A THEORETICAL FRAMEWORK FOR GAIN SCHEDULING

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Abstract

The weighted incremental norm approach was originally introduced as a natural framework for extending well-known H_∞ linear control concepts into the nonlinear context. In this paper, we investigate the numerous links between this new approach and the classical gain-scheduling technique. Althought based on heuristic rules, gain-scheduled control is probably the most widespread nonlinear technique. In this paper, we point out that the control objectives of the gain-scheduled controller design can be expressed as the weighted incremental norm minimization of a nonlinear operator.
The result interest is twofold: it first provides a rigorous mathematical formulation of the gain-scheduling problem. Furthermore, existing gain-scheduling techniques can be interpreted as approximate solutions to the weighted incremental norm minimization of a nonlinear operator.

1 Introduction

The gain scheduling approach is a very classical and widespread nonlinear control technique. The underlying idea is to design at one or more operating points linear time invariant controllers using the associated linearized plant models. The nonlinear control law is then obtained by interpolating (or scheduling) these controllers as a function of the operating point [1, 2, 3, 4, 5, 6, 7].

From a theoretical point of view and from previous works, it seems that the main goal of the gain scheduling approach is to ensure at least the exponential stability of the closed loop system (time varying) linearizations. This point of view is in fact a direct consequence of the Lyapunov's indirect method. Based on a converse theorem due to Lyapunov, which guarantees the existence of a suitable quadratic Lyapunov function for exponentially stable linear systems, it is possible to ensure the existence of an open ball of initial conditions, for which the nonlinear plant is also Lyapunov stable (under regularity assumptions on the nonlinear dynamics). Despite its local nature, the method presents the advantage of being rigorous. Unfortunately, testing the stability of a linear time-varying system which is associated to a nonlinear system linearization is a difficult problem, So, the following conjecture is generally used: if the frozen-time systems associated with the time varying system are stable then the time varying system is also stable. This idea, which is the foundation of most gain scheduling techniques (and which can be compared in some aspects to the famous Aizerman/Kalman's conjecture), is obviously false in the general case. This conjecture is in fact neither sufficient nor necessary. This however explains that most of the work concerning gain scheduling is devoted to the search of conditions, under which this conjecture becomes true [1, 8, 9, 10].

Actually, in the context of nonlinear control, it is well-known that the exponential stability requirement for all the closed loop system linearizations is in general case a very strong requirement. Following this remark, it seems necessary to make crystal clear if this requirement can be justified in a theoretical way.

A second point is linked to a pragmatic point of view. As a matter of fact, when controlling the nonlinear plant, a natural requirement is that the time invariant linearizations along particular constant inputs satisfy the robustness and performance criteria, classically considered in linear control. As a consequence, we focus the following question: is it possible to justify in a theoretical way this pragmatic point of view?

As we will show in the sequel, the answer to these two questions is yes. In fact, a justification can be found in the sensitivity problem. Let us recall that the desensitivity property corresponds to the the fact that the feedback control laws can attenuate the effect of non measurable system perturbations and "shrink" the model uncertainty effects. Desensitivity is in fact the major motivation (maybe the only one) for using feedback strategies versus open loop one [11, 12, 13, 14, 15, 16, 17].

A first paper conclusion is that the classical gain scheduling control objectives (constraints some system linearization properties) can be justified in many control problems, so that the main restriction of the approach is the fact that existing gain scheduling techniques achieve these objectives only in an approximate (or even heuristic) way.

A second difficulty of the gain scheduling approach is that this technique can not be considered as an approximation of any existing nonlinear control design method. This fact has been already pointed out by Wilson J. Rugh [3]:

"What is most striking about gain-scheduling is that, while it is ever more widely used in practice, it has been widely ignored from a theoretical perspective. In particular, it remains unstudied as an explicitly nonlinear control approach. So it seems that gain scheduling is another example of the lamented theory/application gap but in this case application is ahead of theory".

In other words, a global nonlinear framework for the analysis and the design of gain scheduling control systems has to be proposed. The interest of such a framework is clear: assess the mathematical complexity of the control law design, satisfying the gain-scheduling objectives in a nonlinear context.

As illustrated in the sequel, the weighted incremental norm seems to be a natural framework for this problem. First, note that the weighted incremental norm was recently introduced in [18, 19, 20] as a solution for extending well-known H_∞ linear control concepts into the nonlinear context. The nonlinear control problem is transformed into a well-defined optimization problem, which is the weighted incremental norm minimization of a nonlinear operator. This approach allows to simultaneously consider major specifications such as robust stability, sensitivity and attenuation with respect to exogenous perturbations, suitable steady state behaviors associated with step and periodic inputs and Lyapunov stability.

The main objective of this paper is thus to explain why and how this incremental approach provides a natural mathematical framework to the gain scheduling control problem. This paper is the result of our continuing effort on developing the input/output approach for improving engineering practice. One of its roots is the following Michael G. Safonov's remark[21]:

"Sector conditions can provide a rigorous justification of the approximation used in gain scheduling, and, more important from a practical point of view, they can give an engineer the insight needed to make constructive design modifications when simulation results show that the approximations have failed.".

For the sake of completness and clearness, the paper is self-contained, that is, we present some results which were elsewhere published in papers of the first author.

Paper outline Gain scheduling control techniques are shortly overviewed in section 2. In section 3, we investigate the dual nature of the incremental norm approach. In this approach, both local aspects and global aspects of the control problem can simultaneously be captured. In fact, the global behavior (respectively the incremental gain) of a nonlinear operator can be linked to the local behavior (respectively the induced gains) of its (time varying) linearizations. Section 4 focus on feedback systems: contraints on the (nonlinear) feedback system (and its (time varying) linearizations) due to desensitivity to perturbations and model uncertainties are discussed. The link with Lyapunov stability is then made in section 5. Connections between the incremental stability of the nonlinear system and the stability requirement along the system trajectories are presented. In section 6, performance specifications (desensitivy, tracking,..) and robustness are formulated as constraints on weighted incremental norms on closed loop operators of the feedback system and connections with weighted H_∞ norm of (time varying or time invariant) linearizations are developed: testing the weighted incremental norm of a nonlinear operator is equivalent to testing an infinite number of weighted induced norms of linear time-varying operators. We finally point out that incremental stability ensures steady state properties. As a corollary of this discussion, it can be claimed indeed that a gain-scheduling controller may be considered in this context as an approximate solution of the problem of minimizing the weighted incremental norm of a nonlinear operator. In section 7, incremental gain controller design is discussed. Relationships between incremental norm and existing nonlinear concepts are then proposed as a conclusion. We first recall some notations and definitions.

Notations and definition The notations and terminology, here used, are classical in the input-output context (see [22]). The $\mathcal{L}_2$-norm of $f : [t_0, \infty) \mapsto \mathcal{R}^n$ is $\|f\|_2 = \sqrt{\int_{t_0}^{\infty} \|f(t)\|^2 dt}$. The *causal truncation* at $T \in [t_0, \infty)$, denoted by $P_T f$, is defined as $P_T f(t) = f(t)$ for $t \in [t_0, \infty)$ and 0 otherwise. The *extended space*, $\mathcal{L}_2^e$, is composed with the functions whose causal truncations belong to $\mathcal{L}_2$. For convenience, $\|P_T u\|_2$ is denoted by $\|u\|_{2,T}$.
In the sequel, we consider systems with the differential representation

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & h(x(t), u(t)) \\ x(t_0) & = & x_0 \end{array} \right. \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $y(t) \in \mathcal{R}^m$, and $u(t) \in \mathcal{R}^p$. f and h, defined from $\mathcal{R}^n \times \mathcal{R}^p$ into $\mathcal{R}^n$ and $\mathcal{R}^m$ respectively, are assumed to be C^2 and uniformly Lipschitz. Moreover one has $f(x_0, 0) = 0$ and $h(x_0, 0) = 0$. The unique solution $x(t) = \phi(t, t_0, x_0, u)$ belongs to $\mathcal{L}_2^e$ for all $x_0 \in \mathcal{R}^n$ and for all $u \in \mathcal{L}_2^e$. An *unperturbed motion* (of Σ) is a specific motion of Σ associated with an input, $u_r \in \mathcal{L}_2^e$, and with an initial condition, $x_{0r} \in \mathcal{R}^n$, *i.e.* $x_r(t) = \phi(t, t_0, x_{0r}, u_r)$. With reference to this definition, we denote by $x[x_{0r}, u_r]$ this unperturbed motion.

The notion of (incremental) $\mathcal{L}_2$-gain can now be recalled.

Definition 1.1 *Σ is said to be a weakly finite gain stable system if there exist $\gamma \geq 0$ and $\beta \geq 0$ such that $\|\Sigma(u)\|_2 \leq \gamma \|u\|_2 + \beta$ for all $u \in \mathcal{L}_2$. Σ is said to be finite gain stable when $\beta = 0$. The gain of Σ coincides with the minimum value of γ and is denoted by $\|\Sigma\|_i$.*

Definition 1.2 *Σ has a finite incremental gain if there exists $\eta \geq 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta\|u_1 - u_2\|_2$ for all $u_1, u_2 \in \mathcal{L}_2$. The incremental gain of Σ coincides with the minimum value of η and is denoted by $\|\Sigma\|_\Delta$. Σ is said to be incrementally stable if it is stable, i.e. it maps $\mathcal{L}_2$ to $\mathcal{L}_2$, and has a finite incremental gain.*

Remark. The above definitions may appear restrictive from an applicative point of view, since a limited class of possible inputs is considered for the system: as an example, a non-zero constant input does not belong to $\mathcal{L}_2$. This restriction can be nevertheless bypassed using the link between the input-output stability properties on $\mathcal{L}_2$ and its extended space $\mathcal{L}_2^e$ [23, 22]. Indeed, if Σ has a finite incremental gain less or equal to η then for all $T \geq 0$ and for all $u_1, u_2 \in \mathcal{L}_2^e$, the following relation is satisfied:

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \eta\|u_1 - u_2\|_{2,T}.$$

From this inequality, we conclude that the input-output relation, which was already satisfied by the input signals inside $\mathcal{L}_2$, remains valid inside $\mathcal{L}_2^e$.
More generally, when studying the properties of the nonlinear system along a possible motion, the use of the extended space $\mathcal{L}_2^e$ enables to consider a much larger class of possible inputs (e.g. non-zero constant inputs). As an illustration, by introducing:

$$\mathcal{L}_\infty([0,T]) = \{f : \mathcal{R}^+ \mapsto \mathcal{R}^n \mid ess \sup_{t\in[0,T]} \|f(t)\| < \infty\}$$

it can be noted that the following inclusion

$$\mathcal{L}_\infty([0,T]) \subset \mathcal{L}_2([0,T])$$

is true for each value of T [24]. As a consequence, the extended space, which is associated with $\mathcal{L}_2$ for a specific value of T, contains all the signals which have (almost everywhere) a finite amplitude on $[0,T]$.
In conclusion, when analyzing nonlinear system properties, the use of the extended space $\mathcal{L}_2^e$ enables to take into account most of the possible input signals, which are generally considered in an application.

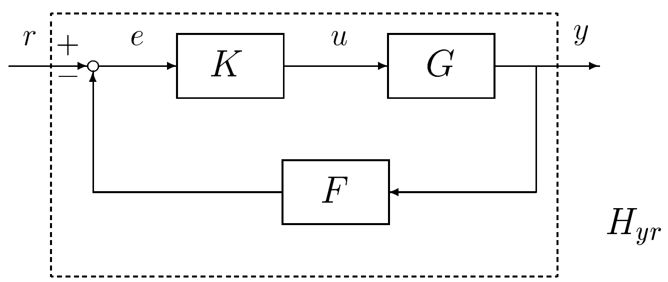

Figure 1: The nonlinear feedback system

We consider, in the sequel, the nonlinear feedback system depicted in figure 1, where G, K, F are nonlinear causal operators from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, representing respectively the plant, the compensator and the feedback, and where r, e, u and y, which belong to $\mathcal{L}_2^e$, denote respectively the system input, the error signal, the plant input, and the system output. The closed-loop system is assumed to be well-posed and the input-output map between the system input and the system output, denoted by H_{yr}, is given by $GK(I + FGK)^{-1}$.

2 Gain-scheduled control systems: a quick overview

We first briefly review various nonlinear control techniques, from a gain scheduling perspective. Two classes of methods can be pointed out. In the first method class, the (time invariant) closed loop linearizations are directly constrained. In the second one, global closed loop specifications are achieved so as the constraints on the closed loop linearizations become indirect. In the first class, the simplest method is first to compute linear controllers at certain operating points and then to interpolate the controllers between these points. Any linear design method can be applied. As a disadvantage of this method, the resulting nonlinear control law does not necessarily satisfy the integrability constraints, *i.e.* there does not necessarily exist a nonlinear controller, such that its linearizations correspond to the controllers obtained with the linear design method [3].

The first class also contains the pseudo-linearization methods (see *e.g.* [25, 26, 3]). A nonlinear controller is now directly designed, so that the integrability constraints are obviously satisfied. The linearizations of this controller on the equilibrium manifold satisfy, for example, an eigenvalue type criterion.

Actually, the main disadvantage of both approaches is their local nature. Restrictive assumptions on the rate of variation of the scheduling parameters have to be done, so as to prove the stability of the nominal nonlinear closed loop system. Moreover, as it was pointed out by Shamma in [1], the linear time invariant models associated with the equilibrium manifold do not need to be too much emphasized in the context nonlinear control. It is also necessary to ensure suitable properties to the associated linear time-varying systems. Following this remark, Shahruz and Behtash [10] introduced quadratic stability concepts which allow to remove the restrictions on the parameter rate of variation in the context of Linear Parameter Varying (LPV) systems. In this context, following the fact that quadratic stability concept can be formulated as an LMI (Linear Matrix Inequality) optimization problem, other constraints can be introduced such as, for example, $\mathcal{L}_2$ condition (see *e.g.* [27, 28, 29]). As a great advantage of these methods, the proposed design process relies on convex optimization involving Linear Matrix Inequality constraints, for which efficient algorithms are now available. Actually, when the full state is measured, the feedback control law reduces to a constant gain matrix and thus it is obviously integrable. Unfortunately, in the case of output feedback, the synthesized LPV controller now depends on the scheduling parameters and its integrability leads to impose strongly restrictions on the initial nonlinear system [29].

The classical input/output linearization method (see *e.g.* [30]) is an other solution to the nonlinear control problem. The idea is now to obtain a linear closed-loop system (from an input/output point of view) using a nonlinear state feedback controller. Restrictive assumptions have to be made on the minimum phase nature of the plant [30]. It is moreover obvious that the system behavior is now linear and identical at all operating points (under some restriction concerning the input-output stability of the zero dynamics, when they exist).

We would like to also mention the optimal methods with quadratic criteria. Two approaches can be considered. In the first approach, the open loop optimal trajectory is computed. The tracking property is then robustified by computing a feedback controller, which minimizes the second variation of the cost [31, 32]. In the same context, with reference to a closed-loop approach, an optimal solution can be directly obtained by solving well-known Hamilton-Jacobi equations[1].

In both cases, using the result of [32] as a basis, it is possible to analyze the properties associated with the system linearization along the optimal trajectory. It is then straightforward to observe the lack of homogeneity of the local criteria derived from the initial global criterion. For instance, the linearized model at the zero trim point satisfies indeed the quadratic criterion [33, 31], which is associated with the initial problem. However, the local criterion at other points blends the terms associated with the linearization of the nonlinear criterion, with other terms involving the system dynamics. Here again, in these methods, global aspects of the problem are focused, ignoring the local ones.

We finally point out the recent works on the nonlinear extension of the H_∞ approach. The nonlinear extension of the H_∞ optimization problem was recently investigated in the $\mathcal{L}_2$ framework through dissipativity techniques / nonlinear differential game arguments (see *e.g.* [34, 35]). In the same context, the LPV approach can be applied in a direct way for computing a nonlinear controller minimizing the nonlinear system $\mathcal{L}_2$ gain. In this specific context, an interesting result was published [35]. It was proved that a nonlinear system can have a finite $\mathcal{L}_2$ gain only if the system linearization at the origin point has also a finite $\mathcal{L}_2$, *i.e.* the system linearization for the null input has a finite H_∞ norm. This first result indicates a possible way to link the gain scheduling approach to a nonlinear framework. Unfortunately, as illustrated in [36], this property is not necessary true for other constant inputs and its associated equilibrium point. In fact, it is possible to built an $\mathcal{L}_2$ gain stable system with, for non zero constant input, an unstable equilibrium point.

[1] The resolution of such a problem is obviously difficult in the general case.

3 Linearization and related results

The main concept in the gain scheduling approach is the linearization one. This concept is classically related to the first order approximation of the dynamics along some specific motion. This point of view is in fact clearly linked to stability considerations and Lyapunov like arguments. In the sequel, we try to convince the reader that the Gâteaux derivative is, in the context of gain scheduling approach, a more interesting notion. Indeed, it is an input/output notion which seems to be a better frame for analyzing the system properties. It allows for example to analyze the asymptotic behavior of the system with respect to specific classes of inputs. Moreover, the introduction of this input/output notion allows to interpret, for any kind of inputs, the behavior of a nonlinear system as the sum of the behavior of linear ones.

Since the Gâteaux derivative is not necessary a well-known notion in the control field, (even if it corresponds to some usual notions in this field), we present in the sequel a self-contained presentation of various results on the differentiability of nonlinear operators on functional spaces. We emphasize a powerful result in this context: the mean value theorem in norm.

3.1 Global vs local: the Gâteaux derivative as a limit

In the following, we characterize the global behavior of a nonlinear system as a sequence of its local variations. Let us consider Δu the input variation and Δy the associated output variation:

$$\Delta y = \Sigma(u + \Delta u) - \Sigma(u). \tag{2}$$

The objective is to characterize in an accurate way the effect of Δu on Δy. Defining the local input variation $\delta u = \Delta u/n$, where n is an integer, the output variation is then rewritten as the sum of the outputs associated with small input increments:

$$\Delta y = \sum_{i=1}^{n} \Sigma(u + i\delta u) - \Sigma(u + (i-1)\delta u). \tag{3}$$

Let us consider equation (3) when the norm of the input increment goes to zero. As in a classical space (e.g. $\mathcal{R}$), the limit can be studied under the introduction of the operator derivative around a specific input. In fact, in the functional analysis frame, there exist, at least, five notions of derivative. The differences between these various notions are related to the fact that we work on spaces of infinity dimension.

The Frchet derivative, which is "similar" to the classical derivative on $\mathcal{R}$, can not be used since it is not defined for many dynamical systems [38, 39]. We then use a weaker notion of the derivative, the Gâteaux one.

Definition 3.1 [37] *Given an operator Σ, defined from $\mathcal{L}_2$ into $\mathcal{L}_2$, let us introduced $u_0 \in \mathcal{L}_2$ and let us assume the existence for any $h \in \mathcal{L}_2$ of a continuous linear operator $D\Sigma_G[u_0]$, from $\mathcal{L}_2$ into $\mathcal{L}_2$, satisfying:*

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma(u_0 + \lambda h) - \Sigma(u_0)}{\lambda} - D\Sigma_G[u_0](h) \right\|_2 = 0$$

then $D\Sigma_G[u_0]$ is called the Gâteaux derivative of Σ at u_0.

So, $D\Sigma_G[u_0]$ is a bounded linear operator defined from $\mathcal{L}_2$ into $\mathcal{L}_2$ and such that for all h belonging to $\mathcal{L}_2$, one has

$$\Sigma(u_0 + \lambda h) = \Sigma(u_0) + \lambda D\Sigma_G[u_0](h) + o(\lambda)$$

when $\lambda \downarrow 0$.

Unfortunately, the definition of the derivative is not necessary relevant in control context since any linear and unstable system is not Gâteaux differentiable. This difficulty is bypassed by the the definition of a causal system derivative on the extended space. With reference to the definition given by Willems in [22], we introduce the following definition.

Definition 3.2 *$D\Sigma_G[u_0]$ from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ is the Gâteaux derivative at u_0 of the causal operator Σ, defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ if it is linear and if for all $T \in [t_0, \infty)$, $P_T D\Sigma_G[u_0]$ is the Gâteaux derivative of $P_T\Sigma$ at $P_T u_0$.*

Let us recall this simple, but essential result concerning causal operators.

Theorem 3.1 [22] *Let us assume that a causal operator Σ defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ has a Gâteaux derivative $D\Sigma_G[u_0]$ at u_0 of $\mathcal{L}_2^e$. Then $D\Sigma_G[u_0]$ is causal on $\mathcal{L}_2^e$.*

When the system is generated by differential equations, definition 3.2 reduces to the usual linearization concept. More precisely, we have the following proposition.

Proposition 3.2 *Let us consider Σ defined by (1) and let us assume that f and h are uniformly Lipschitz and C^2 (that is, it is twice derivable). Then, for any $u_r \in \mathcal{L}_2^e$, the system has a Gâteaux derivative which satisfies the following differential equations:*

$$\bar{y} = D\Sigma_G[u_r](\bar{u}) \begin{cases} \dot{\bar{x}}(t) &= A(t)\bar{x}(t) + B(t)\bar{u}(t) \\ \bar{y}(t) &= C(t)\bar{x}(t) + D(t)\bar{u}(t) \\ \bar{x}(t_0) &= 0 \end{cases} \tag{4}$$

with $A(t) = \dfrac{\partial f}{\partial x}(x_r(t), u_r(t))$, $B(t) = \dfrac{\partial f}{\partial u}(x_r(t), u_r(t))$, $C(t) = \dfrac{\partial h}{\partial x}(x_r(t), u_r(t))$ and $D(t) = \dfrac{\partial h}{\partial u}(x_r(t), u_r(t))$ and where $x_r(t) = \phi(t, t_0, x_0, u_r)$ is the solution of system (1) under input $u_r(t)$ and $x(t_0) = x_0$.

The proof is reported in appendix, page 33.

Remark. We note that it is possible to prove that Σ does not have Frchet derivative on $\mathcal{L}_2^e$ (under the assumption that f or h are not linear functions) (see [38, 39]).

Let us now consider equation (3). We first note that for all $T \geq t_0$:

$$\Sigma(u + \lambda\Delta u) = \Sigma(u) + D\Sigma_G[u](\lambda\Delta u) + o(\lambda).$$

We consequently obtain on any time interval $[t_0, T]$ and for λ sufficiently small:

$$\Sigma(u + \lambda\Delta u) \approx \Sigma(u) + D\Sigma_G[u](\lambda\Delta u).$$

This last expression, with condition (3), enables us to approximate Δy for n sufficiently large [2] by:

$$\Delta y \approx \sum_{i=0}^{n-1} D\Sigma_G[u + i\delta u](\delta u) \tag{5}$$

This relation shows that the output variation can thus be interpreted as the sum of the output signals, which are associated with the response of linear time-varying systems, namely $D\Sigma_G[u + i\delta u]$ to the input signal δu on a finite time interval.

3.2 Global versus local: an exact relation

The approximation given by (5) can be replaced by an integral formula.

Theorem 3.3 *Let us assume that Σ is Gâteaux differentiable on $\mathcal{L}_2^e$. For any $T \in [t_0, \infty)$ and $u_1, u_2 \in \mathcal{L}_2^e$, one has*

$$\Sigma(u_2) - \Sigma(u_1) = \int_0^1 D\Sigma_G[u_1 + \beta(u_2 - u_1)](u_2 - u_1)d\beta$$

The proof is reported in appendix, page 35.

3.3 Global versus local: a norm relation

A second result allows to relate local and global aspects under a simple result which links the Lipschitz constant of the nonlinear operator and the norm of its derivatives.

Before presenting this result, let us introduce $\mathcal{U}^e$, an open and convex subset of $\mathcal{L}_2^e$, *i.e.* if u_1 and $u_2 \in \mathcal{U}^e$, then $\lambda u_1 + (1-\lambda)u_2 \in \mathcal{U}^e$ for all $\lambda \in (0, 1)$.

[2] by taking $\lambda = 1/n$

Theorem 3.4 *Let us assume that a causal operator Σ defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ has a Gâteaux derivative at each point u_0 of $\mathcal{L}_2^e$. Then, there exists a finite constant η such that for any $T \geq t_0$ and for any $u_0, h \in \mathcal{U}^e$, one has*

$$\|D\Sigma_G[u_0](h)\|_{2,T} \leq \eta \|h\|_{2,T} \tag{6}$$

if and only if the nonlinear operator is such that

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \eta \|u_1 - u_2\|_{2,T} \tag{7}$$

for any $T \geq t_0$ and any u_1 and u_2 belonging to $\mathcal{U}^e$.

Proof:

1/. We first prove that (6) implies (7). This implication is a direct consequence of the mean value theorem. Indeed, using the mean value theorem (on $\mathcal{U}^e$) [37], it is clear that the following relation is satisfied[3]:

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \sup_{u \in [P_T(u_1), P_T(u_2)]} \|D\Sigma_G[u]\|_i \|u_1 - u_2\|_{2,T}$$

for any $T \geq t_0$ and for any input signals $u_1, u_2 \in \mathcal{U}^e$. This concludes the first part of the proof.

2/. We now prove that (7) implies (6) by contradiction. To this purpose, let us assume that there exist $T \geq t_0$, u_0 and $h \in \mathcal{U}^e$ and $\rho > 0$ such that:

$$\frac{\|D\Sigma_G[u_0](h)\|_{2,T}}{\|h\|_{2,T}} \geq \eta + \rho.$$

By definition of the Gâteaux derivative, we deduce that there exists $\lambda > 0$ such that

$$\frac{\|\Sigma(u_0 + \lambda h) - \Sigma(u_0) - D\Sigma_G[u_0](\lambda h)\|_{2,T}}{\|\lambda h\|_{2,T}} \leq \frac{\rho}{2}$$

and since

$$\frac{\|D\Sigma_G[u_0](\lambda h)\|_{2,T} - \|\Sigma(u_0 + \lambda h) - \Sigma(u_0) - D\Sigma_G[u_0](\lambda h)\|_{2,T}}{\|\lambda h\|_{2,T}} \leq \frac{\|\Sigma(u_0 + \lambda h) - \Sigma(u_0)\|_{2,T}}{\|\lambda h\|_{2,T}}$$

one has

$$\frac{\|D\Sigma_G[u_0](h)\|_{2,T}}{\|h\|_{2,T}} = \frac{\|D\Sigma_G[u_0](\lambda h)\|_{2,T}}{\|\lambda h\|_{2,T}} \leq \frac{\|\Sigma(u_0 + \lambda h) - \Sigma(u_0)\|_{2,T}}{\|\lambda h\|_{2,T}} + \frac{\rho}{2} \leq \eta + \frac{\rho}{2}$$

which allows to deduce a contradiction.

□

[3] $[P_T(u_1), P_T(u_2)] = \{u \mid u = P_T(u_1) + \alpha(P_T(u_2) - P_T(u_2)), \;\; 0 \leq \alpha \leq 1\}$

3.4 Incremental stability and exponential stability of the linearizations

We present in this section a necessary and sufficient condition for incremental stability. A nonlinear operator is incrementally bounded if and only if its linearizations have exponential stable minimal state-space realizations.

Definition 3.3 [40] *$D\Sigma_G$, a Gâteaux derivative of Σ defined by (4), is said to have a minimal state-space realization if the pair $[A(t), B(t)]$ is uniformly controllable and the pair $[A(t), C(t)]$ is uniformly observable.*

Theorem 3.5 *Let us assume that Σ is Gâteaux differentiable on $\mathcal{L}_2^e$. If the state space representation of each derivative of Σ is minimal then there exists a finite constant η such that for any $T \geq t_0$ and any u_1 and u_2 belonging to $\mathcal{U}^e$ one has:*

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \eta \|u_1 - u_2\|_{2,T}$$

if and only if all its linearizations are exponentially stable.

The proof of this theorem is a direct consequence of theorem 3.4 and of this following classical lemma[4].

Lemma 3.6 [40] *If $D\Sigma_G[u_0]$ has a minimal state-space realization then $\dot{z}(t) = A(t)z(t)$ is exponentially stable if and only if $D\Sigma_G[u_0]$ is $\mathcal{L}_2$ gain stable.*

4 The sensitivity as the key justification

4.1 Introduction

The key justification of gain scheduling techniques can be found in the sensitivity problem. Remember that a deep motivation for using feedback control is the reduction of the effect of non measurable noises and is the "shrinking" of the model uncertainties [11, 12, 13, 16, 17]. The sensitivity concept has been introduced in the linear context as a means to quantify the efficiency of a feedback law, with respect to the effects of small perturbations which are due either to exogenous perturbations, or to parameter variations [11, 12, 13]. The infinitesimal nature of this analysis has however led some authors [14] to introduce a more general concept: the comparison sensitivity function. The idea is now to compare the performance of a closed loop system with the performance of an equivalent open loop system, against small or large disturbances and model perturbations.

The sensitivity concept has been also investigated in the nonlinear context by Kreindler [15], which has extended the sensitivity function introduced in [11] by defining the differential sensitivity function, which characterizes the first order sensitivity. This analysis is

[4] Since f and g is uniformly Lipschitz, $D\Sigma_G[u_0]$ has a "bounded" realization [40].

obtained from properties associated with the linearization of a suitable relation between an input and an output of the nonlinear closed loop system. More recently, the comparison sensitivity properties, associated with a nonlinear closed loop system, were exactly quantified by Desoer and Wang [17], using a Taylor type expansion of a linearizable nonlinear operator. Considering the results by Desoer and Wang (and obviously by Kreindler), it is possible to claim that the performance of the nonlinear system (in terms of the sensitivity properties) clearly depends on the properties of the linearization of the closed-loop operator along the possible trajectories of the system.

As a consequence, the relation between the sensitivity objectives and the properties of the linearizations allows us to give a justification to the local gain scheduling objectives (this point can be considered as the main motivation for using a gain scheduling control law in an adaptive control scheme - see [2] and for a theoretical justification to this fact when a slow time variation of the system is assumed see [9]).

Finally, let us recall that the main limitation of the gain scheduling approach in this problem is due to the fact that the linear time varying objective, associated with the sensitivity requirement, is obtained using the following heuristic: the linear time invariant frozen systems, which are associated with each constant value of the varying parameters, must have "good properties", so as to obtain also "good properties" for the time-varying system.

Unfortunately, this kind of condition is neither sufficient [1] nor necessary in the general case for ensuring suitable properties to the linearization of the nonlinear closed loop system and thus for ensuring suitable nonlinear properties with respect to the sensitivity problem.

4.2 Sensitivity objective: some recalls

In this section, it is shown how the output disturbance problem in nonlinear control implies strong constraints on the closed-loop system linearizations. The reader is referred to [17] for a complete presentation of desensitivity problem in the nonlinear context. In the sequel, we will just consider the output disturbance problem (the other cases presented in [17] can be worked out as well).

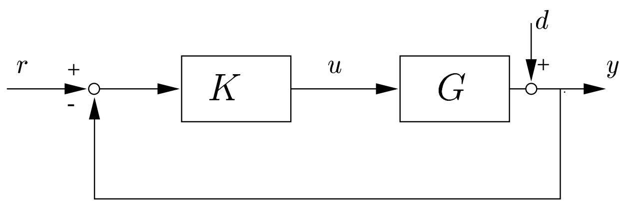

Figure 2: The perturbed closed-loop system

For this purpose, let us assume that $F = I$ and let us associate to the closed-loop system depicted in figure 2 an "equivalent" open-loop map, Ho_{yr}, depicted in figure 3. If the open-loop controller is defined by $K_o = K(I+GK)^{-1}$ then the open-loop system depicted in figure

3, which maps the inputs (r, d) in $\mathcal{L}_2^e \times \mathcal{L}_2^e$ to the output y in $\mathcal{L}_2^e$, satisfies for all $r \in \mathcal{L}_2^e$ and for $d = 0$, the following equality:

$$Ho_{yr}(r, 0) \triangleq H_{yr}(r, 0)$$

where H_{yr} is the system depicted in figure 2 which maps inputs (r, d) which belong to $\mathcal{L}_2^e \times \mathcal{L}_2^e$ to the output y which also belongs to $\mathcal{L}_2^e$.

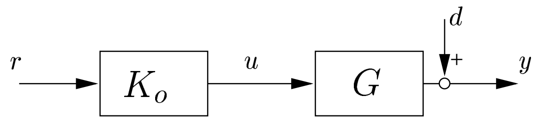

Figure 3: The perturbed equivalent open-loop system

We now calculate the effect induced by the output perturbations on the closed-loop system:

$$\delta H_{yr}(r, d) = d + GK(I + GK)^{-1}(r - d) - GK(I + GK)^{-1}(r)$$

Using the fact $GK(I + GK)^{-1} + (I + GK)^{-1} = I$ allows to rewrite the last relation as:

$$\delta H_{yr}(r, d) = S(r - d) - S(r) \tag{8}$$

with $S = (I + GK)^{-1}$.

In the case of the open-loop configuration:

$$\delta Ho_{yr}(r, d) = GK(I + GK)^{-1}(r) + d - GK(I + GK)^{-1}(r) = d. \tag{9}$$

The main interest of the feedback law strategy is that it allows to achieve a better reduction of effects of the disturbances with respect to the open loop strategy. In mathematical terms, the feedback has a desensitivity effect if the following inequality is satisfied:

$$\|\delta H_{yr}(r, d)\|_{2,T} < \|\delta Ho_{yr}(r, d)\|_{2,T}.$$

Unfortunately for realistic systems, this inequality cannot be satisfied for any input and disturbance in $\mathcal{L}_2^e$. Indeed, as in the linear context (see [16]), one has the following theorem.

Theorem 4.1 [19, 20] *Consider the closed-loop system in figure 1. If the open-loop operator GK is strictly causal then*

$$\|(I + GK)^{-1}\|_\Delta \geq 1.$$

This relation implies that there exist $r, d \in \mathcal{L}_2^e$ such that:

$$\|S(r - d) - S(r)\|_{2,T} \geq \|d\|_{2,T}$$

and thus there necessarily exists, a least, a disturbance such that the feedback law attenuation is not better than the open loop type one. We moreover point out that desensibility implies the incremental stability of S. Actually, if S is not incrementally stable, then for any K, there always exists r and d, such that

$$\|S(r-d) - S(r)\|_{2,T} \geq K\|d\|_{2,T}$$

and thus there exists some perturbation whose effects are arbitrarily amplified.

Following this preliminary remark, the interest of feedback law is necessarily limited to a specific class of perturbations. We then introduce a set of possible perturbations, namely $P^e \subset \mathcal{L}_2^e$, for which we want to ensure that the closed loop strategy performance is better than the open loop one. It is moreover clear that the cost of feedback induced by the closed loop stability problems implies that the use of the feedback control law could be justified if (and only if) there exists an ϵ ($\ll 1$) such that:

$$\|\delta H_{yr}(r,d)\|_{2,T} \leq \epsilon\|\delta Ho_{yr}(r,d)\|_{2,T}$$

for all $d \in P^e \subset \mathcal{L}_2^e$ and for all $r \in \mathcal{L}_2^e$.

4.3 Sensitivity objective strongly constraints the linearizations

We now prove that the sensitivity requirement strongly constraints the properties of the system linearizations. Actually, sensitivity objective constraints the exponential stability of the system linearizations.

Proposition 4.2 *Let us assume that the sensitivity map, i.e.* $S = (I + GK)^{-1}$*, is Gâteaux differentiable on* $\mathcal{L}_2^e$ *and that the set of possible disturbances, i.e.* P^e*, is a convex set which contains the null signal. The densisitivity is achieved with level* $\epsilon > 0$*, i.e for all* $r \in \mathcal{L}_2^e$ *and* $d \in P^e$*, one has:*

$$\|\delta H_{yr}(r,d)\|_{2,T} \leq \epsilon\|\delta Ho_{yr}(r,d)\|_{2,T} \tag{10}$$

if and only if for all $u_0 \in \mathcal{L}_2^e$ *and* $w \in P^e$*, one has:*

$$\|DS_G[u_0](w)\|_{2,T} \leq \epsilon\|w\|_{2,T} \tag{11}$$

Proof:

Note that condition (10) is implied by incremental stability of S. The fact that the stability of the derivative implies the incremental stability of S can be proved as in the first part of the proof of theorem 3.4.

The converse implication must be examined. As a matter of fact, as in the proof of theorem

3.4, we prove by contradiction that condition (10)implies condition (11). Let us assume that condition (11) is not satisfied, that is, there exist $u_0 \in \mathcal{L}_2^e$, $w \in P^e$ and $\rho > 0$ such that

$$\frac{\|DS_G[u_0](w)\|_{2,T}}{\|w\|_{2,T}} \geq \rho + \epsilon$$

The Gâteaux derivative definition ensures that there always exists $\lambda \in (0, 1]$ such that

$$\frac{\|S(u_0 + \lambda w) - S(u_0) - DS_G[u_0](\lambda w)\|_{2,T}}{\|\lambda w\|_{2,T}} \leq \frac{\rho}{2}$$

and then since

$$\frac{\|S(u_0) - S(u_0 + \lambda w)\|_{2,T} + \|S(u_0 + \lambda w) - S(u_0) - DS_G[u_0](\lambda w)\|_{2,T}}{\|\lambda w\|_{2,T}} \geq \frac{\|DS_G[u_0](w)\|_{2,T}}{\|w\|_{2,T}}$$

which allows to conclude that

$$\frac{\|S(u_0) - S(u_0 + \lambda w)\|_{2,T}}{\|\lambda w\|_{2,T}} \geq \epsilon + \frac{\rho}{2}$$

which is a contradiction with condition (10). To conclude the proof, it remains to take $r = u_0 + \lambda w$ which belongs to $\mathcal{L}_2^e$ and $d = \lambda w$ which belongs to P^e since d is a convex combinaison of two elements of P^e (w and 0).

□

We then deduce from this short discussion that the nonlinear system sensitivity is achieved if (and only if) all the system linearizations have a good behavior with respect to perturbations belonging to P_d^e.

The desensitivity requirement is achieved only if the sensitivity map, *i.e.* S, has a finite incremental gain. This ensures that the degradation introduced by the feedback use is finite. Following theorem 3.5, this condition constrains the exponential stability of the linearizations of the system:

Proposition 4.3 *Let us assume that the sensitivity map, i.e. $S = (I + GK)^{-1}$, is Gâteaux differentiable on $\mathcal{L}_2^e$ then desensitivity can be only achieved if and only if the minimal state-space realization of each linearization of S is exponentially stable.*

5 Some discussion about the Lyapunov point of view

5.1 Introduction

This section interest is twofold. We first want to recall classical arguments of the gain scheduling approach ensuring that the unperturbed trajectories are at least asymptotically stable. The gain scheduling approach leads to the exponential stability of the system linearizations. The main interest of this property is then pointed out. Indeed, the exponential stability of system linearizations implies incremental stability.

5.2 The "classical" rules

Let us consider the particular trajectory of system (1), which is associated with a continuous input signal u_r and to x_{0r}, *i.e.* $x_r(t) = \phi(t, t_0, x_{0r}, u_r))$. We first recall the following definition.

Definition 5.1 [33] *An unperturbed motion of Σ, associated with a particular input u_r belonging to $\mathcal{L}_2^e$ and with a particular initial condition x_{0r}, i.e. $x_r(t) = \phi(t, t_0, x_{0r}, u_r)$, is said to be uniformly asymptotically stable in the sense of Lyapunov, if for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $t_1 \geq t_0$ and for all x_{0p} such that $\|x_r(t_1) - x_{0p}\| < \delta(\epsilon)$, one has for all $t \geq t_1$:*

$$\|\phi(t, t_1, x_r(t_1), u_r) - \phi(t, t_1, x_{0p}, u_r)\| \leq \epsilon$$

and

$$\lim_{t\to\infty} \|\phi(t, t_1, x_r(t_1), u_r) - \phi(t, t_1, x_{0p}, u_r)\| = 0$$

If this last property holds for any x_{0p}, the unperturbed motion is called uniformly globally asymptotically stable.
Finally, the unperturbed motion is said to be exponentially stable if there exist two positive constants a and b such that there exists $\delta(\epsilon) > 0$ such that for all $t_1 > t_0$ and for all x_{0p} such that $\|x_r(t_1) - x_{0p}\| < \delta(\epsilon)$, one has for all $t \geq t_1$:

$$\|\phi(t, t_1, x_r(t_1), u_r) - \phi(t, t_1, x_{0p}, u_r)\| \leq \|x_r(t_1) - x_{0p}\| a e^{-b(t-t_1)}.$$

Under the regularity assumption, which has been made on the differential equation (1), the study of the stability of the unperturbed motion can be made on the basis of the stability properties of the linear part of the differential system. Let be $x = x_r + \bar{x}$ and $u = u_r + \bar{u}$. Equation (1) can be rewritten as a sum of linear terms, perturbed by a nonlinear term:

$$\left\{\begin{array}{lcl} \dot{\bar{x}}(t) & = & A(t)\bar{x}(t) + B(t)\bar{u}(t) + R_1(\bar{x}(t), \bar{u}(t)) \\ \bar{y}(t) & = & C(t)\bar{x}(t) + D(t)\bar{u}(t) + R_2(\bar{x}(t), \bar{u}(t)) \\ \bar{x}(t_0) & = & x_{0r} - x_{0p} \end{array}\right. \tag{12}$$

where $A(t) = \dfrac{\partial f}{\partial x}(x_r(t), u_r(t))$, $B(t) = \dfrac{\partial f}{\partial u}(x_r(t), u_r(t))$, $C(t) = \dfrac{\partial h}{\partial x}(x_r(t), u_r(t))$ and $D(t) = \dfrac{\partial h}{\partial u}(x_r(t), u_r(t))$. All these matrices are bounded functions of time (since f and h are assumed uniformly Lipschitz). Moreover, if we assume that $x_r(t)$ and $u_r(t)$ are uniformly bounded[5] on $[t_0, \infty)$, then there exists finite constants L_1 and L_2 such that:

$$\|R_1(\bar{x}, \bar{u})\| \leq L_1(\|\bar{x}\|^2 + \|\bar{u}\|^2) \quad \text{and} \quad \|R_2(\bar{x}, \bar{u})\| \leq L_2(\|\bar{x}\|^2 + \|\bar{u}\|^2).$$

In this case, it is well-known [41, 33] that the exponential stability of the linear part of system (12), *i.e.* $\dot{\bar{z}}(t) = A(t)\bar{z}(t)$, ensures the existence of an open ball of initial conditions around x_{0r}, such that all the trajectories, associated with the particular input u_r and with any initial condition inside this open ball, are locally exponentially stable.

[5]or that the Hessians of f and h are bounded along $x_r(t)$ and $u_r(t)$.

Theorem 5.1 [41, 33] *Let $x[x_0, u_r]$ be a specific unperturbed motion associated with an input signal $u_r \in \mathcal{L}_2^e$ and with a specific initial condition x_0. Let be $A(t) = \frac{\partial f}{\partial x}(x_r(t), u_r(t))$. If the linear differential equation:*

$$\dot{\bar{z}}(t) = A(t)\bar{z}(t)$$

is exponentially stable, then the unperturbed motion $x[x_0, u_r]$ is also exponentially stable.

Despite its local nature, this result presents the advantage of rigorousness. Unfortunately, checking the stability of a linear time-varying system is a difficult problem. In many cases, for bypassing this difficulty, the authors apply a classical heuristic: stabilization of the linear time varying system $\dot{\bar{z}}(t) = A(t)\bar{z}(t)$ is researched by stabilizing the time invariant systems $\dot{z}(t) = A(\tau)z(t)$ for each constant value of τ belonging to $[t_0, \infty)$. This idea, which is the basis of most gain scheduling techniques (and which can be compared in some sense to the famous Aizerman/Kalman's conjecture), is obviously false in the general case (it is fact neither sufficient nor necessary). This however explains that most of the works concerning gain scheduling is dedicated to the search of conditions, under which this conjecture becomes true. A classical condition consists in restricting the time variation of the scheduling parameters [1, 8, 9]. Others conditions can be found using the quadratic stability concept [10] or the averaging concept. This theorem moreover explains the interest of considering LPV systems, *i.e.* Linear systems depending on Varying Parameters, [1, 8].

Finally, from a global point of view, the exponential stability of the time-varying linearization only guarantees that the system motion tends toward a specific unperturbed motion, but nothing is known about the behavior of this specific motion. Actually, this analysis seems only possible in the case of constant inputs and in the existence of equilibrium points.

5.3 The "classical rules" imply incremental stability

By a suitable modification of theorem 3.5, we have the following result:

Proposition 5.2 *Let us assume that Σ is Gâteaux differentiable on $\mathcal{L}_2^e$. If the time-varying linearizations of Σ associated with each input of $\mathcal{U}^e$ are exponentially stable then there exists a finite constant, η, such for any $T \geq t_0$ and any $u_1, u_2 \in \mathcal{U}^e$, one has:*

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \eta \|u_1 - u_2\|_{2,T}$$

In other words, we claim that the underlying design objective of gain scheduling controllers is to obtain a closed loop system at least incrementally bounded on $\mathcal{U}^e$.

6 Connections with the incremental approach

6.1 Introduction

The incremental norm framework was introduced for analyzing the properties of nonlinear closed loop systems from both quantitative and qualitative points of view.

In this framework, the robustness and the performance properties of a nonlinear closed loop system can be analyzed in a quantitative way. The weighted incremental norm approach was indeed originally introduced as an extension of the classical H_∞ control concepts to nonlinear systems [18]. When considering linear time invariant systems, the original idea of [16] was to formulate robustness and performance specifications of a closed loop system as contraints on the H_∞ norm of weighted closed loop transfer functions. The main interest is then to recast the controller design into a well defined optimization problem based on this formulation. In the same way, in the incremental norm approach, the idea is to translate the robustness and performance properties of a (nonlinear) (closed loop) system into constraints on a suitable norm of weighted operator. The incremental norm was proposed as a suitable norm [18].

In a qualitative way, incrementally stable systems possess suitable steady-state properties. Furthermore, the effect of a non zero initial condition is ensured to decay asymptotically to zero. As a first point, a unique steady-state motion corresponds to a given input signal, independently of the initial condition and despite a vanishing perturbation on the input signal. When analyzing the response of the nonlinear closed loop system to a reference input signal, remember that introducing an unknown initial condition can be interesting, since it allows to consider the past unknown values of the reference input signal. As a second point, the steady state response to a constant (resp. periodic) input signal is also constant (resp. periodic).

Note finally that some basic properties of incrementally stable systems will be recalled in the following sections. The reader is referred to [18, 19, 42, 43, 36, 20] for a more complete presentation.

6.2 Tracking and asymptotic properties

The notion of performance is quite difficult to handle in the nonlinear input-output context. We first recall the approach of [17], in which the performance is defined as the ability for the system to minimize “asymptotically” the gain between the inputs of interest r and the error signals e (see figure 4). More precisely, denoting as R_d^e the set of input signals of interest, the following definition is introduced.

Definition 6.1 *The asymptotic performance of the closed loop system of figure 4 is satisfied on R_d^e if there exist $\epsilon > 0$ and $T_0 \geq 0$ such that :*

$$\|(I + FGK)^{-1}r\|_{2,T} \leq \epsilon\|r\|_{2,T} \;\; \forall r \in R_d^e$$

for $T \geq T_0$.

This definition ensures that the relation $FH_{yr} \approx I$ is asymptotically satisfied on R_d^e (see figure 4). In the sequel, the operator $(I + FGK)^{-1}$ is denoted S.

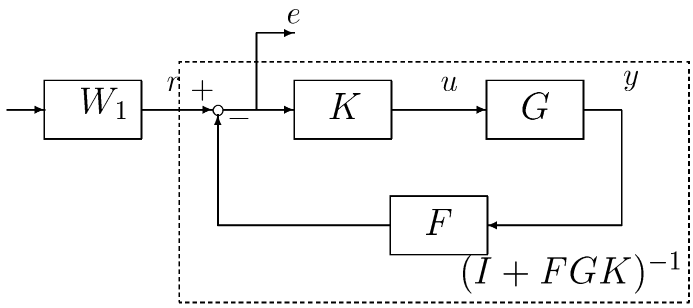

Figure 4: Tracking performance analysis

As in the H_∞ context [16], we will consider

$$R_d^e = \{r \in \mathcal{L}_2^e | \|W_1^{-1}(r)\|_{2,T} \leq \epsilon \|r\|_{2,T}\}$$

where W_1 and W_1^{-1} are two causal and incrementally stable operators. Such an operator W_1 defines the set of input signals R_d^e. We then have the following result.

Theorem 6.1 [18, 20] *The asymptotic performance of the closed system of figure 4 is guaranteed on R_d^e if*

$$\|SW_1\|_\Delta \leq 1$$

6.3 Desensitivity

In this subsection, testing conditions introduced in section 4 is formulated as the computation of the incremental norm of a weighted operator.

To this purpose, as in the H_∞ approach, we now assume that the set of possible disturbances for which desensitivity must be achieved can be defined by:

$$P^e = \{d \in \mathcal{L}_2^e | \|W_p^{-1}(d) - W_p^{-1}(r + d)\| \leq \epsilon \|d\| \text{ for all } r \in \mathcal{L}_2^e\}$$

where W_p and W_p^{-1} are two causal and incrementally stable operators. Note that the set P^e definition slightly differs from the set R_d^e definition. The main reason is to take into account a non zero initial condition (for details, see [20]). By the introduction of this weighting function, the desensitivity is achieved if the "weighted" sensitivity function incremental norm is less than 1. More precisely, we have this following result.

Theorem 6.2 [18, 20] *Consider the nonlinear feedback system depicted in figure 2. If*

$$\|SW_p\|_\Delta \leq 1 \tag{13}$$

then $\|\delta H_{yr}(r,d)\|_{2,T} \leq \epsilon \|\delta Ho_{yr}(r,d)\|_{2,T}$ for all $d \in P^e \subset \mathcal{L}_2^e$ and for all $r \in \mathcal{L}_2^e$.

6.4 Robustness against unstructured uncertainties

We consider the case of an unstructured uncertainties $\widetilde{\Delta}$ on the plant model G. This model perturbation $\widetilde{\Delta}$ may represent uncertainties on the actuator or sensor dynamics, and more generally neglected dynamics. We assume that $\widetilde{\Delta}$ belongs to a set defined as:

$$\Omega_{\widetilde{\Delta}} = \left\{ \widetilde{\Delta} = W_3 \Delta W_2 \mid \|\Delta\|_{\Delta} < 1 \right\}$$

where Δ is a (possibly nonlinear) causal operator from $\mathcal{L}_2^e$ to $\mathcal{L}_2^e$ and W_2 and W_3 are known, causal, incrementally stable input-output maps from $\mathcal{L}_2^e$ to $\mathcal{L}_2^e$.

A stability result is proposed for the interconnected system presented figure 5, where M is a generic nominal closed loop system. In this case, the (internal incremental) stability property corresponds to the incremental stability property of the (well posed) operator defined by the inputs u_1 and u_2 and the outputs y_1 and y_2.

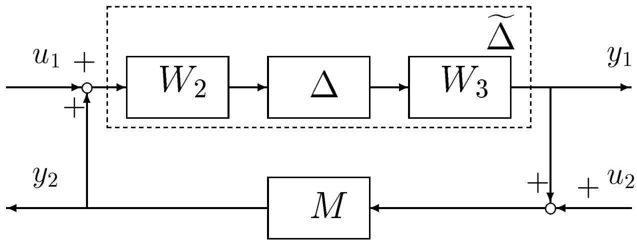

Figure 5: Robustness analysis of the nonlinear feedback system

Theorem 6.3 [18, 20] *If M is incrementally stable and if the following inequality holds:*

$$\|W_2 M W_3\|_{\Delta} \leq 1$$

then the closed loop system of figure 5 is incrementally stable for any uncertainty $\widetilde{\Delta}$ belonging to $\Omega_{\widetilde{\Delta}}$.

6.5 Connections with linear time varying and time invariant control

In this section, we first point out the strong connections between the weighted incremental approach and time varying H_∞ control[6]. Following the previous section, we assume in

[6]Note that, stricto sensus, H_∞ control refers to linear time invariant closed loop systems where the controller is designed by minimizing a weighted transfer function H_∞ norm. The H_∞ norm is not defined in the case of a linear time varying systems: in this case, a possible extension is the $\mathcal{L}_2$ gain ($\mathcal{L}_2$ induced norm). Nevertheless, the $\mathcal{L}_2$ gain control of linear time varying systems is usually referred to as the "H_∞ control". In this paper, we are consistent with this use.

the sequel that performance and robustness specifications are formulated as constraints on the weighted incremental norm of a closed loop operator H, that is, the incremental of the augmented closed loop plan, $M_{zw} = W_o H W_i$. W_i and W_o are the input and output weighting functions associated with robustness and performance requirements. We moreover assume that the augmented system is described by a differential equation with C^2 and globally Lipschitz state and output functions[7]. From theorem 3.4, we deduce

Proposition 6.4 *If the augmented system, $M_{zw} = W_o H W_i$, has a Gâteaux derivative for every input in $\mathcal{L}_2^e$ then $\|M_{zw}\|_\Delta \leq 1$ if and only if*

$$\|DW_{oG}[H(W_i(w_0))]DH_G[W_i(w_0)]DW_{iG}[w_0]\|_i \leq 1 \;\; \forall w_0 \in \mathcal{L}_2^e. \tag{14}$$

Note that $DM_{zwG}[w_0]$ is a linear time-varying operator: from proposition 6.4, solving a weighted incremental problem is thus equivalent to solving an infinite number of linear time-varying weighted induced norm problems. It is worth noting that the constraints (14) are satisfied if (and only if) an infinite number of linear time-varying weighted H_∞ constraints are satisfied.

The non-stationarity of the induced norm condition (14) is now discussed through a simple example described by figure 4, that is, with $H = S$, where $S = (I + FGK)^{-1}$. Given a small variation $\delta r(t)$ of the system input $r(t)$, the corresponding tracking error variation can be then approximated on a finite time interval:

$$e = S(r + \lambda\delta r) - S(r) \approx DS_G[r](\lambda\delta r). \tag{15}$$

Let us here assume that the performance requirements are introduced with a linear weighting input operator, $W_i = W_I$. This weighting function, assumed to be causal and invertible, satisfies the following relation (see subsection 6.2):

$$\|W_I^{-1}(r)\|_{2,T} \leq \epsilon\|r\|_{2,T}. \tag{16}$$

where $\epsilon \ll 1$. If the condition (14) is satisfied, it can be claimed that

$$\|DS_G[W_I(w_0)]W_I\|_i \leq 1. \tag{17}$$

The above relation is a time-varying H_∞ constraint (see figure 6), which ensures for any $\delta r \in R_d^e$ that there exist $\epsilon > 0$ and a time, $T_0 \geq t_0$, such that for all $T \geq T_0$, one has:

$$\|\delta e\|_{2,T} \leq \epsilon\|\delta r\|_{2,T} \tag{18}$$

since

$$\|DS_G[W_I(w_0)](\delta r)\|_{2,T} \leq \|W_I^{-1}(\delta r)\|_{2,T} \leq \epsilon\|\delta r\|_{2,T}. \tag{19}$$

In a performance context, proposition 6.4 can be interpreted in two different ways:

[7]This assumption ensures the existence of the augmented system Gâteaux derivative.

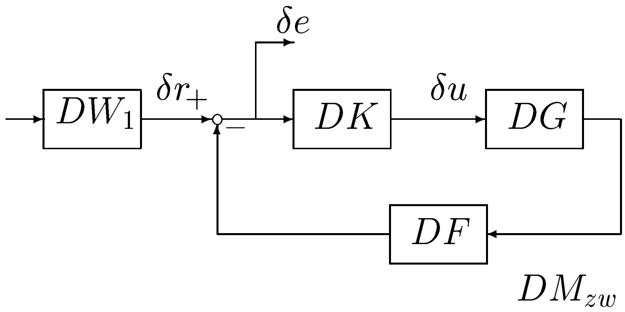

Figure 6: Linearization of the augmented plant

- as a constraint on the system linearizations along the trajectories defined by $W_I(w_0)$. Consequently, this guarantees a good behavior of the nonlinear system along its trajectories despite small perturbations belonging to R_d^e.
- as a constraint on the output variations with respect to small input variations. For example, the output associated with a step input can be interpreted as the sequence of responses to small step-inputs associated with each nonlinear system linearization along the trajectory generated by this step. This output is directly related to the weighting function W_I linearization.

Remark. In the approach proposed in [1], one has to test whether the gain scheduling system satisfies a criterion similar to condition (14).

We now investigate the close connection between the incremental approach and approaches based on time invariant linearizations such as the pseudo-linearization. Note that the class of time invariant linearizations is classically considered in the gain scheduling approach. We then define Z_e, the set of equilibrium points associated with any constant input:

$$Z_e = \{(x_e, u_e) \in \mathcal{R}^n \times \mathcal{R}^p \mid \phi(t, t_0, x_e, u_e) = x_e \; \forall t \geq t_0\} \tag{20}$$

where ϕ is the state transition map of Σ.

Theorem 6.5 [42] *Let Σ be the system given by (1) with the finite incremental gain η. Let u_e be any constant input and x_e be its associated equilibrium point. If x_e is reachable from x_0 then the linearization of Σ, given by the following linear time invariant system:*

$$D\Sigma_G(u_e) \begin{cases} \dot{\bar{x}}(t) &= A\bar{x}(t) + B\bar{u}(t) \\ \bar{y}(t) &= C\bar{x}(t) + D\bar{u}(t) \\ \bar{x}(t_0) &= 0 \end{cases} \tag{21}$$

$A = \frac{\partial f}{\partial x}(x_e, u_e)$, $B = \frac{\partial f}{\partial u}(x_e, u_e)$, $C = \frac{\partial h}{\partial x}(x_e, u_e)$ and $D = \frac{\partial h}{\partial u}(x_e, u_e)$, has a finite $\mathcal{L}_2$ gain less than or equal to η, i.e. $\|D\Sigma_G[u_e]\|_i \leq \eta$.

This result makes crystal clear a direct connection between our nonlinear framework and the classical gain scheduling techniques, especially with the approaches based on the extended linearization (see *e.g.* [3]). In these approaches, some properties of the linear time-invariant linearizations of the system associated with constant inputs are enforced. Finally, with respect to the weighted incremental norm approach and with reference to the augmented system previously defined M_{zw}, (whose norm is less than 1, *i.e.* $\|M_{zw}\|_\Delta \leq 1$), theorem 6.5 ensures that all the time invariant linearizations satisfy a weighted H_∞ norm condition. This constraint is specified at each equilibrium point by the stationary linearization of the nonlinear weighting functions, *i.e.*

$$\|DW_{oG}[H(W_i(w_0))]DH_G[W_i(w_0)]DW_{iG}[w_0]\|_i \leq 1 \tag{22}$$

where $DW_{oG}[H(W_i(w_0))], DH_G[W_i(w_0)]$ and, $DW_{iG}[w_0]$ are linear time invariant systems. This last fact has interesting connections with the work presented in [4].

6.6 Fundamental interests of incremental framework: local behavior versus global behavior

In this subsection, following the results given in [44], we point out that the exponential stability of the linearizations of a nonlinear system ensures suitable global behavior properties. If the system linearizations are exponentially stable then the nonlinear system has the unique steady state property. Moreover, it has a periodic (constant) trajectories for periodic inputs and finally under some reachable assumption of its state space from the initial condition, all its unperturbed trajectories are globally asymptotically stable.

We first consider the analysis of the system behavior with respect to a perturbation on its input which vanishes when the time tends to the infinity.

Theorem 6.6 *Let Σ, be a dynamical system associated with (1). If for all $u_0 \in \mathcal{U}^e$, the linearization $D\Sigma_G[u_0]$ is exponentially stable then for all $u_r, \tilde{u}_r \in \mathcal{U}^e$ such that $u_r - \tilde{u}_r \in \mathcal{L}_2$, one has:*

$$\lim_{t\to\infty} \|\phi(t, t_0, x_0, u_r) - \phi(t, t_0, x_0, \tilde{u}_r)\| = 0.$$

Proof:

Theorem 3.5 implies that there exists a finite constant η such for $T \geq t_0$ and all $u_r, \tilde{u}_r \in \mathcal{U}^e$, one has $\|x[x_0, u_r] - x[x_0, \tilde{u}_r]\|_{2,T} \leq \eta\|u_r - \tilde{u}_r\|_{2,T}$. Since by assumption, $u_r - \tilde{u}_r$ belongs to $\mathcal{L}_2$, one has for any $T \geq t_0$:

$$\|x[x_0, u_r] - x[x_0, \tilde{u}_r]\|_{2,T} \leq \eta\|u_r - \tilde{u}_r\|_2$$

which allows to ensures that $x[x_0, u_r] - x[x_0, \tilde{u}_r]$ belongs to $\mathcal{L}_2$.
By applying Barbalat's lemma, since both motions are two absolute continuous functions of time, the result is proved.

□

We now analyze the effect of persistent perturbations with a finite amplitude: a bounded magnitude input generates a bounded magnitude state.
For sake of briefness, the proofs of theorems presented in this subsection are not developed here (see [44]).

Theorem 6.7 [44, 43] *Let Σ, be a dynamical system associated with (1). If for all $u_0 \in \mathcal{U}^e$, the linearization $D\Sigma_G[u_0]$ is exponentially stable and $0 \in \mathcal{U}^e$ then for any $L \geq 0$, there exists $K \geq 0$ such that*

$$\|\phi(t, t_0, x_0, u_r)\| \leq K$$

for all $u_r \in \mathcal{U}^e$ such that $\|u_r(t)\| \leq L$ a.e.

Incremental stability also ensures desirable response to a periodic input.

Definition 6.2 *A motion, $x(t)$, defined from $[t_0, \infty)$ into $\mathcal{R}^n$ is said to asymptotically periodic if there exists a positive constant $T_\epsilon > 0$ such that $\|x(t+\tau) - x(t)\| \leq \epsilon$ for all $t > T_\epsilon$.*

Theorem 6.8 [44, 43] *Let Σ be a dynamical system associated to (1). If for any $u_r \in \mathcal{U}^e$, the linearization of $D\Sigma_G[u_r]$ is exponentially stable and $u = 0$ belongs to $\mathcal{U}^e$. then for any periodic input belonging to $\mathcal{U}^e$, the associated state space trajectory is asymptotically periodic.*

We now present a result which ensures that the effect of the initial condition vanishes.

Definition 6.3 *The state space of Σ is said to be reachable from x_0 with respect to $\mathcal{U}^e$ if given any $x \in \mathcal{R}^n$ there exist $u \in \mathcal{U}^e$ and $T_r \geq t_0$ such that $x = \phi(t, t - T_r, x_0, u)$ for any $t \geq t_0 + T_r$.*
A set $\Omega \subset \mathcal{R}^n$ is said to be reachable from x_0 with respect to $\mathcal{U}^e$ if given any $x \in \Omega$ there exist $u \in \mathcal{U}^e$ and $T_r \geq t_0$ such that $x = \phi(t_0, t + T_r, x_0, u)$.

Theorem 6.9 [44, 43] *Let Σ, be a dynamical system associated to (1) and Ω an open ball of $\mathcal{R}^n$. If for all $u_r \in \mathcal{U}^e$, the linearization of $D\Sigma_G[u_r]$ is exponentially stable and Ω is reachable from x_0 with respect to $\mathcal{U}^e$ then all the unperturbed motion associated with an initial condition in Ω are asymptotically stable at large, i.e. for any $u_r \in \mathcal{U}^e$ and any $x_p \in \Omega$ one has:*
(i) for any $\epsilon > 0$, there exists $\delta(t_0, \epsilon) > 0$ such that

$$\|\phi(t, t_0, x_p, u_r) - \phi(t, t_0, \tilde{x}_p, u_r)\| \leq \epsilon$$

for any $\tilde{x}_p \in \Omega$ such that $\|x_p - \tilde{x}_p\| < \delta(t_0, \epsilon)$.
(ii) for any $\tilde{x}_p \in \Omega$, one has

$$lim_{t\to\infty} \|\phi(t, t_0, x_p, u_r) - \phi(t, t_0, \tilde{x}_p, u_r)\| = 0$$

7 Incremental gain: analysis and controller design

As pointed out in the introduction, the first interest of the above results is to propose a mathematical framework, which, in some sense, allows to assess the complexity of the classical gain scheduling objectives. We recall in the sequel some results concerning the computation of a nonlinear operator incremental norm for analysis purpose. We first recast the problem of computing an (upper bound)incremental gain of a nonlinear operator as an optimization problem. We then more generally discuss the controller design.

7.1 The incremental real bounded lemma

Let us introduce the system $y = \mathcal{S}(w)$, described by the following equations:

$$\begin{cases} \dot{x}(t) &= f(x(t)) + g(x(t))w(t) \\ y(t) &= h(x(t)) \\ x(t_0) &= x_0 \end{cases} \tag{23}$$

where f and h are assumed smooth enough, so that $\mathcal{S}$ is well-defined for all $x_0 \in \mathcal{R}^n$ and for all $w \in \mathcal{L}_2^e$. Let us associate with this dynamical system, $\mathcal{S}$, defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, a fictitious dynamical system, $\mathcal{S}_f$, defined from $\mathcal{L}_2^e \times \mathcal{L}_2^e$ into $\mathcal{L}_2^e$ by:

$$y_f = \mathcal{S}_f(w_1, w_2) = \mathcal{S}(w_1) - \mathcal{S}(w_2).$$

The available storage function (see [45]), namely S_a, is then introduced as a function from $\mathcal{R}^n \times \mathcal{R}^n$ into $\mathcal{R}^e$:

$$S_a(x_{01}, x_{02}) = \sup_{x \to} - \int_{t_0}^{T} (\eta^2 \|w_1(\tau) - w_2(\tau)\|^2 - \|y_f(\tau)\|^2) d\tau \tag{24}$$

where the supremum is taken over all $w_1, w_2 \in \mathcal{L}_2^e$, and where the notation $\sup\limits_{x \to}$ denotes the supremum over all motions of $\mathcal{S}_f$, which start from state (x_{01}, x_{02}) at time t_0, namely the motions which are associated with the following dynamical system:

$$\mathcal{S}_f \begin{cases} \dot{x}_1(t) &= f(x_1(t)) + g(x_1(t))w_1(t) \\ \dot{x}_2(t) &= f(x_2(t)) + g(x_2(t))w_2(t) \\ y_f(t) &= h(x_1(t)) - h(x_2(t)) \\ x_1(t_0) &= x_{01} \\ x_2(t_0) &= x_{02} \end{cases} \tag{25}$$

Lemma 7.1 [43] *$\mathcal{S}$ has an incremental gain less or equal to η if and only if $S_a(x_0, x_0) = 0$.*

A more classical condition can be formulated as follows.

Lemma 7.2 (Incremental real bounded lemma) [18, 19] *The system (23) has an incremental norm less or equal to* η *if there exists a* C^1 *function* S*, defined from* $\mathcal{R}^n \times \mathcal{R}^n$ *into* $\mathcal{R}$*, such that the following conditions hold true for all* x *and* z *belonging to* $\mathcal{R}^n$*:*

$$(i) \quad S(x,x) = 0$$

$$(ii) \quad S(x,z) \geq 0$$

$$(iii) \quad \frac{\partial S}{\partial x} f(x) + \frac{\partial S}{\partial z} f(z) + \frac{1}{4}\eta^2 \frac{\partial S}{\partial x} g(x) g(x)^T \frac{\partial S^T}{\partial x} + (h(x) - h(z))^T (h(x) - h(z)) \leq 0$$

$$(iv) \quad \frac{\partial S}{\partial x} g(x) + \frac{\partial S}{\partial z} g(z) = 0$$

Remark In this lemma, the proposed condition is only sufficient for incremental boundedness since the available storage function S is assumed C^1. Necessary conditions can be obtained if this assumption is relaxed and if the solutions of the Halmiton Jacobi like equation are defined in the viscosity sense (see [46]).

Using this specific lemma, it is possible to illustrate in a simple way most of the results of the previous section. We enlighten in the sequel the link between the conditions of lemma 7.2 and the Lyapunov stability result. As proved in [43], S parameterizes indeed the Lyapunov-like functions for any system motion. Let us define indeed a specific function V, from $\mathcal{R} \times \mathcal{R}^n$ into $\mathcal{R}$, which is related to S by $V(t,x) = S(x_r(t), x_r(t) + x)$, where $x_r(t) = \phi(t, t_0, x_{0r}, w_r)$ is a specific motion of system (23). The conditions in lemma 7.2 can be used to prove the Lyapunov stability of the unperturbed motion, since $V(t,0) = 0$ and V is decreasing along the perturbed motion of the system:

$$V(t_1, x_1) - V(t_0, x_0) \leq - \int_{t_0}^{t_1} \|y_r(t) - y_2(t))\|^2 dt \leq 0$$

where $y_r(t)$ (resp. $y_2(t)$) is the output of system (23), when applying the input signal $w_r(t)$ and the initial condition x_r (resp. x_2). The stability of the unperturbed motion is then obtained if some uniform irreducibility conditions are assumed (ensuring that V is positive definite, for details see [43]). Asymptotic stability requires the uniform observability of the unperturbed trajectory, which ensures that V is now strictly decreasing along the perturbed motions.

7.2 A computational test

Since computing the (upper bound) incremental gain of a nonlinear operator involves solving Hamilton-Jacobi like equations, it is a difficult problem. In the sequel, we propose a result based on a computational attractive (sufficient) condition. The purpose is to achieve a trade-off between the conservatism and the complexity of the tests associated with (incremental) stability criteria. The interest of such a pragmatic approach was already emphasized by Safonov in [21].

To this purpose, we point out the interest of quadratic type functions S [18, 36]. In this new context, sufficient conditions for the incremental stability of a nonlinear system can be obtained. Testing these sufficient conditions reduce to solve a convex optimization problem involving Linear Matrix Inequality (LMI) constraints. The underlying idea is to use a specific solution to the original Hamilton Jacobi type equation. Remember that efficient numerical algorithms have been proposed [47] for solving LMI-based optimization problems. Actually, we obtain the following result.

Theorem 7.3 [18, 36] *If there exist $\eta > 0$ and a symmetric, positive definite matrix $P \in \mathcal{R}^{n\times n}$ such that*

$$\begin{bmatrix} P\,\frac{\partial f}{\partial x}(x,u) + \frac{\partial f}{\partial x}^T(x,u)P & P\frac{\partial f}{\partial u}(x,u) & \eta^{-1}\frac{\partial h}{\partial u}^T(x,u) \\ \frac{\partial f}{\partial u}^T(x,u)P & -I_{m,m} & \eta^{-1}\frac{\partial h}{\partial u}^T(x,u) \\ \eta^{-1}\frac{\partial h}{\partial u}(x,u) & \eta^{-1}\frac{\partial h}{\partial u}(x,u) & -I_{p,p} \end{bmatrix} < 0$$

is satisfied for all $x \in \mathcal{R}^n$ and any $u \in \mathcal{R}^p$, then system (1) is incrementally stable for any initial condition $x_0 \in \mathcal{R}^n$ and has an incremental gain less than η.

This theorem is a direct consequence of the mean value theorem, since the LMI condition ensures that each linearization of the system is in fact $\mathcal{L}_2$ gain stable.

7.3 Control design problem: singular differential games with incomplete information

The incremental synthesis problem is now reformulated as an optimization problem. To this purpose, we restrict our attention to a specific class of system defined as:

$$\begin{cases} \dot{x}(t) &= f(x(t)) + p(x(t))u(t) + g(x(t))w(t) \\ z(t) &= h(x(t)) + Du(t) \\ x(t_0) &= x_0 \end{cases} \tag{26}$$

where $u(t) \in R^k, w(t) \in R^m, y(t) \in R^p, x(t) \in R^n$, $x_0 \in R^n$ and f, p, g, h are C^2 and uniformly Lipschitz.

The full information control problem consists in finding the (stabilizing) control law $u = K(x,w)$, which minimizes the incremental gain of the nonlinear closed-loop system M_{zw}:

$$\min_K \|M_{zw}\|_\Delta$$

In a classical way, this optimization problem can be reformulated as follows. Let us associated to system (26), this following fictitious system:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & f(x_1) + p(x_1)u_1 + g(x_1)w_1 \\ \dot{x}_2 & = & f(x_2) + p(x_2)u_2 + g(x_2)w_2 \\ z_f & = & h(x_1) - h(x_2) + D(u_1 - u_2) \\ x_1(t_0) & = & x_0 \\ x_2(t_0) & = & x_0 \end{array}\right. \tag{27}$$

By introducing $u = u_1 - u_2$, $w = w_1 - w_2$, $x_t^T = (x_1^T, x_2^T)$, $w_t = (w^T, w_2^T)$ and $u_t^T = (u^T, u_2^T)$, system (27) can be rewritten as:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & f(x_1) + p(x_1)u + p(x_1)u_2 + g(x_1)w + g(x_1)w_2 \\ \dot{x}_2 & = & f(x_2) + p(x_2)u_2 + g(x_2)w_2 \\ z_f & = & h(x_1) - h(x_2) + Du \\ x_1(t_0) & = & x_0 \\ x_2(t_0) & = & x_0 \end{array}\right. \tag{28}$$

We then associate to the previous system, the following cost function:

$$J(u_t, w_t) = \int_{t_0}^{T} \left(\|z_f(t)\|^2 - \eta^2 \|w(t)\|^2 \right) \, dt$$

For a fixed constant $\eta > 0$, the issue is now to determine whether there exists a full information controller $u = K(x, w)$, solution of the problem:

$$\sup_{w_t} \left(\inf_{\{u_t | u = K(x,w)\}} J(u_t, w_t) \right)$$

The resolution of this optimal control problem is quite difficult. Actually, the cost function can be rewritten as:

$$J(u_t, w_t) = \int_{t_0}^{T} \left(\|h(x_1(t)) - h(x_2(t)) + Du(t)\|^2 - \eta^2 \|w(t)\|^2 \right) dt$$

In a differential game interpretation, since the criterion only depends on u and w, it is singular with respect to the inputs associated with both players. Moreover, this "full information" problem becomes an "incomplete information" problem since x_t is not measured (see [34] and references therein) .

A first solution was proposed in [18]. In this solution, the controller design problem is performed in two steps. In the first step, the full information case is considered, *i.e.* all perturbations and states of the fictitious system are assumed to be measured: when solving this first problem, which is already difficult because of the criterion singularity, a full state feedback is obtained, which also depends on the fictitious system states. Since they are not available measured, we add to the static controller a dynamical system which estimates the fictitious system state. Actually, we implicitly apply the equivalence certitude principle (see [34]). On this basis, the obtained controller is now dynamical and the closed loop system is now the connection of two dynamical systems. The obtained controller is then a solution of the synthesis problem if the closed loop system satisfies the incremental real bounded lemma conditions. The conclusion of this short discussion is a new question: even in full information case, has the controller to be dynamical? Clearly, this simple, but essential, question is an open problem in the incremental approach (see [48] for a first answer to this question). In addition, in the gain scheduling approach, the possible answer would allow to assess the necessary structure of a gain scheduling controller.

8 Conclusion

As a conclusion, we first propose figure 7. In this figure, the advantages of the (weighted) incremental norm with respect to existing nonlinear concepts and nonlinear system properties developed in this paper are summarized.

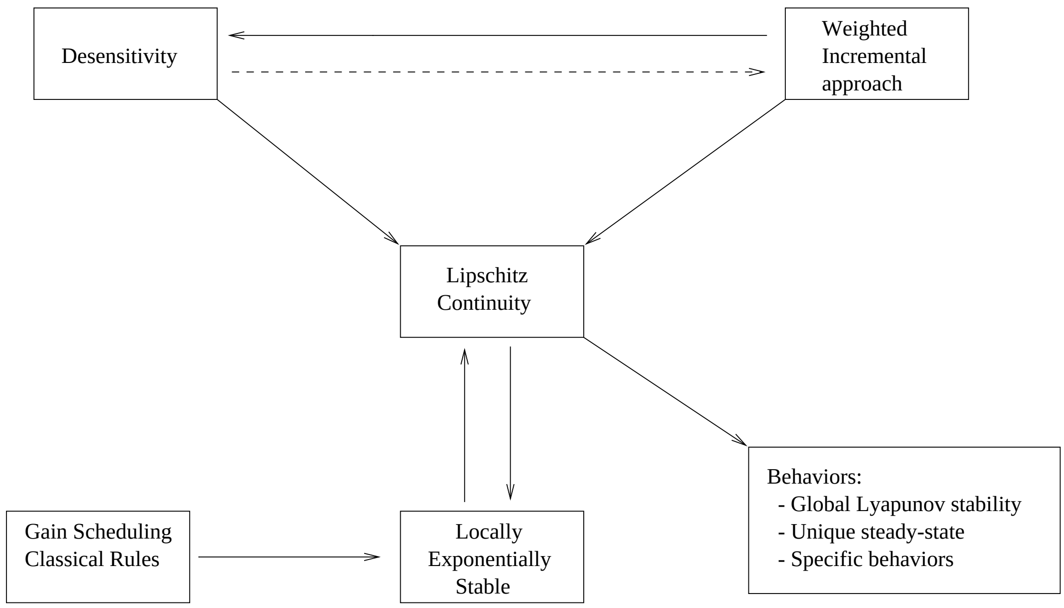

Figure 7: Main implications proved in this paper

Actually, our theoretical framework presents two main advantages.

- It provides a nonlinear framework to the gain-scheduling techniques.
- Solutions to the problem of minimizing the incremental norm of a nonlinear operator can be used to improve the results provided by classical gain-scheduling techniques.

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A Proof of proposition 3.2

Before developing the proof of proposition 3.2, we first focus on the Gâteaux differentiability of a memoryless nonlinearity.

Let $g(u)$ be a nonlinear function defined from $\mathcal{R}^n$ into $\mathcal{R}^n$ such that $g(0) = 0$. The function g is assumed continuous with respect to u. On this basis, we associate to g, a nonlinear operator, noted $\mathcal{N}$, defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ by[8]:

$$\mathcal{N}(u) = g(u).$$

[8]Note that $\mathcal{N}$ is well-defined if (and only if) g satisfies some sector constraints (see [38]).

Lemma A.1 *If g has a partial derivative $g'_u(u) = \frac{\partial g(u)}{\partial u}$ which is bounded and continuous with respect to u then $\mathcal{N}$ has a Gâteaux derivative at any u_0 of $\mathcal{L}_2^e$, referred to as $D\mathcal{N}_G$, which is defined as:*

$$D\mathcal{N}_G[u_0](h) = \frac{\partial g}{\partial u}(u_0)(h).$$

Proof:

We have the following equality:

$$g(u_0 + \lambda h) - g(u_0) - \lambda \frac{\partial g}{\partial u}(u_0)(h) = \int_{u_0}^{u_0+\lambda h} \frac{\partial g}{\partial u}(\xi) d\xi - \lambda \frac{\partial g}{\partial u}(u_0)(h).$$

Let us consider $\xi = (1-\rho)u_0 + \rho(u_0 + \lambda h)$. Then

$$g(u_0 + \lambda h) - g(u_0) - \lambda \frac{\partial g}{\partial u}(u_0)(h) = \lambda \int_0^1 \left(\frac{\partial g}{\partial u}((1-\rho)u_0 + \rho(u_0 + \lambda h)) - \frac{\partial g}{\partial u}(u_0) \right) h d\rho.$$

By taking the norm of the previous expression, we obtain:

$$\left\| \frac{1}{\lambda} \left(g(u_0 + \lambda h) - g(u_0) \right) - \frac{\partial g}{\partial u}(u_0)(h) \right\|^2 = \left\| \int_0^1 \left(\frac{\partial g}{\partial u}(u_0 + \rho\lambda h) - \frac{\partial g}{\partial u}(u_0) \right) h d\rho \right\|^2$$

and by applying the Cauchy-Bunyakovsky-Schwarz inequality (on $\mathcal{R}$), one has:

$$\left\| \int_0^1 \left(\frac{\partial g}{\partial u}(u_0 + \rho\lambda h) - \frac{\partial g}{\partial u}(u_0) \right) h d\rho \right\|^2 \leq \int_0^1 \left\| \left(\frac{\partial g}{\partial u}(u_0 + \rho\lambda h) - \frac{\partial g}{\partial u}(u_0) \right) h \right\|^2 d\rho.$$

Taking the norm on $\mathcal{L}_2^e$ allows to deduce that:

$$\int_{t_0}^{T} \left\| \frac{1}{\lambda} \left(\mathcal{N}(u_0 + \lambda h) - \mathcal{N}(u_0) \right) - D\mathcal{N}_G[u_0](h) \right\|^2 d\tau \leq \int_{t_0}^{T} \int_0^1 \left\| \left(\frac{\partial g}{\partial u}(u_0 + \rho\lambda h) - \frac{\partial g}{\partial u}(u_0) \right) h \right\|^2 d\rho d\tau$$

Interchanging the order of integration by Fubini's Theorem, one obtain:

$$\left\| \frac{1}{\lambda} \left(\mathcal{N}(u_0 + \lambda h) - \mathcal{N}(u_0) \right) - D\mathcal{N}[u_0](h) \right\|_{2,T}^2 \leq \int_0^1 \int_{t_0}^{T} \left\| \left(\frac{\partial g}{\partial u}(u_0 + \rho\lambda h) - \frac{\partial g}{\partial u}(u_0) \right) h \right\|^2 d\tau d\rho.$$

We then apply Theorem 19.1 proposed in [38] which allows to prove that for any fixed $T \in \mathcal{R}$ and any fixed $v \in \mathcal{L}_2^e$, the operator defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ by

$$f_v(z) = \left(\frac{\partial g}{\partial u}(u + z) - \frac{\partial g}{\partial u}(u) \right) v$$

is a continuous and bounded operator on $\mathcal{L}_2^e$, *i.e.* for all $v \in \mathcal{L}_2^e$, for all $T \in \mathcal{R}$ there exists $\eta(v,T) > 0$ such that $\|f_v(z)\|_{2,T} \leq \eta(v,T)\|z\|_{2,T}$. From this result, we finally deduce that:

$$\left\| \frac{1}{\lambda}\left(\mathcal{N}(u_0+\lambda h) - \mathcal{N}(u_0)\right) - D\mathcal{N}_G[u_0](h) \right\|_{2,T} \leq \lambda\eta(h,T)\|h\|_{2,T}\sqrt{\int_0^1 \rho^2 d\rho}$$

which allows to conclude that $\mathcal{N}$ has a Gâteaux derivative on $\mathcal{L}_2^e$.

□

Proof of Proposition 3.2: Let us introduce the following operator:

$$\Phi(x,\dot{x},u,y) = \begin{pmatrix} \dot{x} \\ y \end{pmatrix} - \begin{pmatrix} f(x,u) \\ h(x,u) \end{pmatrix}$$

with $\dot{x}, u$ and y in $\mathcal{L}_2^e$ and with x in AC_2^e, which corresponds to the space of absolute continuous functions (AC) of times equipped with the 2 norm.
This operator defined from $AC_2^e \times \mathcal{L}_2^e \times \mathcal{L}_2^e \times \mathcal{L}_2^e$ into $\mathcal{L}_2^e$, has a Gâteaux derivative on $\mathcal{L}_2^e$ since it is the difference between a linear operator (which has a Gâteaux derivative on AC_2^e) and a nonlinear operator, namely $\mathcal{N} : AC_2^e \times \mathcal{L}_2^e \rightarrow \mathcal{L}_2^e$, defined by

$$\mathcal{N}(x,u) = \begin{pmatrix} f(x,u) \\ h(x,u) \end{pmatrix}$$

which has also a Gâteaux derivative (see lemma A.1). We then deduce that

$$D\Phi_G[x,\dot{x},u,y]\begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \\ \bar{u} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\bar{x}} \\ \bar{y} \end{pmatrix} - \begin{pmatrix} \frac{\partial f(x,u)}{\partial x} & \frac{\partial f(x,u)}{\partial u} & 0 \\ \frac{\partial h(x,u)}{\partial x} & \frac{\partial h(x,u)}{\partial u} & 0 \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{u} \\ \bar{y} \end{pmatrix}$$

which allows to conclude.

B Proof of theorem 3.3

Proof: By introducing a dummy variable, $\beta \in \mathcal{R}$, we define the following abstract function from $\mathcal{R}$ into $\mathcal{L}_2^e$

$$f(\beta) = \Sigma(u + \beta\Delta u)$$

where $\Delta u = u_2 - u_1$. Since Σ is Gâteaux differentiable then f has also a derivative. Indeed,

$$\frac{f(\beta+\Delta\beta) - f(\beta)}{\Delta\beta} = \frac{\Sigma(u+\beta\Delta u + \Delta\beta\Delta u) - \Sigma(u+\beta\Delta u)}{\Delta\beta}$$

and thus (by definition of the Gâteaux derivative)

$$f'_+(\beta) = \lim_{\Delta\beta \downarrow 0} \frac{f(\beta+\Delta\beta) - f(\beta)}{\Delta\beta} = D\Sigma_G[u+\beta\Delta u](\Delta u).$$

In the same way, we can prove that

$$f'_-(\beta) = \lim_{\Delta\beta \downarrow 0} \frac{f(\beta-\Delta\beta) - f(\beta)}{-\Delta\beta} = D\Sigma_G[u+\beta\Delta u](\Delta u)$$

which ensures, since $f_+(\beta) = f_-(\beta)$, that f has a derivative for any β which is given by $f'(\beta) = D\Sigma_G[u+\beta\Delta u](\Delta u)$.

Let us prove that $f'(\beta)$ is a continuous function of its argument. To this purpose, let us denote $u_1 = u + \beta\Delta u$ and $u_2 = u + \beta'\Delta u$ and let us define the following quantity:

$$\begin{aligned}\Pi = \|\lambda(f'(\beta') - f'(\beta))\|_{2,T} - \|\Sigma(u_1+\lambda\Delta u) - \Sigma(u_1) - D\Sigma_G[u_1](\lambda\Delta u) \cdots \\ \cdots - (\Sigma(u_2+\lambda\Delta u) - \Sigma(u_2) - D\Sigma_G[u_2](\lambda\Delta u))\|_{2,T}\end{aligned}$$

Since $\|x\| - \|y\| \le \|y - x\|$, one has

$$\Pi \le \|\Sigma(u_1+\lambda\Delta u) - \Sigma(u_1) - \Sigma(u_2+\lambda\Delta u) + \Sigma(u_2)\|_{2,T}.$$

Σ is Lipschitz continuous on $\mathcal{L}_2^e$ *i.e* for any $T \in [t_0, \infty)$ there exists a finite constant $L_\Sigma(T)$ such that $\|\Sigma(w_1) - \Sigma(w_2)\|_{2,T} \le L_\Sigma(T)\|w_1 - w_2\|_{2,T}$, and thus $\Pi \le 2L_\Sigma(T)\|(\beta-\beta')\Delta u\|_{2,T}$. On this basis, one has:

$$\begin{aligned}\|f'(\beta') - f'(\beta)\| \le \tfrac{2L_\Sigma(T)}{\lambda}\|(\beta-\beta')\Delta u\|_{2,T} + \cdots \\ \cdots \tfrac{\|\Sigma(u_1+\lambda\Delta u) - \Sigma(u_1) - D\Sigma_G[u_1](\lambda\Delta u)\|_{2,T}}{\lambda} + \tfrac{\|\Sigma(u_2+\lambda\Delta u) - \Sigma(u_2) - D\Sigma_G[u_2](\lambda\Delta u)\|_{2,T}}{\lambda}\end{aligned}$$

It is not difficult to conclude that f' is continuous function, *i.e.* for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $\beta' \in \mathcal{R}$ such that $\|\beta' - \beta\| < \delta$, we have $\|f'(\beta') - f'(\beta)\| \le \epsilon$. Indeed, since Σ has a Gâteaux derivative, λ can be chosen such that the sum of the second and third terms of the left side of the previous inequality is less than $\frac{\epsilon}{2}$. On this basis, δ can also be chosen such that the first term is also less than $\frac{\epsilon}{2}$.

Following classical results concerning integration of abstract functions (see [38]), it can be claimed that, since $f(\beta)$ and $f'(\beta)$ are continuous functions of their argument, we have:

$$f(1) - f(0) = \int_0^1 f'(\beta)d\beta$$

and thus

$$\Sigma(u+\Delta u) - \Sigma(u) = \int_0^1 D\Sigma_G[u+\beta\Delta u](\Delta u)d\beta$$

which corresponds to the claimed result.