ANALYSIS OF NONLINEAR SYSTEMS BY THEIR LINEARIZATIONS[1]

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Abstract

In this paper, we present a result allowing to completely analyze the behavior of a nonlinear system only based on the properties of its linearizations. More precisely, we show that the exponential stability of all the linearizations of a nonlinear system ensures suitable global properties. We point out that a nonlinear system for which the linearizations are exponentially stable, has the unique steady state property, and has a periodic (constant) motion for any periodic input. Finally through some reachable assumption of the system state space from the initial condition, we deduce that all its unperturbed motions which are associated to bounded inputs are uniformly globally asymptotically stable.

1 Introduction

The incremental norm approach was proposed as a powerful tool for the analysis and the control of nonlinear systems. Incrementally stable systems were proven indeed to exhibit suitable global and local properties [13]. As an example, the links between incremental stability and Lyapunov stability were presented in [8, 9, 12]. In this context, we propose a result allowing to completely analyze the behavior of a nonlinear system when only the properties of the linearizations (Gâteaux derivatives) of the system are known. More precisely, we consider in this paper a system described by a differential equation:

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & h(x(t), u(t)) \\ x(t_0) & = & x_0 \end{array} \right.$$

and assume that the set of possible inputs is a convex subset of some functional space which is assumed to be a normed vectorial space. We then prove that, if the linearizations associated to each input belonging to the set of possible inputs are exponentially stable, then the behavior of the nonlinear system has the following qualitative properties:

1. for all input signal belonging to the set of possible inputs, there is a unique steady-state motion despite a vanishing perturbation on the input signal;
2. for all constant (resp. periodic) signal belonging to the set of possible inputs, the steady state response is a constant (resp. periodic);
3. for any bounded input belonging to the set of possible inputs, the associated state motion of the system is also bounded.

Moreover, we prove that if the state space of the system has some suitable reachability property with respect to the initial condition then all the unperturbed motions of the system, which are associated to bounded inputs, are uniformly globally asymptotically stable.

It is clear that this result is very interesting in the context of nonlinear control. Actually, some design methods try to achieve some global properties under constraints on the system linearizations only. We now illustrate this point for two well–known techniques: the gain scheduling and the Extended Kalman Filter.

Let us recall that the underlying idea of the gain-scheduling is to design, at one or more operating points, linear time invariant controllers using the associated linearized plant models. The nonlinear control law is then obtained by interpolating (or scheduling) these controllers as a function of the operating point [20, 26, 2, 21]. From a classical point of view, it seems that the main goal of the gain scheduling approach is to ensure at least the exponential stability of the closed loop system linearizations. The result presents in this paper allows in fact to claim that imposing exponential stability to the linearizations leads to ensure global properties to the nonlinear system (see [14, 15] for a complete discussion about the gain scheduling technique and the incremental norm approach).

The importance of the linearizations can be highlighted in the framework of nonlinear observers. This is especially true for the well-known Extended Kalman Filter (EKF). The issue of finding conditions allowing to ensure the stability and robustness of the EKF remains in the general case an open problem. The EKF is nevertheless classically used in many engineering applications, so as to estimate the states or parameters of a plant (estimation of the attitude of a satellite, missile guidance problems, combined estimation of state and

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plant parameters ...- see e.g. [4, 18]).
Even if the EKF works well in many cases, only proofs for specific case have been given in the literature. On the one hand, local properties are often analyzed [18, 3]. These properties are generally (implicitly or explicitly) obtained using the exponential stability of a specific linearization of the system. On the other hand, some specific results can be obtained when assuming a particular structure of this nonlinear system [22, 28].
In the context of the EKF, the result given in this paper allows to point out an interesting fact: a crude application of our result can lead to a wrong conclusion. Indeed, as pointed out in [11], the linearization of the EKF (the Gâteaux derivative of the nonlinear observer along the estimated trajectory) does not correspond to the linearization used to computed the associated Kalman gain: the true linearization (the Gâteaux derivative) of the EKF has to include the linearization of the dynamics of the Ricatti equation. According to the result presented in this paper, only this fact allows to provide a explanation of the EKF divergence even if all the observability conditions of the linearizations are satisfied.
The paper is organized as follows. Notations are introduced in section 2 and some definitions are also recalled. The key result allowing to link the exponential stability of all the linearizations with the incremental boundedness of the system is given in section 3. Section 4 focus on the behavior of the system with respect to perturbed or specific inputs. The results concerning the Lyapunov stability of the unperturbed motion are proposed in section 5. When not presented, the proofs can be found in the technical report [14].

2 Notations and definitions

2.1 Normed functional spaces

The notations and terminology, here used, are classical in the input-output context (see *e.g.* [6]). $\mathcal{L}$ denoted in the sequel a vectorial normed space of functions defined from $[t_0,\infty)$ into $\mathcal{R}^n$, *i.e.*, $\mathcal{L} \triangleq \{f : [t_0,\infty) \rightarrow \mathcal{R}^n | \|f\|_{\mathcal{L}} < \infty\}$.
The *causal truncation* of $f \in \mathcal{L}$ at time $T \in [t_0,\infty)$, denoted by $P_T f$ is given by $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise. For convenience, $\|P_T u\|_{\mathcal{L}}$ is denoted by $\|u\|_{\mathcal{L},T}$. The *extended space* associated to $\mathcal{L}$, denoted by $\mathcal{L}^e$, is composed with the functions whose causal truncations belong to $\mathcal{L}$, *i.e.*, $\mathcal{L}^e \triangleq \{f : [t_0,\infty) \rightarrow \mathcal{R}^n | \forall T \in [t_0,\infty),\ \|P_T f\|_{\mathcal{L}} < \infty\}$. We assume throughout the paper that the norm $\|.\|_{\mathcal{L}}$ used to define $\mathcal{L}$ and $\mathcal{L}^e$ satisfies this two properties:
(*i*) $\forall f \in \mathcal{L}^e$, the map $T \mapsto \|P_T f\|_{\mathcal{L}}$ is monotonically increasing;
(*ii*) $\forall f \in \mathcal{L}$, one has $\lim_{T\rightarrow\infty} \|P_T f\|_{\mathcal{L}} = \|f\|_{\mathcal{L}}$.
We define in the same way the extended open set associated to an open subset, $\mathcal{O}_{\mathcal{L}}$, of $\mathcal{L}$, *i.e.*, $\mathcal{O}^e_{\mathcal{L}} = \{f : [t_0,\infty) \rightarrow \mathcal{R}^n | \forall T \in [t_0,\infty),\ P_T f \in \mathcal{O}\}$.

Some specific functional spaces are especially relevant from a control point of view:
– $\mathcal{L}_2(\mathcal{R}^n,[t_0,\infty))$: The set of $\mathcal{R}^n$-value functions, $f(t)$, which are square integrable on $[t_0,\infty)$, and such that $\int_{t_0}^{\infty} \|f(t)\|^2 dt < \infty$.
– $\mathcal{L}_\infty(\mathcal{R}^n,[t_0,\infty))$: The set of all essentially bounded, measurable $\mathcal{R}^n$-value functions on $[t_0,\infty)$.
– $\mathcal{B}_2(\mathcal{R}^n,[t_0,\infty))$ $(1 \leq p < \infty)$: The set of $\mathcal{R}^n$-value functions, $f(t)$, which are almost everywhere bounded and square integrable on $[t_0,\infty)$, and such that $\int_{t_0}^{\infty} \|f(t)\|^p dt < \infty$, *i.e.*, $\mathcal{B}_2 \triangleq \mathcal{L}_2 \cap \mathcal{L}_\infty$.
– $\mathcal{C}^i_2(\mathcal{R}^n,[t_0,\infty))$: The set of all $\mathcal{R}^n$-value functions on $[t_0,\infty)$, $f(t)$, which are C^i functions of t and such that $\int_{t_0}^{\infty} \|f(t)\|^2 dt < \infty$ (by convention $\mathcal{C}_2 \triangleq \mathcal{C}^0_2$).
– $\mathcal{C}^i_\infty(\mathcal{R}^n,[t_0,\infty))$: The set of all bounded and C^i $\mathcal{R}^n$-value functions on $[t_0,\infty)$.
More generally, in the sequel, the results are deduced for various classes of vectorial normed spaces of functions which are equipped with $\mathcal{L}_2$ or $\mathcal{L}_\infty$ like norms. In the sequel, $\mathcal{U}$ and $\mathcal{Y}$ generally denoted two open subsets of some vectorial functional spaces equipped with the same $\mathcal{L}_p$ norm with $p \in \{2,\infty\}$.

Remark. It can be noted that the following inclusion $\mathcal{L}_\infty([0,T]) \subset \mathcal{L}_2([0,T])$ is true for each value of $T \geq t_0$ [5]. As a consequence, the extended space, which is associated with $\mathcal{L}_2$ for a specific value of T, contains all the signals which have (almost everywhere) a finite amplitude on $[0,T]$.

2.2 Considered system, finite gain and incremental stability

In the sequel, we consider systems exhibiting the differential representation

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t),u(t)) \\ y(t) & = & h(x(t),u(t)) \\ x(t_0) & = & x_0 \end{array} \right. \qquad (1)$$

where $x(t) \in \mathcal{R}^n$, $y(t) \in \mathcal{R}^p$, and $u(t) \in \mathcal{R}^m$. f and h, defined from $\mathcal{R}^n \times \mathcal{R}^m$ into $\mathcal{R}^n$ and $\mathcal{R}^p$ respectively, are assumed such that $f(x_0,0) = 0$ and $h(x_0,0) = 0$. We then assume that Σ is a well–posed system that is a causal operator which associates to each input belonging to an extended open subset of $\mathcal{L}_I$, *i.e.*, $\mathcal{U}^e_{\mathcal{L}_I}$, an output belonging to an extended open subset of $\mathcal{L}_O$, *i.e.*, $\mathcal{Y}^e_{\mathcal{L}_O}$. In the sequel, for sake of simplicity, we denote the input and the output extended open sets as $\mathcal{U}^e$ and $\mathcal{Y}^e$ respectively.
The well–posed assumption implies that the differential equation of system (1) has for all $t \in [t_0,\infty)$ and for all $u \in \mathcal{U}^e$, an unique and absolutely continuous solution denoted by $x(t) = \varphi(t,t_0,x_0,u)$.
An *unperturbed motion* (of Σ) is a specific motion of Σ associated with an input, $u_r \in \mathcal{U}^e$, and with an initial condition, $x_{0r} \in \mathcal{R}^n$, *i.e.*, $x_r(t) = \varphi(t,t_0,x_{0r},u_r)$. With reference to this definition, we denote by $x[x_{0r},u_r]$ this unperturbed motion.
Σ is said to be a *finite gain stable* system from $\mathcal{U}$ into

$\mathcal{Y}$ if there exists $\gamma > 0$ such that $\|\Sigma(u)\|_{\mathcal{Y}} \leq \gamma \|u\|_{\mathcal{U}}$ for all $u \in \mathcal{U}$. The gain of Σ coincides with the minimum value of γ and is denoted by $\|\Sigma\|_{i,\mathcal{U},\mathcal{Y}}$. Σ is *incrementally bounded* from $\mathcal{U}$ into $\mathcal{Y}$ if there exists $\eta \geq 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_{\mathcal{Y}} \leq \eta \|u_1 - u_2\|_{\mathcal{U}}$ for all $u_1, u_2 \in \mathcal{U}$. The incremental gain of Σ coincides with the minimum value of η and is denoted by $\|\Sigma\|_{\Delta,\mathcal{U},\mathcal{Y}}$. Σ is said to be *incrementally stable* if it is incrementally bounded and stable, *i.e.*, it maps $\mathcal{U}$ to $\mathcal{Y}$.

We now make a preliminary remark concerning the class of possible inputs. Indeed, the definitions of input-output stability may appear restrictive from an applicative point of view, since a limited class of possible inputs is considered for the system: as an example, a non-zero constant input does not belong to vectorial normed space equipped to an $\mathcal{L}_2$ type norm. This restriction can be nevertheless bypassed using the link between the input-output stability properties on $\mathcal{L}$ and its extended space $\mathcal{L}^e$ (this result is due to I.W. Sandberg).

Theorem 2.1 [31] *Let Σ be a causal operator, defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ and let η be a positive constant. One has*

$$\sup_{\substack{u_1,\, u_2 \,\in\, \mathcal{U} \\ u_1 - u_2 \neq 0}} \frac{\|\Sigma(u_1) - \Sigma(u_2)\|_{\mathcal{Y}}}{\|u_1 - u_2\|_{\mathcal{U}}} \leq \eta$$

if and only if for any $T \in [t_0, \infty)$, one has

$$\sup_{\substack{u_1,\, u_2 \,\in\, \mathcal{U}^e \\ u_1 - u_2 \neq 0}} \frac{\|\Sigma(u_1) - \Sigma(u_2)\|_{\mathcal{Y},T}}{\|u_1 - u_2\|_{\mathcal{U},T}} \leq \eta$$

This theorem clearly indicates that the input-output relation, which was already satisfied by the input signals inside $\mathcal{L}$, remains valid inside $\mathcal{L}^e$. More generally, when analyzing the properties of the nonlinear system along a possible motion, the use of the extended space $\mathcal{L}^e$ enables to consider a much larger class of possible inputs, *e.g.* non-zero constant inputs.

2.3 Gâteaux derivative

Definition 2.1 [1, 29] *Given an operator Σ, defined from*[1] *$\mathcal{U}$ into $\mathcal{Y}$, let $u_r \in \mathcal{U}$ and assume the existence for any $h \in \mathcal{U}$ of a continuous linear operator $D\Sigma_G[u_r]$, from $\mathcal{U}$ into $\mathcal{Y}$, satisfying:*

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma(u_r + \lambda h) - \Sigma(u_r)}{\lambda} - D\Sigma_G[u_r](h) \right\|_{\mathcal{Y}} = 0$$

then $D\Sigma_G[u_r]$ is called the Gâteaux derivative of Σ at u_r.

Definition 2.2 *$D\Sigma_G[u_r]$ from $\mathcal{U}^e$ into $\mathcal{Y}^e$ is the Gâteaux derivative at u_r of the causal operator Σ, defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$, if it is linear and if for all $T \in [t_0, \infty)$, $P_T D\Sigma_G[u_r]$ is the Gâteaux derivative of $P_T \Sigma$ at $P_T u_r$.*

[1] Let us recall that $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be open subsets of some suitable vectorial normed spaces equipped with the $\mathcal{L}_p$ norm.

In the context of theorem 2.1, we recall that if Σ is a causal operator then $D\Sigma_G[u_r]$ is causal too.

When the system is generated by differential equations, definition 2.2 corresponds to the usual linearization concept. More precisely:

Proposition 2.2 *Let Σ, be a dynamical system associated to (1), defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$. If f and h are uniformly Lipschitz and C^1, then, for any $u_r \in \mathcal{L}_2^e$, the system has a Gâteaux derivative which satisfies the following differential equations:*

$$\bar{y} = D\Sigma_G[u_r](\bar{u}) \begin{cases} \dot{\bar{x}}(t) &= A(t)\bar{x}(t) + B(t)\bar{u}(t) \\ \bar{y}(t) &= C(t)\bar{x}(t) + D(t)\bar{u}(t) \\ \bar{x}(t_0) &= 0 \end{cases} \tag{2}$$

with $A(t) = \frac{\partial f}{\partial x}(x_r(t), u_r(t))$, $B(t) = \frac{\partial f}{\partial u}(x_r(t), u_r(t))$, $C(t) = \frac{\partial h}{\partial x}(x_r(t), u_r(t))$ and $D(t) = \frac{\partial h}{\partial u}(x_r(t), u_r(t))$ and where $x_r(t) = \varphi(t, t_0, x_0, u_r)$ is the solution of system (1) under input $u_r(t)$ and $x(t_0) = x_0$.

Remarks:
(i) We note that it is possible to prove that Σ does not have Fréchet derivative on $\mathcal{L}_2^e$ under the assumption that f or/and h are not linear functions (see *e.g.* [29, 24]).
(ii) The strong requirement concerning the uniform Lipschitz continuity of f and g with respect to its second argument is mainly due to the fact that the inputs belong to $\mathcal{L}_2^e$ space. Actually, it is possible to prove that this condition on f and g is in fact necessary (and sufficient) condition such that the system (1) is a well-defined system.

The strong requirement concerning f and g can be weaken if we restrict our attention to functional spaces of almost everywhere bounded functions. In fact, if we assume that the solution of (1) is well-defined on $[t_0, \infty)$, one has the following result.

Proposition 2.3 *Let Σ, be a dynamical system associated to (1), defined from $\mathcal{L}_\infty^e$ into $\mathcal{L}_\infty^e$. If f and h are two C^1 functions of their arguments then, for any $u_r \in \mathcal{L}_\infty^e$, the system has a Gâteaux derivative which is given by system (2).*

Remark. The previous propositions allow to deduce that Σ is Gâteaux derivable on the normed vectorial spaces which are subset of the $\mathcal{L}_2$ or $\mathcal{L}_\infty$ spaces.

2.4 Means value theorem in norm

The theorem recalled hereafter is a key result in the context of nonlinear control (see *e.g.* [31, 7, 13, 14, 15]). It presents a strong connection between the incremental norm and the local properties associated with the derivative of a nonlinear system.

Theorem 2.4 *Let Σ, be a dynamical system associated to (1), defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ and let η be a positive constant. Let us consider two values u_a and u_b belonging to $\mathcal{U}$ such that the segment $[u_a, u_b] \in \mathcal{U}$. Moreover, let us assume that Σ has a Gâteaux derivative at each point u_0 of $[u_a, u_b]$. The following statements are equivalent:*
(i) For any $u_0, h \in [u_a, u_b]$, one has

$$\|D\Sigma_G[u_0](h)\|_{\mathcal{Y}} \leq \eta \|h\|_{\mathcal{U}}.$$

(ii) For any $u_1, u_2 \in [u_a, u_b]$, one has

$$\|\Sigma(u_1) - \Sigma(u_2)\|_{\mathcal{Y}} \leq \eta \|u_1 - u_2\|_{\mathcal{U}}.$$

If $\mathcal{U}$ is convex set, and all the linearizations of the system are bounded for any $u_0 \in \mathcal{U}$ then the system is in fact incrementally bounded from $\mathcal{U}$ into $\mathcal{Y}$. In the same spirit, through the existence of a path belonging to $\mathcal{U}$ which links u_1 to u_2, we can deduce this following corollary:

Corollary 2.5 *Let Σ be a dynamical system associated to (1), defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ and where $\mathcal{U}$ is assumed to be an arc-connected set. If Σ has a Gâteaux derivative at each point u_0 of $\mathcal{U}$ and if there exists a finite constant $\eta \geq 0$ such that for any $u_0, h \in \mathcal{U}$, one has $\|D\Sigma_G[u_0](h)\|_{\mathcal{Y}} \leq \eta\|h\|_{\mathcal{U}}$ then for any $u_1, u_2 \in \mathcal{U}$, there exists a finite constant $K > 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_{\mathcal{Y}} \leq K$.*

Proof: In fact, let us recall that if $\mathcal{U}$ is an arc-connected subset of a normed space then by definition, there always exists a path belonging to $\mathcal{U}$ which allows to link u_1 to u_2. Moreover, $\mathcal{U}$ is a open subset of a normed space, so it is always possible to built a path linking u_1 to u_2 by a finite series of segments (see remark 13 p. 261 in [25]). On this basis, let us consider the curve linking u_1 to u_2 which is the concatenation of $n-1$ segments, defined by $[\xi(i), \xi(i+1)]$ where by definition $\xi(i) \in \mathcal{U}$ for all $i \in \{1, \cdots, n\}$ and $\xi(1) = u_2$ and $\xi(n) = u_1$. One has $\Sigma(u_1) - \Sigma(u_2) = \sum_{i=1}^{n-1} \Sigma(\xi(i+1)) - \Sigma(\xi(i))$ and thus, we deduce by theorem 2.4, that $\|\Sigma(u_1) - \Sigma(u_2)\|_{\mathcal{Y}} \leq \sum_{i=1}^{n-1} \|\Sigma(\xi(i+1)) - \Sigma(\xi(i))\|_{\mathcal{Y}} \leq \eta \sum_{i=1}^{n-1} \|\xi(i+1) - \xi(i)\|_{\mathcal{U}}$ which allows to conclude the proof since $\xi(i)$ belongs to $\mathcal{U}$.

3 Main result

Let us recall that $\mathcal{U}$ and $\mathcal{Y}$ are two open subsets of some vectorial functional space equipped with the same $\mathcal{L}_p$ norm with $p \in \{2, \infty\}$.

Theorem 3.1 *Let Σ, be a dynamical system associated to (1), defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ where $\mathcal{U}$ is assumed to be a convex set. If f and h are uniformly Lipschitz and C^1 and if for any $u_r \in \mathcal{U}$, the linearization of $D_G\Sigma[u_r]$, i.e., $\dot{z}(t) = A(t)z(t)$, is exponentially stable then there exists a finite constant $\eta \geq 0$ such that for any $T \in [t_0, \infty)$ and $u_1, u_2 \in \mathcal{U}^e$, one has: $\|\Sigma(u_1) - \Sigma(u_2)\|_{\mathcal{Y},T} \leq \eta\|u_1 - u_2\|_{\mathcal{U},T}$.*

The proof is a straightforward consequence of the means value theorem 2.4, theorem 2.1 and of this well-know result:

Lemma 3.2 [27] *Let us assume that the following linear system*

$$\Pi \begin{cases} \dot{\bar{x}}(t) &= A(t)\bar{x}(t) + B(t)\bar{u}(t) \\ \bar{y}(t) &= C(t)\bar{x}(t) + D(t)\bar{u}(t) \\ \bar{x}(t_0) &= 0 \end{cases}$$

has bounded realization, i.e., $A(t), B(t), C(t)$ and $D(t)$ are uniformly bounded matrices of time, then Π has a finite $\mathcal{L}_p$ gain with $p \in [1, \infty]$ if $\dot{\bar{x}}(t) = A(t)\bar{x}(t)$ is exponentially stable.

When $\mathcal{U}$ is only an arc-connected set, we have the following result:

Corollary 3.3 *Let Σ, be a dynamical system associated to (1), defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ and where $\mathcal{U}^e$ is assumed to be an arc-connected set. If f and h are uniformly Lipschitz and C^1 and if for any $u_r \in \mathcal{U}$, the linearization of $D_G\Sigma[u_r]$, i.e., $\dot{z}(t) = A(t)z(t)$, is exponentially stable then for any $u_1, u_2 \in \mathcal{U}$, there exists a finite constant $K \geq 0$ such that $\|\Sigma(u_1) - \Sigma(u_2)\|_{\mathcal{Y}} \leq K$.*

4 Direct consequences of the main result

4.1 Boundedness of motions

We now analyze the effect of persistent perturbations with a finite amplitude.

Theorem 4.1 *Let Σ, be a dynamical system associated to (1) defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ where $\mathcal{U}$ is assumed to be an arc-connected set. If f is uniformly Lipschitz and C^1 and if for any $u_r \in \mathcal{U}$, the linearization of $D\Sigma_G[u_r]$, i.e., $\dot{z}(t) = A(t)z(t)$, is exponentially stable and $0 \in \mathcal{U}$ then for any $L \geq 0$, there exists $M \geq 0$ such that $\|\varphi(t, t_0, x_0, u_r)\| \leq M$ for any $u_r \in \mathcal{U}^e$ such that $\|u_r(t)\| \leq L$ a.e.*

Proof: Corollary 3.3 implies that for any $u_r, \tilde{u}_r \in \mathcal{L}_\infty \cap \mathcal{U}$, there exists a finite constant $K \geq 0$ such that $\|x[x_0, u_r] - x[x_0, \tilde{u}_r]\|_\infty \leq K$.
By assumption $\tilde{u}_r = 0$ belongs to $\mathcal{U}$, which allows to deduce that $\|x[x_0, u_r]\|_{\infty,T} \leq K$ since by definition one has $\varphi(t, t_0, x_0, 0) = 0$. Finally, the motion is an absolute continuous function of time which allows to conclude.

4.2 Steady-state

In this section, we consider the problem of the analysis of the behavior of Σ with respect to a perturbation on its input. We firstly consider the effects of perturbations which "vanish" when the time goes to the infinity.

Theorem 4.2 *Let Σ, be a dynamical system associated to (1) defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ where $\mathcal{U}$ is assumed to be a convex set. If f is uniformly Lipschitz and C^1 and if for any $u_r \in \mathcal{U}$, the linearization of $D_G\Sigma[u_r]$, i.e., $\dot{z}(t) = A(t)z(t)$, is exponentially stable then for any $u_r, \tilde{u}_r \in \mathcal{U}^e$ such that $u_r - \tilde{u}_r \in \mathcal{L}_2$, one has* $\lim_{t\to\infty} \|\varphi(t, t_0, x_0, u_r) - \varphi(t, t_0, x_0, \tilde{u}_r)\| = 0$.

Proof: Theorem 3.1 implies that there exists a finite constant η such for $T \geq t_0$ and any $u_r, \tilde{u}_r \in \mathcal{U}^e$, one has $\|x[x_0, u_r] - x[x_0, \tilde{u}_r]\|_{2,T} \leq \eta\|u_r - \tilde{u}_r\|_{2,T}$. Since by assumption, $u_r - \tilde{u}_r$ belongs to $\mathcal{L}_2$, one has for any $T \geq t_0$: $\|x[x_0, u_r] - x[x_0, \tilde{u}_r]\|_{2,T} \leq \eta\|u_r - \tilde{u}_r\|_2$ which allows to ensures that $x[x_0, u_r] - x[x_0, \tilde{u}_r]$ belongs to $\mathcal{L}_2$. This enables to prove the result on the basis of Barbalat's lemma (see *e.g.* [19] p. 210) since the two motions are two absolute continuous functions of time.

4.3 Periodic and constant inputs

The following theorem claims that the steady state response to a periodic (resp. constant) input signal is a periodic (resp. constant) trajectory.

Definition 4.1 *A motion, $x(t)$, defined from $[t_0, \infty)$ into $\mathcal{R}^n$ is said to asymptotically T–periodic if for any $\epsilon > 0$ there exists a positive constant $T_\epsilon > t_0$ such that for any $K \in \mathcal{N}$ and any $t \geq T_\epsilon$, one has $\|x(t + KT) - x(t)\| \leq \epsilon$.*

Theorem 4.3 *Let Σ, be a dynamical system associated to (1) defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ where $\mathcal{U}$ is assumed to be a convex set. If f is uniformly Lipschitz and C^2 and if for any $u_r \in \mathcal{U}^e$, the linearization of $D\Sigma_G[u_r]$,i.e., $\dot{z}(t) = A(t)z(t)$, is exponentially stable and $0 \in \mathcal{U}$ then for any periodic input belonging to $\mathcal{U}^e$, the associated trajectory is asymptotically periodic.*

5 Lyapunov stability of the unperturbed trajectories

5.1 Preliminary remark

Let us point out the problem of the definition of the Lyapunov stability in the input-output context. Indeed, in the input-output context, the motions with an initial condition different from x_0 at $t = t_0$ are not necessarily defined. This consequently implies that the Lyapunov stability can not be defined without additional assumptions. Moreover, if these motions exist, their properties (with respect to the input-output criteria) are not necessarily related to the properties of the motion starting at x_0 at $t = t_0$.

In the sequel, as in [32], we bypass this problem by the introduction of a backwards extension of Σ which is compatible with the input-output properties of the system (see [31, 32]).

5.2 Global and asymptotic stability of the unperturbed motions

We firstly deduce a simple result which is a direct consequence of theorem 4.2 and of some reachability assumption of the state space of Σ from x_0. Let us recall this following definition.

Definition 5.1 *An open subset of the state space of Σ, i.e. $\Omega \subset \mathcal{R}^n$ is said to be reachable from x_0 with respect to $\mathcal{U}$ if given any $x \in \Omega$ there exist $u \in \mathcal{U}$ and finite constant $T_r \geq 0$ such that $x = \varphi(t_0 + T_r, t_0, x_0, u)$.*

Theorem 5.1 *Let Σ, be a dynamical system associated to (1) defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ where $\mathcal{U}$ is a convex set and Ω is a bounded and open subset of $\mathcal{R}^n$. If f is uniformly Lipschitz and C^2 and if for any $u_r \in \mathcal{U}$, the linearization of $D_G\Sigma[u_r]$, i.e., $\dot{z}(t) = A(t)z(t)$, is exponentially stable, Ω is reachable from x_0 with respect to $\mathcal{U}$ and $0 \in \mathcal{U}$ then all the unperturbed motion associated with bounded inputs belonging to $\mathcal{U}^e$ and initial conditions in Ω are globally (with respect to Ω) asymptotically stable.*

The proof is a straightforward consequence of the exponential stability of the linearizations and of theorem 4.2. Indeed, the system is globally asymptotically stable if the unperturbed motion is asymptotically stable and the unperturbed motion is globally attractive from Ω (see [12]).

Actually, the uniform and global asymptotic stability is obtained if we consider constant or periodic inputs since for periodic systems, the global and asymptotic stability implies the uniform global asymptotic stability (see *e.g.* [17, 30]). Let us firstly introduce the uniform reachability definition:

Definition 5.2 *An open subset of the state space of Σ, i.e. $\Omega \subset \mathcal{R}^n$ is said to be uniformly reachable from x_0 if there is a finite constant $\bar{T}$ such that for any $x \in \Omega$, any $t_1 \geq t_0$ there exist $u \in \mathcal{U}$ and positive constant, T_r with $T_r \leq \bar{T}$ such that $x = \varphi(t_1 + T_r, t_1, x_0, u)$.*

We can now set a result concerning global and uniform asymptotical stability of the unperturbed motion. To this purpose, let us define an open ball of $\mathcal{R}^n$, namely $B_r(x_0, \rho) \triangleq \{x \in \mathcal{R}^n \mid \|x - x_0\| < \rho\}$, which contains all the points of a bounded unperturbed motion, *i.e.* for any $t \in [t_0, \infty)$, one has $x_r(t) \in B_r(x_0, \rho)$. We then now define an open set, namely $\Omega(u_r)$, which contains $B_r(x_0, \rho)$.

Theorem 5.2 *Let Σ, be a dynamical system associated to (1) defined from $\mathcal{U}^e$ into $\mathcal{Y}^e$ where $\mathcal{U}$ is a convex set. If f is uniformly Lipschitz and C^2 and if for any $u_r \in \mathcal{U}$, the linearization of $D_G\Sigma[u_r]$, i.e., $\dot{z}(t) = A(t)z(t)$, is exponentially stable, $\Omega(u_r)$ is uniformly reachable from x_0 with respect to $\mathcal{U}$ and $0 \in \mathcal{U}$ then all the unperturbed motion associated with bounded inputs belonging to $\mathcal{U}^e$ and initial conditions in $\Omega(u_r)$ are globally (with respect to $\Omega(u_r)$) and uniformly asymptotically stable.*

The proof of this theorem is given in [16]. It is in fact a direct consequence of results presented in [12].

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