be proved by a straightforward application of the passivity theorem of [57, 6][3]: the system is indeed the negative connection of a passive linear time invariant system with a positive nonlinearity. However, when applying a step input w of magnitude 31, the output z does not converge to a constant value (see figure 6, left). This fact does not mean that if the system is only $\mathcal{L}_2$ gain

[3]It is also possible to prove that this system is *not* incrementally stable. This proof is omitted for the sake of brevity.

stable, then its performance is poor; it just means that the $\mathcal{L}_2$ gain stability property is not sufficient to ensure a good response to a constant input.

If f is now chosen as $f(p) = 90p^3 + 120p$, the incremental stability of the system is now proved by applying the incremental passivity theorem of [57, 6]. A numerical simulation confirms that the output z converges to a constant value when applying the same step input w as previously (see figure 6, right).

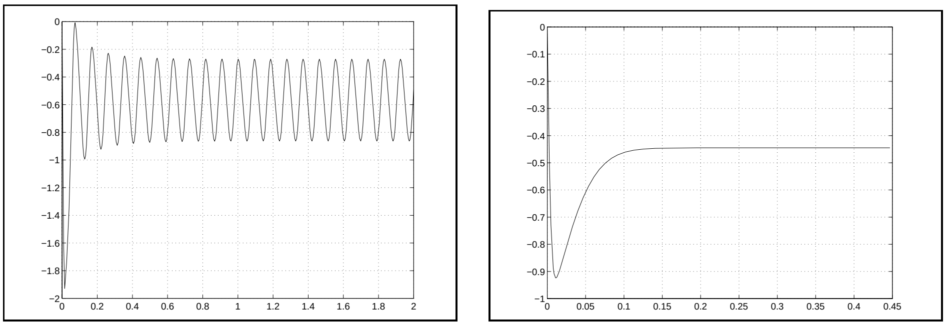


Figure 6: Time response to a step of an $\mathcal{L}_2$ gain stable system (left) and of an incrementally stable system (right)

5.2 Behavior with respect to the initial condition

We focus, in this section, on the Lyapunov property for the unperturbed motions of a quadratically incrementally stable system. Let us recall that an *unperturbed motion* is a specific motion of $\Sigma_{x_{0r}}$ associated with an input, $u_r \in \mathcal{L}_2^e$, and with an initial condition, $x_{0r} \in X$, i.e., $x_r(t) = \phi(t, 0, x_{0r}, u_r)$. In the sequel, and with reference to any specific unperturbed motion associated with Σ, we characterize the perturbation of the motion, which is associated with the same input but which is initialized with a different initial condition.

Theorem 5.1 *If Σ_{x_0} is a quadratically incrementally stable system, then all its unperturbed motions are uniformly globally exponentially stable, i.e. there exist two positive constants a and b such that for any input $u_r \in \mathcal{L}_2^e$ and any initial condition $x_{0r} \in \mathcal{R}^n$, one has*

$$\|\phi(t, t_1, x_r(t_1), u_r) - \phi(t, t_1, x_{0p}, u_r)\| \leq a\|x_r(t_1) - x_{0p}\|e^{-b(t-t_1)}$$

for all $t_1 \geq 0$ and $t \geq t_1$, any $x_{0p} \in \mathcal{R}^n$ where $x_r(t) = \phi(t, 0, x_{0r}, u_r)$.

5.3 Input-output behavior of incrementally bounded systems

In this section, the behavior of the system is characterized with respect to a perturbation on the system input. We first consider the effect on the state behavior of a vanishing perturbation.

Theorem 5.2 *Let Σ_{x_0} be a dynamical system which is quadratically incrementally stable and let us consider a specific unperturbed motion associated with an* *input $u_r \in \mathcal{L}_2^e$ and an initial condition $x_0 \in \mathcal{R}^n$. Then for any $\widetilde{u}_r \in \mathcal{L}_2^e$ such that $u_r - \widetilde{u}_r$ belongs to $\mathcal{L}_2$, the following property is satisfied:*

$$\lim_{t\to\infty} \|\phi(t,0,x_0,u_r) - \phi(t,0,x_0,\widetilde{u}_r)\| = 0$$

As a matter of fact, this theorem implies that the asymptotic behavior of the system depends on the asymptotic behavior of the input. As a consequence, the system has a fading memory property since the effects of the past vanish. We now consider the effect of a persistent perturbation with a finite amplitude.

Theorem 5.3 *Let Σ_{x_0} be a dynamical system which is quadratically incrementally stable. Let us consider a specific unperturbed motion associated with an input $u_r \in \mathcal{L}_2^e$ and an initial condition $x_{0r} \in \mathcal{R}^n$. Then, for any finite constant K_u and for any input $\widetilde{u}_r(t)$ satisfying $\|u_r(t) - \widetilde{u}_r(t)\| \leq K_u$, there exists a finite constant K_x such that*

$$\|\phi(t,0,x_{0r},u_r) - \phi(t,0,x_{0p},\widetilde{u}_r)\| \leq K_x$$

for all $t \geq 0$ and for all $x_{0p} \in \mathcal{R}^n$. Moreover, $\lim_{K_u\to 0} K_x = 0$ if $x_{0r} = x_{0p}$.

In other words, a perturbation with a bounded magnitude generates a perturbation of the motion, which has also a bounded magnitude.

As a consequence of the two above theorems, the nonlinear system has a unique steady state motion for a given input signal.

5.4 Constant and periodic inputs

In the context of quadratically incrementally stable systems, the steady state response to a constant (resp. periodic) input signal is a constant (resp. periodic) output signal.

Theorem 5.4 *Let Σ_{x_0} be a stationary and quadratically incrementally stable system. If the input of the system, namely u_r, is a T-periodic input, then the motion of the system is asymptotically T-periodic. Moreover, there exists at least an initial condition for which the motion is T-periodic.*

Corollary 5.5 *Let Σ_{x_0} be a stationary and quadratically incrementally stable system. The motion which is associated to a constant input tends asymptotically to a constant. Moreover, there exists an equilibrium point for each possible constant input.*

Remarks:
(i) The proofs of all previous results are available in [19]. Some indications concerning these proofs are given in appendix B.
(ii) Theorem 5.4 can be extended to the larger class of almost periodic input signals [18, 19].
(iii) The theorems of this section holds if the considered nonlinear system is incrementally stable, instead of *quadratically* incrementally stable (with some assumptions about the system realization - see [19]).

6 The PI controlled missile is quadratically incrementally stable

Two different methods are used in this section to prove the incremental stability of the PI controlled missile. The first method is a straightforward application of the circle criterion to the LFT model of the time varying linearization of the missile, whereas the second method relies on an LMI formulation combined with a polytopic modeling. The first approach is rendered possible by the special structure of the considered problem. For more general problems, only the second method can be applied. In both approaches, the family of linearizations is modeled in a rather brute way, so as to take into account the uncertainties in the nonlinear aerodynamic model.

6.1 LFT model and the circle criterion

The first point is to note that the linearizations of the nonlinear *closed loop* missile model along any trajectory can be written as the connection of a linear time invariant system $h(s)$ with a time-varying gain $k(t)$ [4]. A direct calculation gives these time varying linearizations as :

$$\delta\dot{x}(t) = (A + bk(t)c)x(t)$$

where $k(t) = (3a_n\alpha_0(t)^2 + 2b_n|\alpha_0(t)|)$ for the missile linearization at $\alpha_0(t)$ and

$$\left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] = \dots$$

$$\dots \left[\begin{array}{ccccc|c} K_\alpha M c_n(2-\frac{M}{3}) & 1 & K_\alpha M d_n & 0 & 0 & K_\alpha M \\ K_q M^2 c_m(-7+8\frac{M}{3}) & 0 & K_q M^2 d_m & 0 & 0 & 2K_q M^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 w_a^2 & -w_a^2 & -2\xi_a w_a & w_a^2 & 0 \\ -k_1 k_2 \frac{K_z}{g} M^2 c_n(2-\frac{M}{3}) & k_2 & -k_1 k_2 \frac{K_z}{g} M^2 d_n & 0 & 0 & -k_1 k_2 \frac{K_z}{g} M^2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

For the a priori chosen interval of variation of $\alpha(t)$ (namely $\pm$ 20 degrees), the corresponding interval of $k(t)$ is $[-15,\, 0]$: $k(t)$ is consequently assumed to belong to the sector $\{-15,\, 0\}$[5], and the closed loop system can be rewritten as the connection of a linear time invariant system, defined by the transfer function $h(s) = d + c(sI - A)^{-1}b$ with the time varying gain $k(t)$.

[4] To obtain in a simple way such an interconnection, we use the fact that $cos(\alpha) \approx 1$, $a_m \approx 2a_n$ and $b_m \approx 2b_n$ in the missile model data. These approximations do not affect the results and simplify the calculations. They particularly allow the use of simple standard criteria to prove the incremental stability property.

[5] A nonlinearity f is in the sector $\{a,\, b\}$, with $a < b$,if

$$\forall q, \quad (f(q) - aq)(f(q) - bq)) \leq 0$$

.

When applying the results of section 4 (Remark (iii)), the closed loop nonlinear missile is quadratically incrementally stable if the conditions of the circle criterion hold. As the real part of the Nyquist diagram of $h(j\omega)$, which is represented on figure 7, is greater than $-\frac{1}{15} \approx -0.066$ [57], the circle criterion guarantees the quadratic stability of the (time varying) linearizations of the missile.

Condition (i) of theorem 4.1 consequently holds [55]. Furthermore, as $\dfrac{\partial f}{\partial u}$ is a constant matrix in our case, condition (ii) of theorem 4.1 is also satisfied. It can thus be claimed that the PI controller ensures the quadratic *incremental* stability of the (nonlinear) closed loop missile.

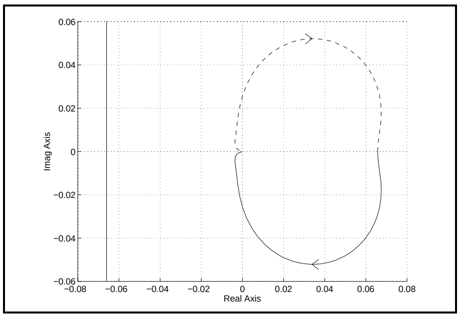


Figure 7: Nyquist diagram of $h(j\omega)$

6.2 An LMI approach

In the case of a more complex example, checking condition (i) of theorem 4.1 would need a more sophisticated approach, based on LMI optimization. To this purpose, a polytopic model can be obtained in the case of our missile example. Let:

$$\frac{\partial f}{\partial x}(t, x, u) \quad = \quad A + k(t)bc$$

where the time varying gain $k(t)$, the state space matrix A and vectors b and c are defined in the above subsection. The nonlinear closed loop missile is then quadratically incrementally stable if there exists a positive definite matrix P satisfying:

$$A^T P + PA < 0 \quad \text{and} \quad (A - 15bc)^T P + P(A - 15bc) < 0.$$

Remember indeed that $k(t)$ belongs to the sector $\{-15,\ 0\}$. Using then the LMI optimization code **SP** of [53] and the Matlab interface for this code, LMITOOL [28], a matrix P satisfying the above equation can be obtained as:

$$P = \begin{bmatrix} 6.5479 & 0.0974 & -0.0927 & -0.0009 & -0.6899 \\ 0.0974 & 0.2843 & -0.3851 & -0.0033 & 0.6318 \\ -0.0927 & -0.3851 & 0.7955 & 0.0056 & -0.7863 \\ -0.0009 & -0.0033 & 0.0056 & 0.0001 & -0.0069 \\ -0.6899 & 0.6318 & -0.7863 & -0.0069 & 2.3722 \end{bmatrix}.$$

7 An incremental approach for nonlinear control

It is illustrated in this section that some robustness and performance problems of multi-input multi-output closed loop nonlinear systems can be formulated, in the same way as in the linear context [59, 8], as well posed optimization problems. These aspects were already presented in [17, 21, 15, 25]. Such an approach is based on the induced norm of a certain augmented system. All results are extracted from [17, 25].

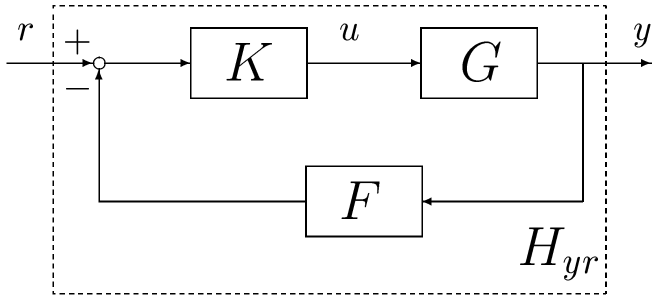


Figure 8: Nonlinear feedback system

The nonlinear feedback system of figure 8 is considered, where F, G and K are nonlinear causal operators from $\mathcal{L}_2^e$ to $\mathcal{L}_2^e$. G can be interpreted as the plant model and F and K as a two-degree-of-freedom controller. The closed loop input-output map $H_{yr} = GK(I + FGK)^{-1}$ is assumed to be well-posed.

We consider, in this section, the following problems:

Nominal incremental performance: how performance can be captured in the incremental framework, when there is no uncertainty in the plant model,

Robust incremental stability: how to guarantee the stability of the closed loop system despite uncertainties,

Robust incremental performance: how to ensure incremental performance despite uncertainties.

The last subsection is dedicated to (LMI based) criteria for the practical check of these properties.

7.1 Tracking and asymptotic properties

The qualitative notion of performance is quite difficult to handle in the non-linear input-output context. We first recall the approach of [7], in which the performance is defined as the ability for the system to minimize "asymptotically" the gain between the inputs of interest r and the error signals e (see figure 9). More precisely, denoting as R_d^e the set of input signals of interest, the following definition is introduced.

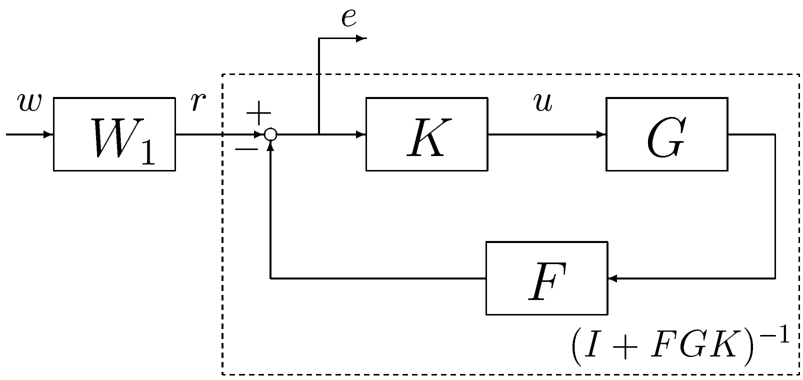


Figure 9: Tracking performance analysis

Definition 7.1 *The asymptotic performance of the closed loop system of figure 9 is satisfied on* R_d^e *if there exists* $T_0 \geq 0$ *such that :*

$$\|P_T(I + FGK)^{-1}r\|_2 \ll \|P_T r\|_2 \;\; \forall r \in R_d^e$$

for $T \geq T_0$.

This definition ensures that the relation $H_{yr} \approx I$ is asymptotically satisfied on R_d^e (see figure 8). As in the H_∞ context [59], we will consider hereafter that there exists an invertible operator W_1 satisfying the following inequality:

$$\|P_T W_1^{-1}(r)\|_2 \ll \|P_T r\|_2$$

when r belongs to R_d^e and where T is taken sufficiently large. Such an operator W_1 characterizes the set of input signals R_d^e. We have then the following result.

Theorem 7.1 *The asymptotic performance of the closed system of figure 8 is guaranteed on* R_d^e *if*

$$\|(I + FGK)^{-1}W_1\|_\Delta \leq 1$$

Our initial aim was to put the asymptotic performance requirement under the form of a weighted criterion: in this context, a *weighted* $\mathcal{L}_2$ *type criterion* appears *a priori* especially suitable (rather than the weighted incremental type criterion of Theorem 7.1). A weighted $\mathcal{L}_2$ type criterion has indeed the following form: the asymptotic performance is guaranteed on R_d^e if

$$\|(I+FGK)^{-1}W_1\|_i \leq 1$$

.

There is however a problem when the initial condition problem is taken into account. Indeed, when the initial condition of an unbiased operator (i.e. $H(0)=0$) changes, the nonlinear operator generally becomes a biased operator (i.e. $H(0)\neq 0$) [6]. It is easily proved in this case that the initial criterion, namely $\|(I+FGK)^{-1}W_1(w)\|_2 \leq \|w\|_2$ boils down to:

$$\|(I+FGK)^{-1}W_1(w)\|_2 \leq \|w\|_2 + \beta$$

where β is a strictly positive constant. As a consequence, the performance of the system is now given by:

$$\|(I+FGK)^{-1}(r)\|_2 \leq \|W_1^{-1}(r)\|_2 + \beta$$

Therefore, the value of β limits the performance of the system, and this means that an $\mathcal{L}_2$ type criterion does not guarantee the robustness of the asymptotic performance property with respect to the initial condition problem: as illustrated in section 5, the use of an incremental type criterion allows to bypass this problem.

7.2 Robustness against unstructured uncertainties

We consider the case of unstructured uncertainties $\widetilde{\Delta}$, which are inserted at the input or output of the plant model G. This model perturbation $\widetilde{\Delta}$ may represent uncertainties on the actuator (for an example, see figure 10) or sensor dynamics, and more generally neglected dynamics.
We assume that $\widetilde{\Delta}$ belongs to a set defined as:

$$\Omega_{\widetilde{\Delta}} = \left\{ \widetilde{\Delta} = W_3 \Delta W_2 \mid \|\Delta\|_\Delta < 1 \right\}$$

where Δ is a (possibly nonlinear) causal operator from $\mathcal{L}_2^e$ to $\mathcal{L}_2^e$ and W_2 and W_3 are known, causal, incrementally stable input-output maps from $\mathcal{L}_2^e$ to $\mathcal{L}_2^e$.

A stability result is proposed for the interconnected system of figure 11 (M is a generic nominal closed loop system - see for instance figure 10). In this case, the (internal incremental) stability property corresponds to the incremental stability property of the (well posed) operator defined by the inputs u_1 and u_2 and the outputs y_1 and y_2.

Theorem 7.2 *[17, 25] If M is incrementally stable and if the following inequality holds:*

$$\|W_2 M W_3\|_\Delta \leq 1$$

then the closed loop system of figure 11 is incrementally stable for any uncertainty $\widetilde{\Delta}$ belonging to $\Omega_{\widetilde{\Delta}}$.

[6] with some assumptions concerning the reachability of the new initial condition from the previous one.

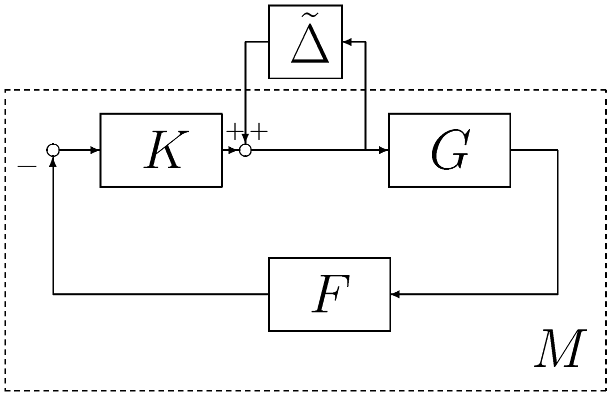


Figure 10: Input (actuator) uncertainty

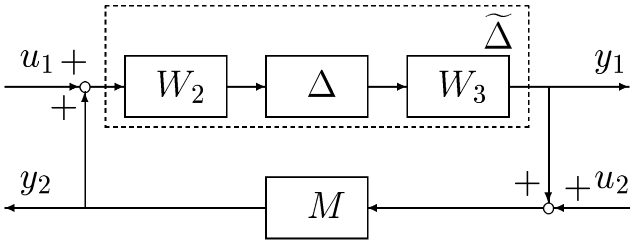


Figure 11: Robustness analysis of the nonlinear feedback system

7.3 Performance with uncertainties

We first define the notion of robust performance.

Definition 7.2 *The performance of the closed system of figure 12 is incrementally robust if it holds for every $\widetilde{\Delta} \in \Omega_{\widetilde{\Delta}}$, that is:*

$$\forall \widetilde{\Delta} \in \Omega_{\widetilde{\Delta}}, \quad \|(I + FG(I - \widetilde{\Delta})^{-1}K)^{-1}W_1\|_\Delta \leq 1$$

In the same way as in the linear case [17], Theorems 7.1 and 7.2 can be combined to capture both performance and uncertainties in a single statement.

8 How to (practically) check the conditions presented in section 7 ?

A class of interconnected systems is introduced in this section. The closed loop PI controlled missile will be indeed rewritten under this form in section 9. Note that the stability analysis of a related class of interconnected systems was recently considered in [33]. In this section, we adapt classical [58, 54, 6] as well as more recent results, which were developed for the stability analysis

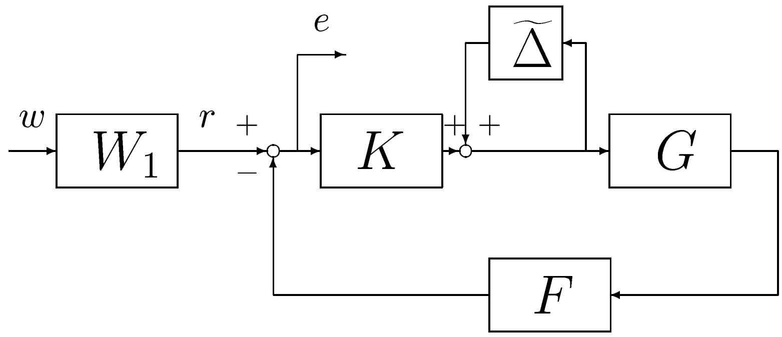


Figure 12: Robust incremental performance

of uncertain *Linear Time Invariant* systems, in order to obtain directly applicable conditions, ensuring the robust *incremental* stability and performance properties described in the previous section.

8.1 Interconnected systems

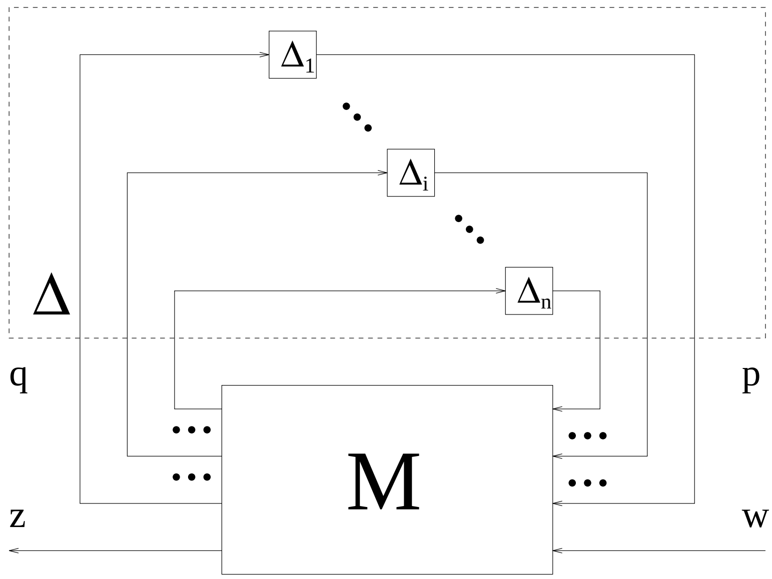


Figure 13: Interconnected system

In order to obtain computable conditions, we focus on the case when the closed loop system can be rewritten as the connection of a stable linear time invariant (LTI) system M (whose Laplace transform is $M(s)$) with a possibly nonlinear and time varying operator Δ:

$$\begin{aligned} p &= \Delta(q) \\ \begin{bmatrix} q \\ z \end{bmatrix} &= M\left(\begin{bmatrix} p \\ w \end{bmatrix}\right) \end{aligned} \tag{6}$$

where e is the output of the system and w its input. p and q are internal signals. The LTI system M models the part of the system without uncertainties, whereas the global system (6) is perturbed by nonlinearities and (dynamical) uncertainties in Δ. M can be split into four sub-blocks, with respect to the signals dimensions, in the following way:

$$M(s) = \begin{bmatrix} M_{qp}(s) & M_{qw}(s) \\ M_{zp}(s) & M_{zw}(s) \end{bmatrix}.$$

To this system, is associated the feedback connection M_{qp}-Δ defined by:

$$p = \Delta(q) \quad \text{and} \quad q = M_{qp}(p).$$

We assume that the closed loop is well-posed[7]. The operator Δ, which thus models the uncertain and nonlinear part of the system, has the following structure:

$$\Delta \triangleq \mathbf{diag}\left(\Delta_1, \ldots, \Delta_i, \ldots, \Delta_r\right). \tag{7}$$

Each sub-block of Δ belongs to a certain class of uncertain and nonlinear systems.

- Δ_i may be an (unstructured) model uncertainty: it is thus a multi-input multi-output operator, which is only known, either to have an incremental norm less than 1, or to be incrementally passive; depending on the considered problem, Δ_i can be nonlinear time variant or linear time invariant;

- Δ_i may be otherwise a single input, single output static nonlinearity, i.e. $\Delta_i = f_i$, where f_i is a nonlinearity which is known to be incrementally passive, or to have an incremental gain less than one.

In the sequel, we thus consider the case where the Δ operator is incrementally passive or has an incremental gain less than one. Several additional classes could be nevertheless considered, such as uncertain time invariant gains (i.e. LTI parametric uncertainties) or uncertain time delays (see *e.g.* [46]).

Note finally that an interconnected system, with incrementally passive subsystems, can be transformed into an interconnected system, with sub systems whose incremental gains are less one, via the so-called *loop shifting* transformation (see *e.g.* [6]).

[7] This assumption is mild. It can be proved that the conditions obtained in the sequel guarantee the well-posedness (see the book [54]).

8.2 Interconnected system analysis

The system (6) is said to be *incrementally stable* if the associated feedback connection M_{qp}-Δ is internally incrementally stable, that is, the operator from $(w_1,\, w_2)$ to $(q,\, p)$ defined in figure 14 is incrementally stable.

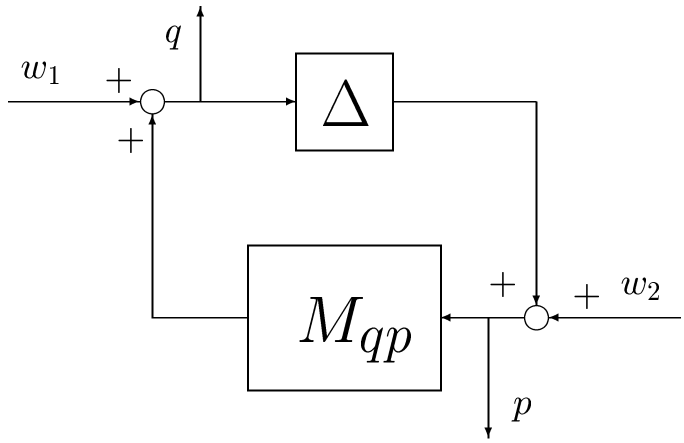


Figure 14: Internal stability

One way to prove the incremental stability of this interconnection is to directly apply theorem 4.2 or theorem 4.3. Nevertheless, such an approach can lead to a conservative criterion. As it was already pointed out in subsection 4.1, one way to reduce this conservatism is to introduce *multipliers* and/or *scalings*. Multipliers are usually considered when the sub-systems are (incrementally) passive (see *e.g.* [54, 6, 1, 30]), whereas scalings are introduced in the context of the (incremental) small gain theorem (see *e.g.* [9]). The basic idea is to transform the closed loop M_{qp}-Δ into a new closed loop $\widetilde{M}_{qp}$-$\widetilde{\Delta}$, which has the same stability properties, i.e. M_{qp}-Δ is stable if and only if $\widetilde{M}_{qp}$-$\widetilde{\Delta}$ is also stable. Furthermore, if Δ is incrementally passive (respectively with an incremental gain less than one), then $\widetilde{\Delta}$ must also be chosen incrementally passive (respectively with an incremental gain less than one).

Families of multipliers or scalings are now proposed for each Δ_i sub-block class. Assume

first that Δ_i is a multi-input multi-output nonlinear time variant model uncertainty or

a static nonlinearity. If its incremental norm is less than 1, then the incremental norm of the operator defined by $p = \frac{1}{\lambda}\Delta(\lambda q)$ is also less than 1, for any real $\lambda \neq 0$. λ is thus a *scaling*. On the other hand, if Δ_i is passive, then the operator $m\Delta$ is also passive for any real $m > 0$. m is thus a *multiplier*. In addition, if Δ_i is linear time invariant, then λ and m can be taken as frequency dependent, with no pole or zero on the imaginary axis [5, 31]. When considering the overall operator Δ of equation (7), and when assuming that Δ has an incremental gain less than unity, the corresponding scaling set $\mathcal{D}(\Delta)$ has then the following structure:

$$D \triangleq \mathbf{diag}\,(d_1 I, \ldots, d_i I, \ldots, d_r I)$$

where the scalars $d_i = \lambda_i^2$ possibly depend on the frequency, if Δ_i is linear time invariant.

Theorem 8.1 *System (6) is incrementally stable, for any Δ whose incremental norm is less than 1, if there exists a scaling $D \in \mathcal{D}(\Delta)$ satisfying:*

$$\textit{for all } \omega, \qquad M_{qp}(j\omega)^* D(j\omega) M_{qp}(j\omega) - D(j\omega) < 0. \tag{8}$$

Proof: The proof is derived from the previous discussion and from the application of theorem 4.3.

When Δ is incrementally passive, let us now introduce the set of multipliers $\mathcal{M}_{mul}(\Delta)$ with the following structure:

$$M_{mul} \triangleq \mathbf{diag}\,(m_1 I, \ldots, m_i I, \ldots, m_r I)$$

where m_i is possibly frequency dependent if Δ_i is linear time invariant.

Theorem 8.2 *System (6) is incrementally stable, for any Δ which is incrementally passive, if there exists a multiplier $M_{mul} \in \mathcal{M}_{mul}(\Delta)$ satisfying:*

$$\textit{for all } \omega, \qquad M_{mul}(j\omega) M_{qp}(j\omega) + (M_{mul}(j\omega) M_{qp}(j\omega))^* < 0. \tag{9}$$

Proof: The proof is derived here again from the previous discussion and from the application of theorem 4.2. See also the interpretation in section 4.1 of theorem 4.1.

Both theorems (8.1) and (8.2) can be applied to prove the robust incremental stability property, which was defined in section 7.

We now focus on the *incremental performance* evaluation. We propose conditions for ensuring that the interconnected system (6) has an incremental gain less than one. If Δ is incrementally passive, then system (6) is first transformed into a new interconnected system, whose incremental passivity ensures that the system (6) has an incremental gain less than one. Such a transformation is detailed in [6, page 216].

Let us introduce the set $\mathcal{D}^{perf}(\Delta)$ of scalings of the form:

$$D^{perf} \triangleq \mathbf{diag}\left(D, d^{perf} I\right)$$

where $D \in \mathcal{D}(\Delta)$ and d^{perf} is a positive scalar.

Theorem 8.3 *System (6) has an incremental gain less than* 1*, for any Δ whose incremental norm is less than 1, if there exists a scaling $D^{perf} \in \mathcal{D}^{perf}(\Delta)$ satisfying:*

$$\textit{for all } \omega, \qquad M(j\omega)^* D^{perf}(j\omega) M(j\omega) - D^{perf}(j\omega) < 0. \tag{10}$$

Assume now that Δ is incrementally passive and that the incremental passivity of the interconnected system (6) is to be proved. Let us introduce the set of multipliers $\mathcal{M}_{mul}^{perf}(\Delta)$ of the form: $M_{mul}^{perf} \triangleq \mathbf{diag}\left(M_{mul}, m_{perf} I\right)$ where $M_{mul} \in \mathcal{M}_{mul}(\Delta)$ and m_{perf} is a positive real scalar.

Theorem 8.4 *System (6) is incrementally passive, for any Δ which is incrementally passive, if there exists a multiplier $M_{mul}^{perf} \in \mathcal{M}_{mul}^{perf}(\Delta)$ satisfying:*

$$\textit{for all } \omega, \qquad M_{mul}^{perf}(j\omega)M(j\omega) + \left(M_{mul}^{perf}(j\omega)M(j\omega)\right)^* < 0. \tag{11}$$

The proof of both theorems 8.3 and 8.4 is beyond the scope of this paper (noting that these proofs can be obtained in a straightforward way using the $\mathcal{S}$-procedure - see [34]). Note simply that these theorems can be applied to prove the robust incremental performance property defined in section 7, as illustrated in the next section on the problem of analyzing the PI controlled missile.

9 Quantitative analysis of the PI controlled missile

9.1 Nominal (incremental) performance analysis.

We prove the incremental performance of the missile control system. This performance is defined through the use of a linear weighting function W_1, which classically defines a template between the reference input r and the tracking error e:

$$W_1(s) = 0.1\frac{s+25}{s+0.25}.$$

The purpose of this weighting is to ensure the tracking of reference signals r at low frequencies (see figure 15).

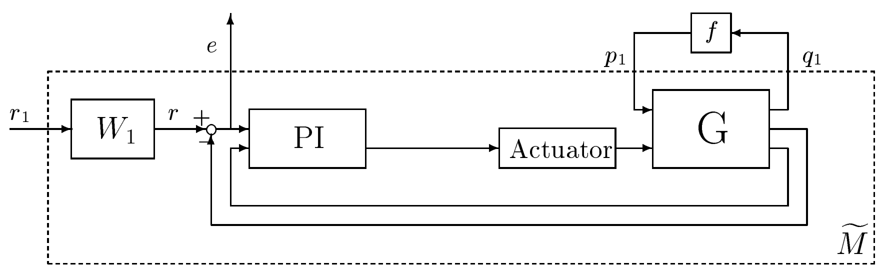


Figure 15: Nominal incremental performance analysis.

The first step is to rewrite the closed loop system as an interconnected system $\widetilde{M}$ - $\widetilde{\Delta}$, of the form (6). See figure 15 for the definition of the transfer $\widetilde{M}$, while $\widetilde{\Delta} = f$. The system is thus rewritten as the connection of an LTI system $M(s)$ with the nonlinearity $f(\alpha) = a_m\alpha^3 + b_m|\alpha|$. For the a priori chosen interval of variation of $\alpha(t)$ (namely $\pm$ 20 degrees), f is in the incremental sector

$\{-15,\, 0\}$[8]. G is the following LTI system on the figure[9]:

$$\begin{aligned} \dot{x_G} &= \begin{bmatrix} K_\alpha M c_n(2-\frac{M}{3}) & 1 \\ K_q M^2 c_m(-7+8\frac{M}{3}) & 0 \end{bmatrix} x_G + \begin{bmatrix} K_\alpha M & K_\alpha M d_n \\ 2K_q M^2 & K_q M^2 d_m \end{bmatrix} \begin{bmatrix} p_1 \\ \delta \end{bmatrix} \\ \begin{bmatrix} q_1 \\ \eta \\ q \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ \frac{K_z}{grav} M^2 c_n (2-M/3) & 0 \\ 0 & 1 \end{bmatrix} x_G + \begin{bmatrix} 0 & 0 \\ \frac{K_z}{grav} M^2 & \frac{K_z}{grav} M^2 d_n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ \delta \end{bmatrix}. \end{aligned}$$

As explained in the previous section, the obtained system $\widetilde{M}$-$\widetilde{\Delta}$ is transformed by loop shifting into a system M-Δ, where Δ is passive and the performance property corresponds to an incremental passivity property. The second step is the application of Theorem 8.2, which allows to prove the incremental stability of the closed loop for any nonlinearity f in the (incremental) sector $\{-15,\, 0\}$. Remember that function f (which summarizes the nonlinear aerodynamic model) is poorly known : this fact is explicitly taken into account by considering the whole set of nonlinearities inside the incremental sector $\{-15,\, 0\}$.

The incremental performance of the nonlinear system is then proved if condition (11) is satisfied. Note that this condition is a frequency dependent Linear Matrix Inequality constraint. Searching for $M_{mul}^{perf}(j\omega)$ satisfying condition (11) is thus an infinite dimensional optimization problem. Nevertheless, by the Kalman-Yakubovitch-Popov lemma [36], the frequency-dependent condition (11) can be rewritten as an LMI constraint with an additional "multiplier" P. The corresponding optimization problem becomes thus finite dimensional and appears computationally more tractable. As a matter of fact, this LMI condition is equivalent to a scaled real bounded lemma condition [4].

Let $(A_M,\, B_M,\, C_M,\, D_M)$ be the state-space representation of the LTI system $M(s)$. The considered problem boils down to the following optimization problem. The interconnected system is incrementally performant if there exist a positive definite matrix P and two positive scalars β and λ satisfying:

$$\left[\begin{array}{c|cc} P & 0 & 0 \\ \hline 0 & \lambda & 0 \\ 0 & 0 & \beta \end{array}\right] \left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & D_M \end{array}\right] + \left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & D_M \end{array}\right]^T \left[\begin{array}{c|cc} P & 0 & 0 \\ \hline 0 & \lambda & 0 \\ 0 & 0 & \beta \end{array}\right]^T < 0. \qquad (12)$$

Using the Matlab LMI Control Toolbox [26], we find that condition (12) is

[8] A nonlinearity f is in the incremental sector $\{a,\, b\}$, with $a < b$,if

$$\forall q_1 \neq q_2, \quad (f(q_1) - f(q_2) - a(q_1 - q_2))(f(q_1) - f(q_2) - b(q_1 - q_2)) \leq 0$$

.

[9] We use here again the fact that $cos(\alpha) \approx 1$, $a_m \approx 2a_n$ and $b_m \approx 2b_n$.

satisfied for $\lambda = 0.372$, $\beta = 0.22$ and P:

$$\begin{bmatrix} -9.86 & 0.357 & 71.2 & -1.05 & 9.74 & -8.04 \times 10^{-4} \\ 0.357 & -2.14 & -442 & 9.20 \times 10^{-3} & 0.35 & 1.66 \times 10^{-2} \\ 71.2 & -442 & -9.99 \times 10^{4} & 1.75 & 80.3 & 3.80 \\ -1.05 & 9.20 \times 10^{-3} & 1.75 & -0.114 & 1.05 & 2.67 \times 10^{-6} \\ 9.74 & 0.349 & 80.3 & 1.05 & -9.97 & -6.87 \times 10^{-3} \\ -8.04 \times 10^{-4} & 1.66 \times 10^{-2} & 3.80 & 2.67 \times 10^{-6} & -6.87 \times 10^{-3} & -2.01 \times 10^{-4} \end{bmatrix}.$$

9.2 Robust (incremental) stability analysis.

Neglected dynamics $\bar{\Delta}$ are added to the closed loop system so as to take into account model uncertainties at the actuator input (e.g. unmodeled actuator dynamics or time-delays - see figure 16). $\bar{\Delta}$ is a linear time invariant system, which is defined by:

$$\|\bar{\Delta}\|_\infty \leq 0.5. \tag{13}$$

The above relation is a generalization of the (linear) modulus margin of 0.5, which is itself a generalization of the classical 6 dB gain margin. This gain margin requirement is standard (and stringent) in the context of missile control.

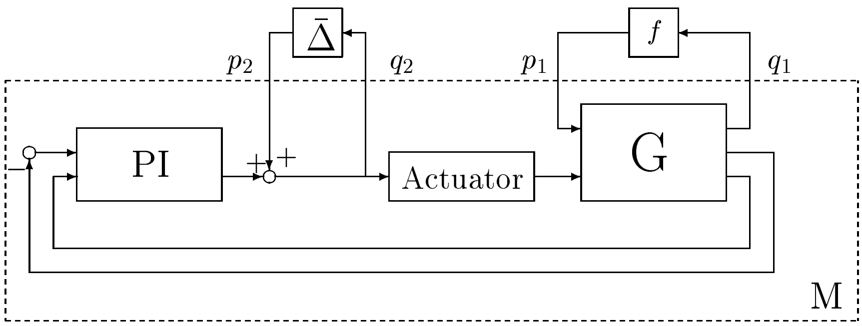


Figure 16: Robust incremental stability analysis.

As in the previous subsection, the system can be rewritten as the connection of an LTI system $M(s)$ with both the model uncertainty $\bar{\Delta}$ and the nonlinearity $f(\alpha) = a_m \alpha^3 + b_m |\alpha|$ (see figure 16). Δ thus becomes $\Delta = \mathbf{diag}(f, \bar{\Delta})$.

We would like to check the incremental stability property of the nonlinear closed loop, for all model uncertainties $\bar{\Delta}$ satisfying equation (13) and for all nonlinearities inside the incremental sector $\{-15,\ 0\}$. A sufficient condition was proposed in theorem 8.2. In fact, we consider a little more complicated problem. We would like indeed to compute an upper bound ν_{est} of the inverse input margin ν, which is defined as the smallest ν such that the system is incrementally stable, for any $\bar{\Delta}$ satisfying $\|\tilde{\Delta}\|_\infty \leq \frac{0.5}{\nu}$ and for any nonlinearity in $\{-15,\ 0\}$.

This problem appears to be a generalization [58, 37, 32] to the nonlinear context of a skewed μ problem [11, 12], noting that skewed μ analysis was originally introduced in the linear context (see for instance [42, 43] for other non-linear formulations of μ problems). For the sake of brievity, we simply indicate that, from condition (8), our problem boils down to the following optimization problem (see [47] for more details):

$$\begin{array}{rcll} \nu^2_{est}(\omega) & = & \text{minimize} & \nu(\omega)^2 \\ & & \text{over the variables} & s_1(j\omega) = s_1(j\omega)^* > 0,\ \lambda > 0 \\ & & \text{subject to} & \end{array}$$

$$\text{for all } \omega, \qquad M(j\omega)^* \mathbf{diag}(\lambda, 0.25 s_1(j\omega)) M(j\omega) - \mathbf{diag}(\lambda, \nu(\omega)^2 s_1(j\omega)) < 0 \tag{14}$$

with $\nu_{est} = \max_\omega \nu_{est}(\omega)$. Condition $\nu_{est}(\omega) < 1$ is to be satisfied for every ω. $s_1(j\omega)$ is the (frequency dependent) multiplier, which is associated to the linear uncertainty (namely the neglected dynamics), whereas λ is the (real constant) scaling, which is associated to the nonlinearity f.

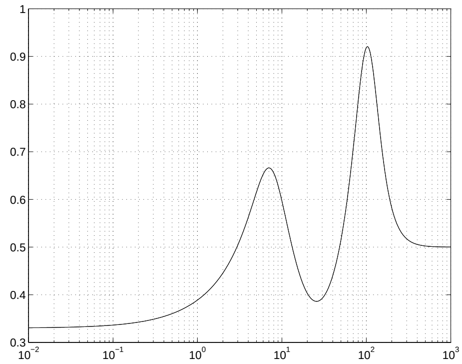


Figure 17: Generalized ν analysis.

As pointed out in [14], the previous LMI conditions are homogeneous. As a consequence, in the constraint (14), λ can be set to 1 without loss of generality. It then becomes possible to solve the above optimization problem using a conventional frequency gridding approach, since the values of $s_1(j\omega)$ at various frequencies are independent.

The problem is consequently solved in an approximate way, that is at each point of a frequency gridding (in the same way as in μ analysis problems). A generalized eigenvalue problem is solved at each frequency using the solver of the Matlab LMI Control Toolbox [26].

Figure 17 presents the values of $\nu(\omega)$ as a function of frequency ω (at the zero frequency, one obtains $\nu(0) = 0.33$). The maximal value of $\nu(\omega)$ is obtained as $\nu = 0.92$. Since this value is less than 1, it can be claimed that the robust incremental stability is guaranteed despite the unmodeled dynamics at the plant input.

9.3 Robust (incremental) performance analysis.

We now prove the *robust* incremental performance of the missile control system, with respect to the uncertainty considered in the previous subsection. We first define in the sequel a new linear weighting function W_1, which characterizes the worst case performance level (i.e. the performance level, which is still considered as satisfactory in the worst case - the weighting function W_1 of the previous subsections corresponded to a nominal performance requirement, which could thus be chosen more stringent) :

$$W_1(s) = 0.1\frac{s+25}{s+0.25}.$$

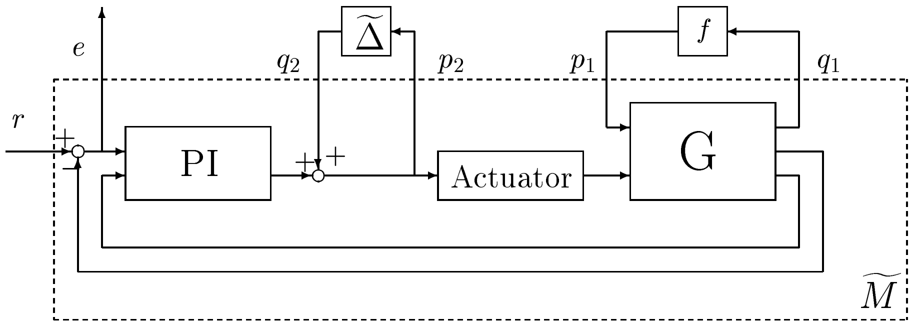


Figure 18: Robust incremental performance analysis.

Robust incremental performance is proved by checking condition (11) of theorem 8.4. In the same way as in the previous subsections, we transform the problem to fit it in the passivity framework. As previously, the problem boils down to the following optimization problem: find two positive scalars m_1 and m_3, a transfer function $m_2(j\omega) = m_2(j\omega)^*$, satisfying for all ω:

$$\mathbf{diag}(m_1, m_2(j\omega), m_3)M(j\omega) + M(j\omega)^*\mathbf{diag}(m_1, m_2(j\omega), m_3) < 0. \qquad (15)$$

where M is obtained from $\widetilde{M}$ (see figure 18) in the same way as previously, using a loop shifting.

As in the previous case (robust incremental stability), the to be tested condition corresponds to an infinite dimensional optimization problem. In contrast with the previous case and for the sake of completeness, we use a different approach. This direct (state space) approach is in the spirit of the one proposed in [44, 31, 30]. These references propose indeed a method for computing the robustness margin without frequency gridding, in the context of an LTI closed loop subject to LTI structured model perturbations: in the method of [44, 31, 30], which basically uses the passivity theorem with multipliers, these multipliers have to be restricted in practice to belonging to finite dimensional subspaces (for computational requirements). Using the Matlab LMI Control Toolbox [26], we prove the robust incremental performance. Condition (15) is satisfied for $m_1 = 1.508$, $m_2(j\omega) = 0.9069\frac{(j\omega+\sqrt{2})(j\omega-\sqrt{2}))((j\omega)^2+52.4j\omega+18911)((j\omega)^2-52.4j\omega+18911)}{(j\omega+1)(j\omega-1)(j\omega+45)(j\omega-45)(j\omega+300)(j\omega-300)}$ and $m_3 = 0.0024$.

10 Conclusion and perspectives

In this paper, we have justified through a realistic example the use of the incremental norm approach for the control of nonlinear dynamical systems.

To this aim, we have first proved that a realistic PI controlled missile is quadratically incrementally stable. The PI controller was synthesized in a conventional way. As a consequence of this result, the nonlinear missile control system was proved to exhibit many interesting qualitative and quantitative properties:

- A suitable stability property with respect to the initial condition.
- A steady state property when applying specific classes of input signals: the steady state response to a constant (resp. periodic) input signal is a constant (resp. periodic) output signal. Using the incremental norm framework, it was possible to prove this property, which was otherwise experimentally observed in nonlinear simulations and which corresponds to a design requirement.
- An incremental robustness and performance property: our framework allowed us to quantify the (nonlinear) incremental performance property. This one was simultaneously defined as the minimization of the tracking error and as the classical (linear) requirement of a 6 dB gain margin at the missile input. This last requirement was obtained in this paper in a nonlinear context.

Note that other control laws could have been used in our example. A PI controller was chosen, because of its wide use in the engineering community.

As a matter of fact, the contribution of this paper is also to present methods, which allow to understand the good results, which can be obtained when applying to a nonlinear plant a linear controller designed on the basis of LTI considerations (at least in the specific context of a missile control system).

Another contribution of this paper is to illustrate the interest of various LMI-based criteria for the analysis of realistic nonlinear systems. As a matter of fact, because of this successful application of the weighted incremental norm approach to a realistic missile example, it would be interesting now to investigate in the same way the application of this framework to other classes of nonlinear plants.

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Appendix

A Numerical data of the missile model

a_n	$1.0286\ 10^{-4}$	deg^{-3}
b_n	$-0.94457\ 10^{-2}$	deg^{-2}
c_n	-0.1696	deg^{-1}
d_n	-0.034	deg^{-1}
a_m	$2.1524\ 10^{-4}$	deg^{-3}
b_m	$-1.9546\ 10^{-2}$	deg^{-2}
c_m	0.051	deg^{-1}
d_m	-0.206	deg^{-1}
ω_a	150	
ξ_a	0.7	

P_0	$973.3\ lb/ft^2$
S	$0.44\ ft^2$
m	$13.98\ slugs$
V	$1036.4\ ft/s$
d	$0.75\ ft$
I_y	$182.5\ slug.ft^2$
K_α	$0.7PoS/m/V$
K_q	$0.7PoSd/I_y$
K_z	$0.7PoS/m$
g	32.2

Table 2: Missile parameters.

B Additional comments - sketches of proof

We further explain the content of sections 4 and 5. We first consider the incremental stability property. To this purpose, with reference to section 3, the nonlinear system:

$$\Sigma_{x_0} \ : \ \dot{x}(t) \ = \ f(t, x(t), u(t)) \quad with \quad x(0) \ = \ x_0 \tag{16}$$

is associated with the fictitious dynamical system described by equations:

$$\left\{ \begin{array}{lllll} \dot{x}_1(t) & = & f(t, x_1(t), u_1(t)) & with & x_1(0) = x_{01} \\ \dot{x}_2(t) & = & f(t, x_2(t), u_2(t)) & with & x_2(0) = x_{02} \end{array} \right. \tag{17}$$

System (16) is incrementally stable if and only if there exists a scalar γ such that, for system (17), $\|\Delta x\|_2 \leq \gamma^2 \|\Delta u\|$ with $\Delta x = x_1 - x_2$ and $\Delta u = u_1 - u_2$ [17]. All results in the sequel are based on the following lemma.

Lemma B.1 *Assume that conditions (i) and (ii) of theorem 4.1 hold for P. Let us introduce the function S, which is defined from $\mathcal{R}^n \times \mathcal{R}^n$ into $\mathcal{R}$ by:*

$$S(x_1, x_2) = x_1^T P x_1 + x_2^T P x_2 - 2x_1^T P x_2$$

where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *is the state vector of system (17).*

There exist then two positive constants ϵ_1 and β satisfying:

$$\frac{dS(x_1,x_2)}{dt} \leq -\epsilon_1 \|\Delta x\|^2 + \beta \|\Delta u\|^2. \tag{18}$$

The results of this paper can be derived from the properties of S and of its derivative along the motions of the fictitious system.

Proof of the Lemma.
The derivative of S with respect to time is given along the possible motions of system (17) by:

$$\frac{dS(x_1,x_2)}{dt} = \frac{\partial S(x_1,x_2)}{\partial x_1} f(t,x_1,u_1) + \frac{\partial S(x_1,x_2)}{\partial x_2} f(t,x_2,u_2).$$

Let us introduce $\Delta x = x_1 - x_2$, $\Delta f_x = f(t,x_1,u_1) - f(t,x_2,u_1)$ and $\Delta f_u = f(t,x_2,u_1) - f(t,x_2,u_2)$. After some straightforward manipulations, the derivative of S can be rewritten as:

$$\frac{dS(x_1,x_2)}{dt} = 2\Delta x^T P \Delta f_x + 2\Delta x^T P \Delta f_u$$

On the one hand, noting that:

$$\Delta f_x = \int_{x_2}^{x_1} \frac{\partial f}{\partial x}(t,z,u_1)\, dz = \int_0^1 \frac{\partial f}{\partial x}(t,(x_1-x_2)\lambda + x_2, u_1)\Delta x \, d\lambda,$$

one obtains:

$$\Delta x^T P \Delta f_x = \int_0^1 \Delta x^T P \frac{\partial f}{\partial x}(t,(x_1-x_2)\lambda + x_2, u_1)\Delta x \, d\lambda$$

and hence, using point (i) of theorem 4.1, one obtains:

$$\Delta x^T P \Delta f_x + \Delta f_x^T P \Delta x \leq -\epsilon \|\Delta x\|^2. \tag{19}$$

On the other hand, the Cauchy-Schwarz inequality and condition (ii) of theorem 4.1 ensure that:

$$\Delta x^T P \Delta f_u \leq \|\Delta x^T P\| \|\Delta f_u\| \leq \sigma_{f_u} \sigma_P \|\Delta x\| \|\Delta u\| \tag{20}$$

where $\Delta u = u_1 - u_2$ and σ_P is the maximal singular value of matrix P.

Using the completion of the square, inequality (18) is deduced from inequalities (19) and (20):

$$\begin{aligned}\frac{dS(x_1,x_2)}{dt} &\leq \tfrac{\epsilon}{2}\left(-\|\Delta x\|^2 - \|\Delta x\|^2 + \frac{2\sigma_{f_u}\sigma_P}{\epsilon}\|\Delta x\|\|\Delta u\|\right) \\ &\leq -\tfrac{\epsilon}{2}\|\Delta x\|^2 + \frac{\sigma_{f_u}\sigma_P}{2\epsilon}\|\Delta u\|^2.\end{aligned}$$

□

B.1 Incremental stability.

Since $P > 0$, $S(x_1, x_2) \geq 0$ for all x_1 and x_2. Using inequality (18), it can be claimed then that there exists a scalar γ satisfying for system (17) $\|\Delta x\|_2 \leq \gamma^2 \|\Delta u\|$. System (16) is thus incrementally bounded. $f(0,0,0) = 0$ allows to prove that the system is $\mathcal{L}_2$ stable and thus incrementally stable.

B.2 Behavior with respect to the initial condition.

We can now prove the exponential stability of the unperturbed motion of system (16). To this purpose, we associate with a specific motion of the system, a function V, from $\mathcal{R}^+ \times \mathcal{R}^n$ into $\mathcal{R}$, which is related to S by:

$$V(t, \Delta x(t)) \quad = \quad S(x_r(t), x_r(t) + \Delta x(t)) \quad = \quad \Delta x(t)^T P \Delta x(t) \tag{21}$$

where $x_r(t) = \phi(t, 0, x_{0r}, u_r)$ and $\Delta x(t) = \phi(t, 0, x_{0r}, u_r) - \phi(t, 0, x_{0p}, u_r)$.
Let $u_1(t) = u_2(t) = u_r(t)$, $x_{01} = x_{0r}$ and $x_{02} = x_{0p}$. We deduce from equation (18) that:

$$\frac{dV}{dt} = \frac{dS}{dt} \leq -\epsilon_1 \|\Delta x\|^2$$

Classical arguments can then be used to conclude that the unperturbed motion is exponentially stable.

B.3 Input-output behavior of incrementally bounded systems.

In the same way, the function V, which is defined by equation (21), can also be used to prove the qualitative behavior of incrementally bounded systems. Δx is now defined as the difference between the motion associated with u_r and the one associated with $\tilde{u}_r$, i.e. $\Delta x(t) = \phi(t, 0, x_{0r}, u_r) - \phi(t, 0, x_{0r}, \tilde{u}_r)$. Using inequality (18), one obtains:

$$\frac{dV}{dt} \leq -\epsilon_1 \|\Delta x\|^2 + \gamma^2 \|\Delta u\|^2$$

This last inequality is the basis of the proofs of the properties, which characterize the behavior of the system with respect to particular classes of input signals. See [22, 18] for more details.