Necessary conditions for incremental stability and the second order variations

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***Abstract*— The main goal of this paper is to investigate the links between two incremental stability tests. First conditions, based on the dissipativity framework, leads to test the existence of a suitable available storage function satisfying Hamilton–Jacobi–Bellman type equations. Second conditions, deduced of the mean value theorem in norm, leads to test the finite gain stability of all the linearizations (Gâteaux derivatives) of the nonlinear system. The main contribution of this paper is to point out how the Jacobi like necessary conditions, *i.e.* the second order variations of the dissipativity criteria, allows to connect the test based on the mean value theorem in norm to the one based on the dissipativity framework.**

I. Introduction

Incremental stability was recently proposed as a powerful tool for analyzing qualitative properties [9], [10] and quantitative properties [8], [11] of nonlinear systems. Using two approaches, *necessary and sufficient* conditions were proposed for ensuring incremental stability and computing the incremental norm of nonlinear operators. Nevertheless, due to the problem complexity, deriving computationally efficient tests from these conditions is a difficult problem. This is the motivation of proposing alternative conditions in order to obtain a maximum number of possible tests. Nevertheless, if different conditions are obtained, the question of their connections arises.

A first approach is based on the dissipativity theory and leads to conditions involving Hamilton–Jacobi–Bellman inequalities [8], [7] (an equivalent result was obtained independently in [20]). The second approach is based on a mean value theorem which presents the strong connection between the incremental stability of a nonlinear operator and the uniform $\mathcal{L}_2$ gain stability of its time varying linearizations [14]. Based on this connection, a necessary and sufficient condition of incremental stability was recently proposed in [13] for the Lur'e nonlinear system analysis.

In order to get insight on the relationship between both approaches, we focus on the necessity of uniform $\mathcal{L}_2$ gain stability of its time varying linearizations for ensuring the incremental stability of a nonlinear operator. In this paper, we prove that necessity can be obtained by application of the dissipativity theory. Actually, we reveal how the singularity of the optimal incremental stability criterion implies the uniform $\mathcal{L}_2$ gain stability of its time varying linearizations by a second order variation condition.

II. Notations, definitions and some recalls

A. Considered system, finite gain and incremental stability

The notations and terminology are classical [6]. $\mathcal{L}_p$ denoted the space of $\mathcal{R}^n$ valued functions defined on $[t_0,\infty)$, for which the pth power of the norm is integrable. The $\mathcal{L}_p$ norm is defined by $\|f\|_p = \sqrt[p]{\int \|f(t)\|^p dt}$. The *causal truncation* of $f \in \mathcal{L}_p$ at time $T \in [t_0,\infty)$, denoted by $P_T f$ is given by $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise. For convenience, $\|P_T u\|_p$ is denoted by $\|u\|_{p,T}$. The *extended space* associated to $\mathcal{L}_p$, denoted by $\mathcal{L}_p^e$, is composed with the functions whose causal truncations belong to $\mathcal{L}_p$, *i.e.*, $\mathcal{L}_p^e \triangleq \{f : [t_0,\infty) \to \mathcal{R}^n | \forall T \in [t_0,\infty),\ \|P_T f\|_p < \infty\}$. With $p \geq q \geq 1$ and $T \in [t_0,\infty)$, there exists $C_T \geq 0$ such that if $u \in \mathcal{L}_p^e$ then $\|u\|_{q,T} \leq C_T \|u\|_{p,T}$ and $u \in \mathcal{L}_q^e$.

In the sequel, we consider systems defined by:

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t),u(t)) \\ y(t) & = & h(x(t),u(t)) \\ x(t_0) & = & x_0 \end{array} \right. \qquad (1)$$

where $x(t) \in \mathcal{R}^n$, $y(t) \in \mathcal{R}^m$, and $u(t) \in \mathcal{R}^m$. f and h, from $\mathcal{R}^n \times \mathcal{R}^m$ into $\mathcal{R}^n$ and $\mathcal{R}^m$ resp., are assumed at least C^2 with $f(x_0,0)=0$ and $h(x_0,0)=0$. f and h and their gradients are uniformly Lipschitz continuous. Σ is assumed well–posed, that is, a causal operator from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$. Σ is *finite gain stable* (*incrementally bounded*) from $\mathcal{L}_2$ into $\mathcal{L}_2$ if there exists $\gamma > 0$ ($\eta \geq 0$) such that $\|\Sigma(u)\|_2 \leq \gamma \|u\|_2$ for all $u \in \mathcal{L}_2$ ($\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta \|u_1 - u_2\|_2$ for all $u_1, u_2 \in \mathcal{L}_2$). The gain (incremental gain) of Σ, denoted by $\|\Sigma\|_i$ ($\|\Sigma\|_\Delta$), is the minimum value of γ (η). Σ is said to be *incrementally stable* if it is incrementally bounded and stable, *i.e.*, it maps $\mathcal{L}_2$ to $\mathcal{L}_2$. Theorem 2.1 links the input-output stability properties on $\mathcal{L}_2$ to its properties on the extended space $\mathcal{L}_2^e$.

Theorem 2.1 ([25]): Let Σ be a causal operator, from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ and let be $\eta > 0$. $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta \|u_1 - u_2\|_2$ for any $u_1, u_2 \in \mathcal{L}_2$ if and only if for any $T \in [t_0,\infty)$ $\|\Sigma(u_1) - \Sigma(u_2)\|_{2,T} \leq \eta \|u_1 - u_2\|_{2,T}$ for any $u_1, u_2 \in \mathcal{L}_2^e$.

Remark: There exists a similar theorem for *the finite gain stability*: $\|\Sigma(u))\|_2 \leq \gamma \|u\|_2$ for any $u \in \mathcal{L}_2$ if and only if for any $T \geq t_0$, $\|\Sigma(u))\|_{2,T} \leq \gamma \|u\|_{2,T}$ for any $u \in \mathcal{L}_2^e$.

B. Gâteaux derivative and the mean value theorem [14]

Here, p and q are real numbers greater than or equal to 1.

Definition 2.1: Let Σ be an operator from $\mathcal{L}_p$ into $\mathcal{L}_q$ and $u_r \in \mathcal{L}_p$. If there exists, for any $h \in \mathcal{L}_p$, a continuous linear operator $D\Sigma_G[u_r]$, from $\mathcal{L}_p$ into $\mathcal{L}_q$, such that:

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma(u_r + \lambda h) - \Sigma(u_r)}{\lambda} - D\Sigma_G[u_r](h) \right\|_q = 0$$

then $D\Sigma_G[u_r]$ is called the Gâteaux derivative of Σ at u_r.

The Gâteaux derivative definition is now extended to a larger class of operators.

Definition 2.2: $D\Sigma_G[u_r]$ from $\mathcal{L}_p^e$ into $\mathcal{L}_q^e$ is the Gâteaux derivative at u_r of the causal operator Σ from $\mathcal{L}_p^e$ into $\mathcal{L}_q^e$ if it is linear and for all $T \in [t_0, \infty)$, $P_T D\Sigma_G[u_r]$ is the Gâteaux derivative of $P_T\Sigma$ at $P_T u_r$.

Proposition 2.1: Let Σ be a dynamical system associated to (1) from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$. If f and h are uniformly Lipschitz and C^1 then, for any $u_r \in \mathcal{L}_2^e$, the system has a Gâteaux derivative defined by the following differential equations:

$$\bar{y} = D\Sigma_G[u_r](\bar{u}) \left\{ \begin{array}{rcl} \dot{\bar{x}}(t) & = & A(t)\bar{x}(t) + B(t)\bar{u}(t) \\ \bar{y}(t) & = & C(t)\bar{x}(t) + D(t)\bar{u}(t) \\ \bar{x}(t_0) & = & 0 \end{array} \right. \tag{2}$$

with $A(t) = \dfrac{\partial f}{\partial x}(x_r(t), u_r(t))$, $B(t) = \dfrac{\partial f}{\partial u}(x_r(t), u_r(t))$, $C(t) = \dfrac{\partial h}{\partial x}(x_r(t), u_r(t))$ and $D(t) = \dfrac{\partial h}{\partial u}(x_r(t), u_r(t))$ and where $x_r(t) = \varphi(t, t_0, x_0, u_r)$ is the solution of system (1) under input $u_r(t)$ and $x(t_0) = x_0$.

Theorem 2.2 is a key result in nonlinear control since it links the nonlinear system incremental norm to its derivatives norms.

Theorem 2.2: Let Σ be a dynamical system associated to (1) from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$, with a Gâteaux derivative at each point u_r of $\mathcal{L}_2^e$. Let be $\eta > 0$. For any $u_r, h \in \mathcal{L}_2$, one has $\|D\Sigma_G[u_r](h)\|_2 \leq \eta \|h\|_2$ if and only if for any $u_1, u_2 \in \mathcal{L}_2$, one has $\|\Sigma(u_1) - \Sigma(u_2)\|_2 \leq \eta \|u_1 - u_2\|_2$.

C. Incremental stability and dissipativity

Dissipatitivity is a powerful framework for nonlinear system analysis from the input-output point of view and from the internal stability (Lyapunov like) point of view. It is now clear that Willems, by introducing dissipativity in [28], proposed a theoretical framework which unified these two fundamental aspects of stability, see e.g. [26], [17]. Another major interest of dissipativity is to formulate input-output properties as optimization problems (the paper results are based on this). In general, the resolution of these optimization problems reduce to the resolution of Hamilton–Jacobi–Bellman like equations [28], [16], [18]. This approach is now usual. One of the most striking results is, *e.g.*, the computation of the $\mathcal{L}_2$ gain (H_∞ norm) of an LTI system by the bounded real lemma (LMI optimization) or its associated Riccati equation. This approach had be successfully applied to nonlinear system analysis, see [28], [16], [17], despite numerous technical problems, *e.g.* assumptions on the storage function differentiability, computational complexity, etc. In nonlinear, the Riccati equations or the Linear Matrix Inequalities are replaced by Hamilton–Jacobi–Bellman equations or inequations. Recent results on the resolution of Hamilton–Jacobi–Bellman equations lead to necessary and sufficient conditions with weak assumptions on the system [18]. For the incremental stability, we proved in [8], [7] that the incremental stability analysis boils down to the resolution of Hamilton–Jacobi–Bellman equalities (a similar result is in [20]).

Let us now consider our problem. From our previous works [7], [10], we know that the incremental stability problem can be recast as a dissipativity problem by introducing a fictitious dynamical system Σ_f. This system, from $\mathcal{L}_2^e \times \mathcal{L}_2^e$ into $\mathcal{L}_2^e$, is related to Σ by : $y_f = \Sigma_f(u_1, u_2) \triangleq \Sigma(u_1) - \Sigma(u_2)$. A state-space realization is then given by

$$\Sigma_f \left\{ \begin{array}{rcl} \dot{x}_1(t) & = & f(x_1(t), u_1(t)) \\ \dot{x}_2(t) & = & f(x_2(t), u_2(t)) \\ y_f(t) & = & h(x_1(t), u_1(t)) - h(x_2(t), u_2(t)) \\ x_1(t_0) & = & x_{10}, \quad\quad x_2(t_0) \;=\; x_{20} \end{array} \right. \tag{3}$$

Let us associated to Σ_f a specific *supply rate function* $w_f(t) = \eta^2\|u_1(t) - u_2(t)\|^2 - \|y_f(t)\|^2$ and a cost function:

$$J(t_0, T, x_{10}, x_{20}, u_1, u_2) = -\int_{t_0}^{T} w_f(\tau) d\tau$$

with its associated optimal cost:

$$S_a(T, t_0, x_1, x_2) \triangleq \sup_{u_1, u_2} J(t_0, T, x_{10}, x_{20}, u_1, u_2).$$

Proposition 2.2 ([10]): Let Σ be a dynamical system (1) from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ and let $\eta > 0$. Σ has a incremental gain less or equal η if and only if $S_a(T, t_0, x_0, x_0) = 0$.

Proof: By Theorem 2.1, if the system has a incremental gain less than or equal to η then for any $T \geq t_0$ and any $u_1, u_2 \in \mathcal{L}_2^e$, one has $J(t_0, T, x_0, x_0, u_1, u_2) \leq 0$. We thus have prove that $S_a(T, t_0, x_1, x_2) \leq 0$. In the other hand, when $u_1(t) = u_2(t)$, one has $x_1(t) = x_2(t) \triangleq x(t)$ for any $t \in [t_0, T]$ since $x_1(t_0) = x_2(t_0) = x_0$. Finally $y_f(t) = 0$ in $[t_0, T]$ and thus necessarily $S_a(T, t_0, x_1, x_2) \geq 0$ which allows to conclude the proof. ■

Remark: $S_a(t_0, x_1, x_2) = \lim_{T \to \infty} S_a(T, t_0, x_1, x_2)$, when the limit exists, can be proved to the *available storage function* associated to Σ_f and w_f, see [28].

Necessary condition of Proposition 2.2 is very weak since it is satisfied by any system. In order to propose an alternative proof of the mean value theorem necessary condition, the first and second order conditions are now considered.

III. Necessary conditions based on first and second order variations

In this section, an alternative proof of the mean value theorem necessary condition is proposed by investigating the optimality necessary conditions of the dissipative-like criterion. We first consider the conditions deduced from the Pontryagin's maximum principle, *i.e.* conditions deduced from the first order variations. We emphasize that the incremental criterion leads to a total singular optimal control problem, even if the generalized Legendre-Clebsch conditions are considered. We then deduce the main paper contribution by considering the second order variations *i.e.* Jacobi like conditions.

A. First order conditions

Let us now introduce $\lambda_1 \in \mathcal{R}^n$ and $\lambda_2 \in \mathcal{R}^n$, an $(2n+1)$-vector $\tilde{\lambda}^T = (\lambda_0, \lambda_1^T, \lambda_2^T)$ with $\lambda_0 \in \mathcal{R}$ and the Hamiltonian $H(x_1, x_2, u_1, u_2, \tilde{\lambda}) \triangleq \lambda_0 \left(\|y_f\|^2 - \eta^2 \|u_1 - u_2\|^2\right) + \lambda_1^T f(x_1, u_1) + \lambda_2^T f(x_2, u_2)$. We moreover define: $M(x_1, x_2, \tilde{\lambda}) = \sup_{u_1, u_2} H(x_1, x_2, u_1, u_2, \tilde{\lambda})$. Following [5], the necessary conditions deduced from the Pontryagin's maximum principle can be formulated as follows.

Proposition 3.1 ([5]): If (u_1^*, u_2^*) is an optimal solution, that is for any u_1, u_2 belonging to $\mathcal{L}_2^e$, one has

$$J(t_0, T, x_{10}, x_{20}, u_1^*, u_2^*) \geq J(t_0, T, x_{10}, x_{20}, u_1, u_2)$$

then the optimal solution has the following properties.

- (i) There exists an absolute continuous vector function $\tilde{\lambda}(t)^T = (\lambda_0, \lambda_1(t)^T, \lambda_2(t)^T) \neq 0$, $t \in [t_0, T]$ where $\lambda_0 > 0$ does not depend on t and where $\lambda_1(t)$ and $\lambda_2(t)$ are given for almost $t \in [t_0, T]$ by

$$\begin{aligned} \frac{d\lambda_1(t)}{dt} &= -\frac{\partial H}{\partial x_1}(x_1^*(t), x_2^*(t), u_1^*(t), u_2^*(t), \tilde{\lambda}(t)) \\ \frac{d\lambda_2(t)}{dt} &= -\frac{\partial H}{\partial x_2}(x_1^*(t), x_2^*(t), u_1^*(t), u_2^*(t), \tilde{\lambda}(t)) \end{aligned}$$

- (ii) For all $t \in [t_0, T]$, the Hamiltonian as a function of u_1 and u_2 takes its maximum at (u_1^*, u_2^*), *i.e.* almost everywhere in $[t_0, T]$, one has $H(x_1^*, x_2^*, u_1^*, u_2^*, \tilde{\lambda}) = M(x_1^*, x_2^*, \tilde{\lambda})$.
- (iii) The function $M(t) = M(x_1^*(t), x_2^*(t), \tilde{\lambda}(t))$ is constant on $[t_0, T]$.
- (iv) We have the transversality conditions: $\lambda_1(T) = 0$ and $\lambda_2(T) = 0$.

This proposition can be proved from [5], section 7. Indeed, the Bolza problem and the Meyer problem can be transformed one into another. The necessary conditions for Bolza problem can be thus easily deduced from the necessary conditions of the Meyer problem. Proposition 3.1 proof is thus related to Theorem 7.1.i proof in [5]. It is moreover necessary to use arguments presented in[5], sections 7.4 and 7.3.H to take into account, respectively, the lack of dynamics convexity and the unboundedness of the inputs.

Remark: The problem is not an abnormal optimal problem since necessarily $\lambda_0 \neq 0$. Indeed, condition (i) implies that $\tilde{\lambda} \neq 0$ for any $t \in [t_0, T]$. As, from condition (iv), $\lambda_1(T) = 0$ and $\lambda_2(T) = 0$, we have $\lambda_0 \neq 0$. We thus assume in the sequel that $\lambda_0 = 1$ without loss of generality.

Let us now focus on the incremental stability problem. We seek conditions satisfied by the worst inputs of Σ_f when its initial condition is such that $x_1(0) = x_2(0) = x_0$. Any pair of inputs which is such that $u_1 = u_2$ is a natural candidate for optimality. Let us prove that all the conditions of Proposition 3.1 are satisfied when $u_1 = u_2 \triangleq u(t)$ for any $u \in \mathcal{L}_2^e$. Indeed, $x_1(t) = x_2(t) = x(t)$, by condition (i), $\lambda_1(t)$ and $\lambda_2(t)$ are equal and they are the solution of the following backward differential equation:

$$\dot{\lambda}(t) = -\lambda(t) \frac{\partial f}{\partial x}(x(t), u(t))$$

which is initialized at $t = T$ to zero by condition (iv). We then deduce that $\lambda_1(t) = \lambda_2(t) = 0$ in $[t_0, T]$ and then $M(t) = 0$ in $[t_0, T]$ and thus that condition (iii) is fulfilled. Finally, the pair (u, u) maximizes the Hamiltonian a.e. since $H(x, x, u_1, u_2, \tilde{\lambda}) = -\eta^2 \|u_1 - u_2\|^2$. All the conditions of Proposition 3.1 are thus satisfied when $u_1 = u_2$.

In the general case, the main interest of the Pontryagin's maximum principle is to restrict the number of the possible optimal trajectories. It is not longer true in the incremental case since the maximization of the Hamiltonian does not really restrict the number of possible optimal solutions: the optimal problem is said to be singular [2], [5]. Classically, when a singular optimal control problem is considered, it is possible to obtain (and to restrict) possible optimal solutions by considering more sophisticated necessary conditions, *the generalized Legendre-Clebsch necessary conditions*. We show in the next section that these conditions does not give information on the possible optimal solutions. In fact, the incremental stability leads to consider "a totally singular optimal problem".

B. The generalized Legendre-Clebsch necessary conditions

In order to simplify the conditions deduced from the generalized Legendre-Clebsch necessary conditions, we consider an equivalent partially singular problem obtained by considering that the pair of inputs is $\Delta u \triangleq u_1 - u_2$ and u_2. Following this modification, $\Delta u = 0$ is the unique solution which maximizes the Hamiltonian: $\tilde{H}(x_1, x_2, \Delta u, u_2, \tilde{\lambda}) \triangleq \lambda_0 \left(\|y_f\|^2 - \eta^2 \|\Delta u\|^2\right) + \lambda_1^T f(x_1, \Delta u + u_2) + \lambda_2^T f(x_2, u_2)$. Indeed, by Proposition 3.1, one has $\lambda_0 \neq 0$ and $\lambda_1(t) = \lambda_2(t) = 0$ in $[t_0, T]$ and thus

$$\frac{\partial \tilde{H}}{\partial \Delta u}(0) = 0 \text{ and } \frac{\partial^2 \tilde{H}}{\partial^2 \Delta u} = -\eta^2 < 0.$$

In contrast, u_2 remains singular since the Hamiltonian does not depend on it. We then have defined a partially singular problems [2]. Following a classical approach of partially singular problems, the successive derivatives of

$\partial\tilde{H}/\partial u_2$, denoted in the sequel $\tilde{H}_{u2}$ with respect to time are considered in order to obtain u_2. The so-called *generalized Legendre-Clebsch conditions* are then obtained. These conditions are satisfied at the order q if for any $t \in [t_0, T]$, one has, with q is a positive integer:

$$\frac{\partial}{\partial u_2}\frac{d^{2q+1}}{dt^{2q+1}}\tilde{H}_{u_2} = 0 \quad \text{and} \quad (-1)^q \frac{\partial}{\partial u_2}\frac{d^{2q}}{dt^{2q}}\tilde{H}_{u_2} \geq 0.$$

In our case, since $\tilde{H}_{u_2} = 0$, the two previous quantities are always equal to zero for any q. No condition involving u_2 is thus obtained.
We then conclude that necessary conditions deduced from the Pontryagin's maximum principle or from the more sophisticated generalized Legendre-Clebsch conditions are unable to restrict the infinity number of trajectories which are candidates for the optimality. This result is clearly a direct consequence of the singularity of the cost function associated to the incremental stability.

C. Second order variations: Jacobi like conditions

In the previous section, we have deduce that there exists an infinite number of inputs which are compatible with the first order necessary conditions. We then investigate in this section the necessary condition deduced from the second order type arguments, *i.e.*, Jacobi like conditions[1]. The necessary part of the mean value theorem is then obtained (see Theorem 2.2). The main goal of this section is to prove the following proposition.

Proposition 3.2: Let Σ be a dynamical system (1) from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ and let $\eta > 0$. Σ has a incremental gain less or equal η only if the $\mathcal{L}_2$ gain of all its linearization are necessarily less than or equal to η.

Let T be a fixed value of time in $[t_0, \infty)$ and let u be an input belonging to $\mathcal{L}_2^e$. Let us first consider a small variation of the first input of Σ_f defined by $u_1(t) = u(t) + \delta\ \bar{u}(t)$ where $\delta > 0$ and $\bar{u} \in \mathcal{L}_2^e$. We prove in the sequel that the second order necessary condition associated to a small variation of δ, $\bar{u}$ in $\mathcal{L}_2^e$, implies that the linearization of the system at u $D_G\Sigma[u](\bar{u})$ has necessarily an $\mathcal{L}_2$ gain less than or equal to η. To prove this, we introduce a function, denoted $\psi(\delta)$, defined from $\mathcal{R}$ into $\mathcal{R}$ and related to the cost function by: $\psi(\delta) = J(t_0, T, x_0, x_0, u + \delta\ \bar{u}, u)$. The proposition proof is a direct consequence of the following second order characterization of a function maximum.

Theorem 3.1: If ψ has a second order derivative then 0 is a local maximum only if $\psi'(0) = 0$ and $\psi''(0) \leq 0$.

Let us apply it to our problem. We first compute the first and the second order derivatives of $\psi(\delta)$.

Lemma 3.1: Let Σ be a dynamical system associated to (1), from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$. Then, for any $u \in \mathcal{L}_2^e$, ψ has a first and second order derivatives given by:
(i) – The first order derivative of ψ is given by

$$\psi'(\delta) = 2\int_{t_0}^{T} \left[y_f(\tau)^T \bar{y}(\tau) - \eta^2(\delta\bar{u}(\tau))^T\bar{u}(\tau)\right] d\tau \tag{4}$$

$$\text{where} \quad \left\{\begin{array}{rcl} \dot{\bar{x}}(t) &=& A(t)\bar{x}(t) + B(t)\bar{u}(t) \\ \bar{y}(t) &=& C(t)\bar{x}(t) + D(t)\bar{u}(t) \\ \bar{x}(t_0) &=& 0 \end{array}\right. \tag{5}$$

with $A(t) = \dfrac{\partial f}{\partial x}(x_1(t), u_1(t))$, $B(t) = \dfrac{\partial f}{\partial u}(x_1(t), u_1(t))$, $C(t) = \dfrac{\partial h}{\partial x}(x_1(t), u_1(t))$ and $D(t) = \dfrac{\partial h}{\partial u}(x_1(t), u_1(t))$ and, by definition, $x_1(t), x_2(t)$ and $y_f(t)$ given by

$$\left\{\begin{array}{rcl} \dot{x}_1(t) &=& f(x_1(t), u(t) + \delta\bar{u}(t)) \\ \dot{x}_2(t) &=& f(x_2(t), u(t)) \\ y_f(t) &=& h(x_1(t), u(t) + \delta\bar{u}(t)) - h(x_2(t), u(t)) \\ x_1(t_0) &=& x_2(t_0) = x_0 \end{array}\right.$$

(ii) – The second order derivative of $\psi(\delta)$ is given by

$$\psi''(\delta) = 2\int_{t_0}^{T} \left[y_f(\tau)^T\zeta(\tau) + \|\bar{y}((\tau)\|^2 - \eta^2\|\bar{u}(\tau)\|^2\right] d\tau \tag{6}$$

$$\text{where} \left\{\begin{array}{rcl} \dot{\xi}(t) &=& A(t)\xi(t)\bar{x}(t) + A^2(t)\bar{x}(t)\bar{x}(t) + \cdots \\ && +2E(t)\bar{x}(t)\bar{u}(t) + B^2(t)\bar{u}(t)\bar{u}(t) \\ \zeta(t) &=& C(t)\xi(t)\bar{x}(t) + C^2(t)\bar{x}(t)\bar{x}(t) + \cdots \\ && +2F(t)\bar{x}(t)\bar{u}(t) + D^2(t)\bar{u}(t)\bar{u}(t) \\ \xi(t_0) &=& 0 \end{array}\right. \tag{7}$$

where

$$\begin{array}{ll} A^2(t) = \dfrac{\partial^2 f}{\partial^2 x}(x_1(t), u_1(t)), & E(t) = \dfrac{\partial^2 f}{\partial x \partial u}(x_1(t), u_1(t)) \\ B^2(t) = \dfrac{\partial^2 f}{\partial^2 u}(x_1(t), u_1(t)), & C^2(t) = \dfrac{\partial^2 h}{\partial^2 x}(x_1(t), u_1(t)) \\ F(t) = \dfrac{\partial^2 h}{\partial x \partial u}(x_1(t), u_1(t)), & D^2(t) = \dfrac{\partial^2 h}{\partial^2 u}(x_1(t), u_1(t)). \end{array}$$

Proof of Proposition 3.2: From section II-C, Σ is incrementally stable only if $J(t_0, T, x_0, x_0, u, u) \leq 0$ for any $u \in \mathcal{L}_2^e$. Moreover, since $J(t_0, T, x_0, x_0, u, u) = 0$ then $\psi(0)$ has to be a maximum. The proof of the proposition is then a direct consequence of Theorem 3.1 and Lemma 3.1. Indeed, we easily deduce from Lemma 3.1 that $\psi'(0) = 0$. Indeed, when $\delta = 0$, one has $u_1(t) = u_2(t)$ and then $x_1(t) = x_2(t)$ in $[t_0, T]$. We then deduce that $y_f(t) = 0$ in $[t_0, T]$, and thus, by (4) that $\psi'(0) = 0$. Let us now consider the consequence of the second order condition: $\psi''(0) \leq 0$. Since $y_f(t) = 0$ in $[t_0, T]$, (6) can be rewritten has: $\psi''(0) = 2\int_{t_0}^{T}\left[\|\bar{y}((\tau)\|^2 - \eta^2\|\bar{u}(\tau)\|^2\right] d\tau$ which does not depend of time-varying variables defined by (7) but depends of time-varying variables defined by (5). Moreover, $\delta = 0$ and it is thus not difficult to see that the linear system defined by (5) is in fact the linearization of Σ at u. We have then proved that for any $T \in [t_0, \infty)$ and for any $u \in \mathcal{L}_2^e$, the linearization of Σ at u, $\bar{y} = D_G\Sigma[u](\bar{u})$, has to satisfy: $\|\bar{y}\|_{2,T} \leq \eta\|\bar{u}\|_{2,T}$. Theorem 2.1 allows to conclude that $\bar{y} = D_G\Sigma[u](\bar{u})$ has thus necessarily an $\mathcal{L}_2$ gain less than η. ∎

[1]Strictly speaking, the Jacobi's conditions are usually only associated to Lagrange problems and we consider a Bolza one.

IV. Conclusion

Obtaining an alternative proof to a powerful result, useful in control is not questionable. The classical proof of the mean value theorem in norm is made in the functional analysis framework. In this paper, we propose to use the dissipativity framework in order to obtain, based on the classical calculus of variations, the necessary part of the mean value theorem. Finally, we emphasize that it is also possible to obtain the sufficient part of the mean value theorem in norm using the same way. Indeed, Jacobi like conditions allow in some cases to obtain also sufficient conditions for a local maximum, see [19][Theorem 9 p. 358][2]. From this preliminary local result, it is possible to develop a complete and different proof of the mean value theorem.

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Appendix

Proof of Lemma 3.1: $J = \mathcal{J}(w_1, w_2)$ is an operator defined from $\mathcal{L}_2^e \times \mathcal{L}_2^e$ to $\mathcal{L}_1^e$ by

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& f(x_1, w_1) \\ \dot{x}_2 &=& f(x_2, w_2) \\ \dot{x}_3 &=& \|h(x_1, w_1) - h(x_2, w_2)\|^2 - \eta^2 \|w_1 - w_2\|^2 \\ J &=& x_3 \\ x_1(t_0) &=& x_2(t_0) = x_0 \\ x_3(t_0) &=& 0 \end{array} \right. \tag{8}$$

which is such that $\psi(\delta) = \mathcal{J}(u, u + \delta\bar{u})$. Let us prove that if $\mathcal{J}$ is Gâteaux differentiable at $(u + \delta_0\bar{u}, u)$ then ψ has a first order derivative at $\delta = \delta_0$. Indeed, the right derivative of ψ exists since it is given by

$$\begin{array}{rcl} \psi'_+(\delta_0) &=& \lim\limits_{\delta \downarrow 0} \dfrac{\mathcal{J}(u + \delta_0\bar{u} + \delta\bar{u}, u) - \mathcal{J}(u + \delta_0\bar{u}, u)}{\delta} \\ &=& D_G\mathcal{J}[u + \delta_0\bar{u}, u](\bar{u}, 0). \end{array}$$

The left derivative of ψ exists also since

$$\begin{array}{rcl} \psi'_-(\delta_0) &=& \lim\limits_{\delta \downarrow 0} \dfrac{\mathcal{J}(u + \delta_0\bar{u} - \delta\bar{u}, u) - \mathcal{J}(u + \delta_0\bar{u}, u)}{-\delta} \\ &=& D_G\mathcal{J}[u + \delta_0\bar{u}, u](\bar{u}, 0). \end{array}$$

Finally, since $\psi'_+(\delta_0) = \psi'_-(\delta_0)$, we deduce that ψ has a first order derivative when $\mathcal{J}$ is Gâteaux differentiable. We can similarly prove that the second order derivative of ψ exists if the operator $(w_1, w_2) \mapsto D_G\mathcal{J}[w_1, w_2](\bar{w_1}, \bar{w_2})$ has a Gâteaux derivative at $(u + \delta_0\bar{u}, u)$. We now prove that $\mathcal{J}$ has a first and second order Gâteaux derivatives and we compute them.

Lemma 1.1: Let g be defined from $[t_0, \infty) \times \mathcal{R}$ into $\mathcal{R}$ and such that $g(t, 0) = 0$ for almost every $t \in [t_0, \infty)$ and such that there exist $b \geq 0$ and $a \in \mathcal{L}_1^e$ such that $\|g(t, u)\| \leq b\|u\|^2 + a(t)$ and let $\mathcal{N}$ be the associated memoryless nonlinear system defined from $\mathcal{L}_2^e$ into $\mathcal{L}_1^e$ by $\mathcal{N}(u) = g(t, u(t))$. If g and $\frac{\partial g}{\partial u}(t, u)$ are continuous functions with respect to u for almost every $t \in [t_0, \infty)$ and measurable on $[t_0, \infty)$ with respect to t for every fixed valued $u \in \mathcal{R}$ and there exist $a' \in \mathcal{L}_1^e$ and $b' \geq 0$ such that $\|\frac{\partial g}{\partial u}(t, u)\| \leq b'\|u\| + a'(t)$ then $\mathcal{N}$ admits a Gâteaux derivative at any u_0 of $\mathcal{L}_2^e$ given by

$D\mathcal{N}_G[u_0](h) = \frac{\partial g}{\partial u}(t, u_0)h.$

Proof: For any $T \in [t_0, \infty)$, $u_0, h \in \mathcal{L}_2^e$ and $\lambda > 0$,

$$\begin{aligned}\Pi(t) &= g(t, u_0(t) + \lambda h(t)) - g(t, u_0(t)) - \lambda \frac{\partial g}{\partial u}(t, u_0(t))h(t) \\ &= \int_{u_0(t)}^{u_0(t)+\lambda h(t)} \frac{\partial g}{\partial u}(t, \xi)d\xi - \lambda \frac{\partial g}{\partial u}(t, u_0(t))h(t)\end{aligned}$$

with $\xi = (1-\rho)u_0(t) + \rho(u_0(t) + \lambda h(t))$. $\frac{\Pi(t)}{\lambda}$ can be rewritten as

$$\int_0^1 \underbrace{\left(\frac{\partial g}{\partial u}(t, u_0(t) + \rho\lambda h(t)) - \frac{\partial g}{\partial u}(t, u_0(t))\right) h(t)}_{\Xi_\lambda(t,h,\rho)} d\rho \tag{9}$$

where

$$\begin{aligned}&\int_0^1 \frac{\partial g}{\partial u}(t, u_0(t) + \rho\lambda h(t)))h(t)d\rho = \\ &\qquad = \lim_{m\to\infty} \sum_{i=1}^{m} \frac{1}{m}\frac{\partial g}{\partial u}\left(t, u_0(t) + \frac{i}{m}\lambda h(t)\right) h(t)\end{aligned}$$

exists and is a measurable function of t. Indeed, this is the limit of a sequence of measurable functions [4]. Moreover, since $|\int_0^1 f(\rho)d\rho| \leq \int_0^1 |f(\rho)|d\rho$ then

$$\left\|\frac{\Pi(t)}{\lambda}\right\| \leq \int_0^1 \|\Xi_\lambda(t, h, \rho)\| \, d\rho$$

By Fubini's theorem, we deduce that for any $T \in [t_0, \infty)$ one has

$$\begin{aligned}&\left\|\frac{1}{\lambda}\left[\mathcal{N}(u_0 + \lambda h) - \mathcal{N}(u_0)\right] - \delta\mathcal{N}(u_0, h)\right\|_{1,T} \\ &\qquad \leq \int_0^1 \int_{t_0}^{T} \|\Xi_\lambda(\tau, h, \rho)\| d\tau d\rho\end{aligned}$$

where $\delta\mathcal{N}(u_0, h) \triangleq \frac{\partial g}{\partial u}(t, u_0)h$. From the last inequality, the proof is achieved if $\int_0^1 \int_{t_0}^T \|\Xi_\lambda(\tau, h, \rho)\| d\tau d\rho$ goes to zero when λ goes to zero.

Let us pick $v(t) \in \mathcal{L}_2^e$ and let us introduce this following nonlinear function defined from $[t_0, \infty) \times \mathcal{R}$ into $\mathcal{R}$ by

$$f(t, u) = \left[\frac{\partial g}{\partial u}(t, u_0(t) + u) - \frac{\partial g}{\partial u}(t, u_0(t))\right] v(t)$$

and its associated operator, $\mathcal{F}$, defined from $\mathcal{L}_2^e$ into $\mathcal{L}_1^e$. We then prove that $\mathcal{F}$ is a continuous operator. To this purpose, we use a classical theorem (see [23] or Theorem 10.9iv in [5]) which ensures that the operator $\mathcal{F}$ is continuous from $\mathcal{L}_2^e$ into $\mathcal{L}_1^e$ if (and only if) there exists $a \in \mathcal{L}_1^e$ and $b \geq 0$ such that $\|f(t, u)\| \leq b\|u\|^2 + a(t)$ for any u. Since it is has been assumed that $\left\|\frac{\partial g}{\partial u}(t, u)\right\| \leq b'\|u\| + a'(t)$, one has

$$\left\|\frac{\partial g}{\partial u}(t, u_0 + u)v\right\| \leq (b'\|u\| + b'\|u_0\| + a') \|v\|$$

From the last inequality and since $xy = x^2 + (y/2)^2 - (x - y/2)^2$, on has

$$\left\|\frac{\partial g}{\partial u}(t, u_0(t) + u)v(t)\right\| \leq b'\|u\|^2 + a''(t)$$

where $a''(t) = b'\|v(t)/2\|^2 + b'\|u_0(t)\|\|v(t)\| + a'(t)\|v(t)\|$. Since v, u_0 and a' belong to $\mathcal{L}_2^e$ thus $a''(t)$ belongs to $\mathcal{L}_1^e$. Finally, since $\left\|\frac{\partial g}{\partial u}(t, u_0(t))v(t)\right\|$ belongs to $\mathcal{L}_1^e$, we have thus proved that $\mathcal{F}$ is continuous from $\mathcal{L}_2^e$ into $\mathcal{L}_1^e$. Moreover, since $\mathcal{F}(0) = 0$ then for any $\epsilon > 0$ there exists $\lambda_0 > 0$ such that for any $\rho \in [0, 1]$ one has $\|\mathcal{F}(\rho\lambda h)\|_{1,T} \leq \epsilon$ for any $\lambda \leq \lambda_0$. We finally deduce that for any $\epsilon > 0$, there exists $\lambda_0 > 0$ such that $\left\|\frac{1}{\lambda}\left[\mathcal{N}(u_0 + \lambda h) - \mathcal{N}(u_0)\right] - \delta\mathcal{N}(u_0, h)\right\|_{1,T} \leq \epsilon$ which allows to prove that $\delta\mathcal{N}(u_0, h) = \frac{\partial g}{\partial x}(t, u_0)h$ is the first variation of $\mathcal{N}$. Since $\delta\mathcal{N}$ is a linear and bounded operator from $\mathcal{L}_2^e$ into $\mathcal{L}_1^e$, $\mathcal{N}$ has a Gâteaux derivative at any point u_0 on $\mathcal{L}_2^e$ given by $D\mathcal{N}_G[u_0](h) = \frac{\partial g}{\partial u}(t, u_0)h$. ■

We now deduce that $\mathcal{J}$ is Gâteaux differentiable. Let us introduce a function $\Phi(x, \dot{x}, w, J)$ which is equal to:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ J \end{bmatrix} - \begin{bmatrix} f(x_1, w_1) \\ f(x_2, w_2) \\ \|h(x_1, w_1) - h(x_2, w_2)\|^2 - \eta^2\|w_1 - w_2\|^2 \\ x_3 \end{bmatrix}$$

with $x^T = (x_1, x_2, x_3)$ and $w^T = (w_1, w_2)$. When $\Phi(x, \dot{x}, w, J) = 0$, we recover system (8). The dynamical system defined by (8) is well-defined then $\dot{x}$ and J belong to $\mathcal{L}_1^e$ when w belongs to $\mathcal{L}_2^e$. Moreover, x belongs to AC_1^e which is the space of absolute continuous functions (AC) of times endowed with the $\mathcal{L}_1$ norm and Φ is then an operator defined from $AC_1^e \times \mathcal{L}_1^e \times \mathcal{L}_2^e \times \mathcal{L}_1^e$ into $\mathcal{L}_1^e$. Φ has a Gâteaux derivative since it is the difference between a linear operator (which has a Gâteaux derivative on AC_1^e) and a nonlinear operator, namely $\mathcal{N} : AC_1^e \times \mathcal{L}_2^e \to \mathcal{L}_1^e$, associated to this nonlinear function

$$g(x, w) = \begin{bmatrix} f(x_1, w_1) \\ f(x_2, w_2) \\ \|h(x_1, w_1) - h(x_2, w_2)\|^2 - \eta^2\|w_1 - w_2\|^2 \\ x_3 \end{bmatrix}$$

which has a Gâteaux derivative by the previous lemma. Since $D\Phi_G[x, \dot{x}, w, J](\bar{x}, \dot{\bar{x}}, \bar{w}, \bar{J}) = 0$ then $(\dot{\bar{x}}_1, \dot{\bar{x}}_2, \dot{\bar{x}}_3, \bar{J} = J_g[u, u](\bar{x}, \bar{w})$ with J_g the Jacobian of g:

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & 0 & 0 & \frac{\partial f}{\partial w_1} & 0 \\ 0 & \frac{\partial f}{\partial x_2} & 0 & 0 & \frac{\partial f}{\partial w_2} \\ 2y\frac{\partial h}{\partial x_1} & 2y\frac{\partial h}{\partial x_2} & 0 & 2w' + 2y\frac{\partial h}{\partial w_1} & -2w' - 2y\frac{\partial h}{\partial w_2} \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

where $y = h(x_1, u_1) - h(x_2, w_2)$ and $w' = w_1 - w_2$.

For the second order Gâteaux derivative of Σ, similar arguments can be used. The assumptions of Lemma 1.1 are fulfilled in this context since f (resp. h) and its gradient are assumed uniformly Lipschitz continuous.