# WEIGHTED INCREMENTAL NORM: A NEW APPROACH TO GAIN SCHEDULING[1]

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**Abstract**

The weighted incremental norm approach was originally introduced as a natural framework for extending well-known $H_\infty$ linear control concepts into the nonlinear context. In this paper, we investigate the numerous links between this new approach and the classical gain-scheduling technique: more precisely, we show that the control objectives of the gain-scheduled controller design can be expressed as the weighted incremental norm minimization of a nonlinear operator. The result interest is twofold: it first provides a rigorous mathematical formulation of the gain-scheduling problem. Furthermore, existing gain-scheduling techniques can be interpreted as approximate solutions to the weighted incremental norm minimization of a nonlinear operator. This paper is a shortened version of the technical report [13].

## 1 Introduction

The gain scheduling approach is a very classical and widespread nonlinear control technique. The underlying idea is to design at one or more operating points linear time invariant controllers using the associated linearized plant models. The nonlinear control law is then obtained by interpolating (or scheduling) these controllers as a function of the operating point [21, 2, 17, 15].

Actually, from a theoretical point of view and from previous works, it seems that the main goal of the gain scheduling approach is to ensure at least the exponential stability of the closed loop system linearizations. This point of view is in fact a direct consequence of the Lyapunov's indirect method. Despite its local nature, the method presents the advantage of being rigorous. Unfortunately, testing the stability of a linear time-varying system which is associated to a nonlinear system linearization, is a difficult problem, so that the following conjecture is generally used: if the frozen-time systems associated with the time varying system are stable, then the time varying system is also stable. This idea, which is the basis of most gain scheduling techniques (and which can be compared in some sense to the famous Aizerman/Kalman's conjecture), is obviously false in the general case. This conjecture is in fact neither sufficient nor necessary. This however explains that most of the work concerning gain scheduling is devoted to the search of conditions, under which this conjecture becomes true [21, 9, 25, 20].

Actually, in the context of nonlinear control, it is well-known that the exponential stability requirement for all the closed loop system linearizations is in the general case a very strong requirement. Following this remark, it seems necessary to make crystal clear if this requirement can be justified in a theoretical way.

A second point is linked to a pragmatic point of view. Actually, when controlling the nonlinear plant, a natural requirement is that the time invariant linearizations along particular constant inputs satisfy the robustness and performance criteria, classically considered in linear control. As a consequence, we focus the following question: is it possible to justify in a theoretical way this pragmatic point of view?

As we will show in the sequel, the answer to these two questions is yes. In fact, a justification can be found in the sensitivity problem. Let us recall that the desensitivity property corresponds to the fact that the feedback control laws can attenuate the effect of non measurable system perturbations and "shrink" of the model uncertainty effects. Desensitivity is in fact the major motivation (maybe the only one) for using feedback strategies versus open loop one [3, 4, 14, 24, 6]. A first paper conclusion is that the classical gain scheduling control objectives (constraint some system linearization properties) can be justified in many control problems, so that the main restriction of the approach is the fact that existing gain scheduling techniques achieve these objectives only in an approximate (or even heuristic) way.

A second difficulty of the gain scheduling approach is that this technique can not be considered as an approximation of any existing nonlinear control design method. This fact has been already pointed out by Wilson J. Rugh [17]: *"What is most striking about gain-scheduling is that, while it is ever more widely used in practice, it has been widely ignored from a theoretical perspective. In particular, it remains unstudied as an explicitly nonlinear control approach. So it seems that gain scheduling is another example of the lamented theory/application gap but in this case application is ahead of theory"*.

In other words, a global nonlinear framework for the

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analysis and the design of gain scheduling control systems has to be proposed. The interest of such a framework is clear: assess the mathematical complexity of the control law design, satisfying the gain-scheduling objectives in a nonlinear context.

As we will show in the sequel, the weighted incremental norm seems to be a natural framework for this problem. First, note that the weighted incremental norm was recently introduced in [7, 12] as a solution for extending well-known $H_\infty$ linear control concepts into the nonlinear context. The nonlinear control problem is in a non-stationary way transformed into a well-defined optimization problem, which is the weighted incremental norm minimization of a nonlinear operator. This approach allows to simultaneously consider major specifications such as robust stability, sensitivity and attenuation with respect to exogenous perturbations, suitable steady state behaviors associated with step and periodic inputs and Lyapunov stability.

The main objective of this paper is thus to explain why and how this incremental approach provides a natural mathematical framework to the gain scheduling control problem. For the sake of shortness, only some aspect are developed in this paper. The reader is referred to [13] for a far more complete discussion.

**Notations and definition.** The notations and terminology are classical in the input-output context (see [23]). The $\mathcal{L}_2$-norm of $f : [t_0, \infty) \mapsto \mathcal{R}^n$ is $\|f\|_2 = (\int_{t_0}^{\infty} \|f(t)\|^2 dt)^{1/2}$. The *causal truncation* at $T \in [t_0, \infty)$, denoted by $P_T f$ gives $P_T f(t) = f(t)$ for $t \in [t_0, \infty)$ and 0 otherwise. The *extended space*, $\mathcal{L}_2^e$ is composed with the functions whose causal truncations belong to $\mathcal{L}_2$. For convenience, $\|P_T u\|_2$ is denoted by $\|u\|_{2,T}$.

In the sequel, we consider systems with the differential representation

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & h(x(t), u(t)) \\ x(t_0) & = & x_0 \end{array} \right. \quad (1)$$

where $x(t) \in \mathcal{R}^n$, $y(t) \in \mathcal{R}^m$, and $u(t) \in \mathcal{R}^p$. $f$ and $h$, defined from $\mathcal{R}^n \times \mathcal{R}^p$ into $\mathcal{R}^n$ and $\mathcal{R}^m$ respectively, are assumed to be $C^2$ and uniformly Lipschitz. Moreover one has $f(x_0, 0) = 0$ and $h(x_0, 0) = 0$. The unique solution $x(t) = \phi(t, t_0, x_0, u)$ belongs to $\mathcal{L}_2^e$ for all $x_0 \in \mathcal{R}^n$ and for any $u \in \mathcal{L}_2^e$. An *unperturbed motion* (of $\Sigma$) is a specific motion of $\Sigma$ associated with an input, $u_r \in \mathcal{L}_2^e$, and with an initial condition, $x_{0r} \in \mathcal{R}^n$, *i.e.* $x_r(t) = \phi(t, t_0, x_{0r}, u_r)$. With reference to this definition, we denote by $x[x_{0r}, u_r]$ this unperturbed motion. The notion of (incremental) $\mathcal{L}_2$-gain can now be recalled. $\Sigma$ is said to be a finite gain stable system if there exists $\gamma \geq 0$ such that $\|\Sigma(u)\|_2 \leq \gamma \|u\|_2$ for all $u \in \mathcal{L}_2$. The gain of $\Sigma$ coincides with the minimum value of $\gamma$ and is denoted by $\|\Sigma\|_i$. $\Sigma$ has a finite incremental gain if there exists $\eta \geq 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta \|u_1 - u_2\|_2$ for all $u_1, u_2 \in \mathcal{L}_2$. The incremental gain of $\Sigma$ coincides with the minimum value of $\eta$ and is denoted by $\|\Sigma\|_\Delta$. $\Sigma$ is said to be incrementally stable if it is stable, *i.e.* it maps $\mathcal{L}_2$ to $\mathcal{L}_2$, and has a finite incremental gain. We consider, in the sequel, the nonlinear feedback system depicted in figure 1, where $G, K, F$ are nonlinear causal operators from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, representing respectively the plant, the compensator and the feedback, and where $r, e, u$ and $y$, which belong to $\mathcal{L}_2^e$, denote respectively the system input, the error signal, the plant input, and the system output. The closed-loop system is assumed to be well-posed and the input-output map between the system input and the system output, denoted by $H_{yr}$, is given by $GK(I + FGK)^{-1}$.

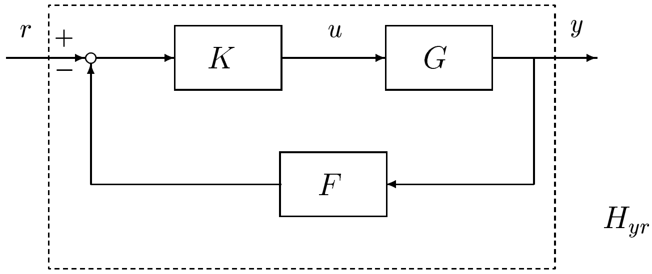


**Figure 1:** The nonlinear feedback system

## 2 Linearization and related results

The main concept in the gain scheduling approach is the linearization one. This concept is classically related to the first order approximation of the system along some specific motion. As a matter of fact, this point of view is clearly linked to stability considerations and Lyapunov like arguments. In the sequel, we try to convince the reader that the Gâteaux derivative is, in the context of gain scheduling approach, a more interesting notion. Indeed, it is an input/output notion which seems to be a better frame for analyzing the system properties.

**Global vs local: the Gâteaux derivative as a limit** In the following, we characterize the global behavior of a nonlinear system as a sequence of its local variations. Let us consider $\Delta u$ the input variation and $\Delta y$ the associated output variation:

$$\Delta y = \Sigma(u + \Delta u) - \Sigma(u). \quad (2)$$

The objective is to characterize in an accurate way the effect of $\Delta u$ on $\Delta y$. Defining the local input variation $\delta u = \Delta u / n$, where $n$ is an integer, the output variation is then rewritten as the sum of the outputs associated with small input increments:

$$\Delta y = \sum_{i=1}^{n} \Sigma(u + i\delta u) - \Sigma(u + (i-1)\delta u). \quad (3)$$

Let us consider equation (3) when the norm of the input increment goes to zero. As in a classical space (e.g. $\mathcal{R}$), the limit can be studied under the introduction of the operator derivative around a specific input. In fact, in the functional analysis frame, there exist, at least, five notions of derivative. The Fréchet derivative, which is "similar" to the classical derivative on $\mathcal{R}$, can not be used since it is not defined for many dynamical systems [22]. We then use a weaker notion of the derivative, the Gâteaux one.

**Definition 2.1** [1, 23] *Given a causal operator $\Sigma$, defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, let us introduced $u_0 \in \mathcal{L}_2^e$ and* *let us assume the existence for any $T \in [t_0, \infty)$ and any $h \in \mathcal{L}_2^e$ of a continuous linear operator $D\Sigma_G[u_0]$, from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, satisfying:*

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma(u_0 + \lambda h) - \Sigma(u_0)}{\lambda} - D\Sigma_G[u_0](h) \right\|_{2,T} = 0$$

*then $D\Sigma_G[u_0]$ is called the Gâteaux derivative of $\Sigma$ at $u_0$.*

When the system is generated by differential equations, definition 2.1 reduces to the usual linearization concept. More precisely, we have the following proposition.

**Proposition 2.1** [13] *Let us consider $\Sigma$ defined by (1) and let us assume that $f$ and $h$ are uniformly Lipschitz and $C^2$ (that is, it is twice derivable). Then, for any $u_r \in \mathcal{L}_2^e$, the system has a Gâteaux derivative which satisfies the following differential equations:*

$$\bar{y} = D\Sigma_G[u_r](\bar{u}) \begin{cases} \dot{\bar{x}}(t) &= A(t)\bar{x}(t) + B(t)\bar{u}(t) \\ \bar{y}(t) &= C(t)\bar{x}(t) + D(t)\bar{u}(t) \\ \bar{x}(t_0) &= 0 \end{cases} \quad (4)$$

*with $A(t) = \frac{\partial f}{\partial x}(x_r(t), u_r(t))$, $B(t) = \frac{\partial f}{\partial u}(x_r(t), u_r(t))$, $C(t) = \frac{\partial h}{\partial x}(x_r(t), u_r(t))$ and $D(t) = \frac{\partial h}{\partial u}(x_r(t), u_r(t))$ and where $x_r(t) = \phi(t, t_0, x_0, u_r)$ is the solution of system (1) under input $u_r(t)$ and $x(t_0) = x_0$.*

Let us now consider equation (3). We first note that for all $T \geq t_0$:

$$\Sigma(u + \lambda\Delta u) = \Sigma(u) + D\Sigma_G[u](\lambda\Delta u) + o(\lambda).$$

We consequently deduce that (3), enables us to approximate $\Delta y$ for $n$ sufficiently large by:

$$\Delta y \approx \sum_{i=0}^{n-1} D\Sigma_G[u + i\delta u](\delta u) \quad (5)$$

This relation shows that the output variation can thus be interpreted as the sum of the output signals, which are associated with the response of linear time-varying systems, namely $D\Sigma_G[u + i\delta u]$ to the input signal $\delta u$ on a finite time interval.

The approximation given by (5) can be replaced by an integral formula.

**Theorem 2.2** [13] *Let us assume that $\Sigma$ is Gâteaux differentiable on $\mathcal{L}_2^e$. For any $T \in [t_0, \infty)$ and $u_1, u_2 \in \mathcal{L}_2^e$, one has*

$$\Sigma(u_2) - \Sigma(u_1) = \int_0^1 D\Sigma_G[u_1 + \beta(u_2 - u_1)](u_2 - u_1)d\beta$$

**Global versus local: the means value theorem** A second result allows to relate local and global aspects under a simple result which links the Lipschitz constant of the nonlinear operator and the norm of its derivatives.

**Theorem 2.3** [23, 13] *Let us assume that a causal operator $\Sigma$ defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ has a Gâteaux derivative at each point $u_0$ of $\mathcal{L}_2^e$. Then, there exists a finite constant $\eta$ such that for any $T \geq t_0$ and for any $u_0, h \in \mathcal{L}_2^e$, one has*

$$\|D\Sigma_G[u_0](h)\|_{2,T} \leq \eta \|h\|_{2,T} \quad (6)$$

*if and only if the nonlinear operator is such that*

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \eta \|u_1 - u_2\|_{2,T} \quad (7)$$

*for any $T \geq t_0$ and any $u_1$ and $u_2$ belonging to $\mathcal{L}_2^e$.*

**Incremental stability and exponential stability of the linearizations** We present in this section a necessary and sufficient condition for incremental stability. A nonlinear operator is incrementally bounded if and only if its linearizations have exponential stable minimal state-space realizations.

**Definition 2.2** [19] *$D\Sigma_G$, a Gâteaux derivative of $\Sigma$ defined by (4), is said to have a minimal state-space realization if the pair $[A(t), B(t)]$ is uniformly controllable and the pair $[A(t), C(t)]$ is uniformly observable.*

**Theorem 2.4** [13] *Let us assume that $\Sigma$ is Gâteaux differentiable on $\mathcal{L}_2^e$. If the state space representation of each derivative of $\Sigma$ is minimal then there exists a finite constant $\eta$ such that for any $T \geq t_0$ and any $u_1$ and $u_2$ belonging to $\mathcal{L}_2^e$ one has:*

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \eta \|u_1 - u_2\|_{2,T}$$

*if and only if all its linearizations are exponentially stable.*

## 3 The sensitivity as the key justification

The key justification of gain scheduling techniques can be found in the sensitivity problem. Remember that a deep motivation for using feedback control is the reduction of the effect of non measurable noises and the "shrinking" of the model uncertainties [3, 4, 14, 24, 6]. The sensitivity concept has been introduced in the linear context as a means to quantify the efficiency of a feedback law, with respect to the effects of small perturbations which are due either to exogenous perturbations, or to parameter variations [3, 4, 14]. The infinitesimal nature of this characterization has led however some authors [5] to introduce a more general concept: the comparison sensitivity function. The idea is now to compare the performance of a closed loop system with the performance of an equivalent open loop system, against small or large disturbances and model perturbations.

The sensitivity concept has been also studied in the nonlinear context by Kreindler [16], which has extended the sensitivity function introduced in [3] by defining the differential sensitivity function, which characterizes the first order sensitivity. This characterization is obtained on the basis of properties associated with the linearization of a suitable relation between an input and an output of the nonlinear closed loop system. More recently, the comparison sensitivity properties, associated with a nonlinear closed loop system, were exactly quantified by Desoer and Wang, using a Taylor type expansion of a linearizable nonlinear operator [6]. Considering the results by Desoer and Wang (and obviously by Kreindler), it is possible to claim that the performance of the nonlinear system (in terms of the sensitivity properties) clearly depends on the properties of the linearization of the closed-loop operator along the possible trajectories of the system.

As a consequence, the relation between the sensitivity objectives and the properties of the linearizations allows us to give a justification to the local gain scheduling objectives (this point can be considered as the main motivation for using a gain scheduling control law in an adaptive control scheme - see [2] and for a theoretical justification to this fact when a slow time variation of the system is assumed see [25]).

Finally, let us recall that the main limitation of the gain scheduling approach in this problem is due to the fact that the linear time varying objective, associated with the sensitivity requirement, is obtained using the following heuristic: the linear time invariant frozen systems, which are associated with each constant value of the varying parameters, must have "good properties", so as to obtain also "good properties" for the time-varying system.

Unfortunately, this kind of condition is neither sufficient [21] nor necessary in the general case for ensuring suitable properties to the linearization of the nonlinear closed loop system and thus for ensuring suitable nonlinear properties with respect to the sensitivity problem.

**Sensitivity objective: some recalls** we now investigate how the output disturbance problem in nonlinear control implies strong constraints on the closed-loop system linearizations. The reader is referred to [6] for a complete presentation of desensitivity problem in the nonlinear context. In the sequel, we will just consider the output disturbance problem (the other cases presented in [6] can be worked out as well). For this

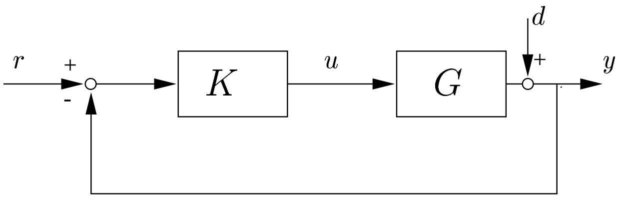


**Figure 2:** The perturbed closed-loop system

purpose, let us assume that $F = I$ and let us associate to the closed-loop system depicted in figure 2 an "equivalent" open-loop map, $Ho_{yr}$, depicted in figure 3. If the open-loop controller is defined by $K_o = K(I + GK)^{-1}$ then the open-loop system depicted in figure 3, which maps the inputs $(r, d)$ in $\mathcal{L}_2^e \times \mathcal{L}_2^e$ to the output $y$ in $\mathcal{L}_2^e$, satisfies for all $r \in \mathcal{L}_2^e$ and for $d = 0$, the following equality: $Ho_{yr}(r, 0) \stackrel{\Delta}{=} H_{yr}(r, 0)$ where $H_{yr}$ is the system depicted in figure 2 which maps inputs $(r, d)$ which belong to $\mathcal{L}_2^e \times \mathcal{L}_2^e$ to the output $y$ which also belongs to $\mathcal{L}_2^e$.

The effect induced by the output perturbations on the closed-loop system is given by $\delta H_{yr}(r, d) = d + GK(I + GK)^{-1}(r - d) - GK(I + GK)^{-1}(r)$. Since $GK(I + GK)^{-1} + (I + GK)^{-1} = I$, one deduces that $\delta H_{yr}(r, d) = S(r - d) - S(r)$ with $S = (I + GK)^{-1}$. The effect induced by the output perturbations on the open-loop system is given by $\delta Ho_{yr}(r, d) = GK(I + GK)^{-1}(r) + d - GK(I + GK)^{-1}(r) = d$.

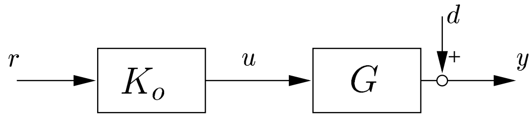


**Figure 3:** The perturbed equivalent open-loop system

The main interest of the feedback law strategy is that it allows to achieve a better reduction of effects of the disturbances with respect to the open loop strategy. In mathematical terms, the feedback has a desensitivity effect if the following inequality is satisfied:

$$\|\delta H_{yr}(r, d)\|_{2,T} < \|\delta Ho_{yr}(r, d)\|_{2,T}.$$

Unfortunately for realistic systems, this inequality cannot be satisfied for any input and disturbance in $\mathcal{L}_2^e$. Indeed, as in the linear context (see [24]), one has the following theorem.

**Theorem 3.1** [7, 12] *Consider the closed-loop system in figure 1. If the open-loop operator $GK$ is strictly causal then*

$$\|(I + GK)^{-1}\|_\Delta \geq 1.$$

This relation implies that there exist $r, d \in \mathcal{L}_2^e$ such that:

$$\|S(r - d) - S(r)\|_{2,T} \geq \|d\|_{2,T}$$

and thus there necessarily exists, a least, a disturbance such that the feedback law attenuation is not better than the open loop type one. We moreover point out that the use of the feedback implies the incremental stability of $S$. Actually, if $S$ is not incrementally stable, then for any $K$, there always exist $r$ and $d$, such that

$$\|S(r - d) - S(r)\|_{2,T} \geq K\|d\|_{2,T}$$

and thus there exists some perturbation whose effects are arbitrarily amplified.

Following this preliminary remark, the interest of feedback law is necessarily limited to a specific class of perturbations. We then introduce a set of possible perturbations, namely $P^e \subset \mathcal{L}_2^e$, for which we want ensure that the closed loop strategy performance is better than the open loop one. It is moreover clear that the cost of feedback induced by the stability problems implies that the use of the feedback control law could be justified if (and only if) there exists an $\epsilon(\ll 1)$ such that:

$$\|\delta H_{yr}(r, d)\|_{2,T} \leq \epsilon\|\delta Ho_{yr}(r, d)\|_{2,T}$$

for any $d \in P^e \subset \mathcal{L}_2^e$ and any $r \in \mathcal{L}_2^e$.

**Sensitivity objective strongly constraints the linearizations** We now prove that the sensitivity requirement strongly constraints the properties of the system linearizations. Actually, sensitivity objective constraints the exponential stability of the system linearizations.

**Proposition 3.2** [13] *Let us assume that the sensitivity map, i.e. $S = (I + GK)^{-1}$, is Gâteaux differentiable on $\mathcal{L}_2^e$ and that the set of possible disturbances,* *i.e.* $P^e$, *is a convex set which contains the null signal. The desensitivity is achieved with level* $\epsilon > 0$, *i.e for any* $r \in \mathcal{L}_2^e$ *and* $d \in P^e$, *one has:*

$$\|\delta H_{yr}(r,d)\|_{2,T} \leq \epsilon \|\delta Ho_{yr}(r,d)\|_{2,T}$$

*if and only if for any* $u_0 \in \mathcal{L}_2^e$ *and* $w \in P^e$, *one has:*

$$\|DS_G[u_0](w)\|_{2,T} \leq \epsilon \|w\|_{2,T}$$

We then deduce from this short discussion that the nonlinear system sensitivity is achieved if (and only if) all the system linearizations have a good behavior with respect to perturbations belonging to $P_d^e$.

The desensitivity requirement is achieved only if the sensitivity map, *i.e.* $S$, has a finite incremental gain. This ensures that the degradation introduced by the feedback use is finite. Following theorem 2.4, this condition constrains the exponential stability of the linearizations of the system:

**Proposition 3.3** [13] *Let us assume that the sensitivity map, i.e.* $S = (I+GK)^{-1}$, *is Gâteaux differentiable on* $\mathcal{L}_2^e$ *then desensitivity can be only achieved if and only if the minimal state-space realization of each linearization of* $S$ *is exponentially stable.*

## 4 Connections with the incremental approach

The incremental norm framework is useful for analyzing the properties of nonlinear closed loop systems from both quantitative and qualitative points of view.

In a quantitative way, it is possible to analyze the robustness and performance properties of a nonlinear closed loop system. The weighted incremental norm approach was indeed originally introduced as an extension of the classical $H_\infty$ control concepts into a nonlinear context: in a linear context, the original idea of [24] was to recast the initial design problem into a well defined optimization problem, involving the minimization of a *weighted* $H_\infty$ norm. In the same way, in the incremental norm approach, the idea is to define the robustness and performance properties of a (nonlinear) system by adding a suitable weighting function, which reflects the desired properties for the closed loop.

In a qualitative way, incrementally stable systems possess suitable steady-state properties, and the effect of a non zero initial condition is guaranteed to decay asymptotically to zero (in fact incremental stability implies asymptotic stability in sense of Lyapunov of any unperturbed trajectory of the system). As a first point, a unique steady-state motion corresponds to a given input signal, independently of the initial condition and despite a vanishing perturbation on the input signal. As a second point, the steady state response to a constant (resp. periodic) input signal is also constant (resp. periodic).

Note finally that some basic properties of incrementally stable systems will be recalled in the following sections. The reader is referred to [7, 11, 12] for a more complete presentation. Note that an extended version of this paper is available as a technical report [13].

**Desensitivity** In the sequel, it is shown how the conditions given in section 3 can be reformulated as the minimization of the incremental norm of a suitable weighted operator.

As a matter of fact, as in the $H_\infty$ approach, we now assume that the set of possible disturbances for which desensitivity must be achieved can be defined by the set of disturbances belonging to $\mathcal{L}_2^e$ such that

$$\|W_p^{-1}(d) - W_p^{-1}(r+d)\|_{2,T} \leq \epsilon \|d\|_{2,T}$$

for any $r \in \mathcal{L}_2^e$ where $W_p$ and $W_p^{-1}$ are two causal and incrementally stable operators. Under the introduction of this weighting function, the desensitivity is achieved if the "weighted" incremental norm of the sensitivity function is less than 1. More precisely, we have this following result.

**Theorem 4.1** [12] *Consider the nonlinear feedback system depicted in figure 2. If*

$$\|SW_p\|_\Delta \leq 1$$

*then* $\|\delta H_{yr}(r,d)\|_{2,T} \leq \epsilon \|\delta Ho_{yr}(r,d)\|_{2,T}$ *for any* $d \in P^e \subset \mathcal{L}_2^e$ *and any* $r \in \mathcal{L}_2^e$.

**Stationary linearizations** We now investigate the close connection between the incremental approach and the pseudo-linearization type criteria. To this purpose, we restrict our attention to a specific class of linearizations, namely the time invariant ones. Note that this class was classically considered in the gain scheduling approach. We then define $Z_e$, the set of equilibrium points associated with any constant input:

$$Z_e = \{(x_e, u_e) \in \mathcal{R}^n \times \mathcal{R}^p \mid \phi(t, t_0, x_e, u_e) = x_e \ \forall t \geq t_0\}$$

where $\phi$ is the state transition map of $\Sigma$.

**Theorem 4.2** [8] *Let* $\Sigma$ *be the system given by (1) with a finite incremental gain* $\eta$. *Let* $u_e$ *be any constant input and* $x_e$ *be its associated equilibrium point. If* $x_e$ *is reachable from* $x_0$ *then the linearization of* $\Sigma$, *given by the following linear time invariant system:*

$$D\Sigma_G(u_e) \left\{ \begin{array}{rcl} \dot{\bar{x}}(t) & = & A\bar{x}(t) + B\bar{u}(t) \\ \bar{y}(t) & = & C\bar{x}(t) + D\bar{u}(t) \\ \bar{x}(t_0) & = & 0 \end{array} \right. \tag{8}$$

$A = \frac{\partial f}{\partial x}(x_e, u_e)$, $B = \frac{\partial f}{\partial u}(x_e, u_e)$, $C = \frac{\partial h}{\partial x}(x_e, u_e)$ *and* $D = \frac{\partial h}{\partial u}(x_e, u_e)$, *has a finite* $\mathcal{L}_2$ *gain less than or equal to* $\eta$, *i.e.* $\|D\Sigma_G[u_e]\|_i \leq \eta$.

This result makes crystal clear a direct connection between our nonlinear framework and the classical gain scheduling techniques, especially with the approaches based on the extended linearization (see *e.g.* [17]). In these approaches, some properties are imposed to the linear time-invariant linearizations of the system associated with constant inputs. Finally, let $M_{zw} = W_o H W_i$ be the augmented plant where $W_i$ and $W_o$ are the input and output weighting functions associated with robustness and performance requirements (see [12]). We moreover assume that the augmented system is described by a differential equation with $C^2$ and globally Lipschitz drift and output functions (this ensures the existence of the Gâteaux derivative of the augmented system). With respect to the weighted incremental norm approach and with reference to the augmented system (whose norm is assumed less than 1, *i.e.* $\|M_{zw}\|_\Delta \leq 1$), theorem 4.2 ensures that all the linearizations satisfy a weighted $H_\infty$ criterion. This criterion is specified at each equilibrium point by the stationary linearization of the nonlinear weighting functions, *i.e.*

$$\|DW_{oG}[H(W_i(w_0))]DH_G[W_i(w_0)]DW_{iG}[w_0]\|_i \leq 1$$

where $DW_{oG}[H(W_i(w_0))]$ and $DW_{iG}[w_0]$ are respectively the input and the output linear time invariant weighting functions associated to the $H_\infty$ criterion. This last fact has interesting connections with the work presented in [15].

## 5 Conclusion

As a conclusion, we first propose figure 4. In this figure, the advantages of the (weighted) incremental norm with respect to existing nonlinear concepts and nonlinear system properties are summarized. Actually, our theoretical framework presents two main advantages:

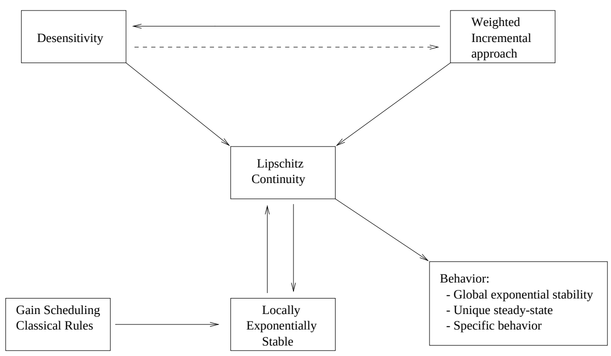


**Figure 4:** Main implications proved in this paper

1. It provides a nonlinear framework to the gain-scheduling techniques.
2. Solutions to the problem of minimizing the incremental norm of a nonlinear operator can be used to improve the results provided by classical gain-scheduling techniques.

A much more detailed discussion is proposed in the technical report [13].


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