Algebraic multigrid based on De Rham complexes on graphs with applications to Maxwell's equations

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Formulation

Our aim is the resolution of linear systems coming from the edge finite element discretisation of:

$$\begin{cases} \text{To find } \mathbf{E} \in V \text{ such that: } a(\mathbf{E}, \mathbf{E}') = F(\mathbf{E}'),\ \forall \mathbf{E}' \in V_0, \\ \text{with } a(\mathbf{E}, \mathbf{E}') = \int_\Omega \nu \,\mathrm{curl}\, \mathbf{E} \cdot \mathrm{curl}\, \mathbf{E}' + \int_\Omega \gamma \mathbf{E} \cdot \mathbf{E}'. \end{cases}$$

with $V \subset \mathbf{H}(\mathrm{curl}, \Omega)$. We consider ν and $\gamma > 0$.

- Straightforward applications: transient vector wave equation or eddy current formulation.
- Possible extensions: magnetostatics, time-harmonic problems.

Finite element spaces

Two finite element spaces considered :

- Edge element space $\mathcal{W}^1(\mathcal{T})$ included in $\mathbf{H}(\mathrm{curl}, \Omega)$: the lowest order Nedelec elements. Degrees of freedom: path integral along the mesh edges. Basis $(\lambda_i^h)_{i\in\{1,\ldots,E^h\}}$.
- Nodal element space $\mathcal{W}^0(\mathcal{T})$ included in $\mathrm{H}^1(\Omega)$: P_1-Lagrange finite elements. Degrees of freedom: values at the mesh vertices. Basis $(\phi_p^h)_{p\in\{1,\ldots,N^h\}}$.

Coming from continuous space properties, both discrete spaces belong to a De Rham complex with complete sequence property.
In particular:

$$\mathrm{grad}\,\phi_p^h = \sum_{i=1}^{E^h} G_{ip}^h \lambda_i^h,\ \forall p \in \{1,\ldots,N^h\}.$$

and $\mathrm{range}(\mathrm{grad}) = \ker(\mathrm{curl})$.

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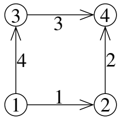


Coming from continuous space properties, both discrete spaces belong to a De Rham complex with complete sequence property.
In particular:

$$G^h = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{grad}\, \phi_p^h = \sum_{i=1}^{E^h} G_{ip}^h \lambda_i^h, \ \forall p \in \{1, \dots, N^h\}.$$

and range(grad) = ker(curl).

Smoother choice

- Laplacian case: a few iterations of damped Jacobi or Gauss-Seidel provide an efficient smoothing.
- curl ν curl case: classical algorithms do not deal with the component in the kernel of curl, even if the dimension of this subspace is proportional to the number of mesh nodes.
- In order to supress this difficulty, a hybrid Gauss-Seidel algorithm has been proposed by Hiptmair[1]:
  - Classical smoothing for the global system.
  - Specific smoothing on the kernel of curl.

[1] Multigrid method for Maxwell's equations. *SIAM J. Numer. Anal.*, 36(1), 1999.

Algebraic construction of prolongation operators

- The coarse nodal and edge spaces are included in the fine spaces:

$$\phi_n^H = \sum_{p=1}^{N^h} P_{pn}^{\mathrm{nod}} \phi_p^h, \ \forall n \in \{1, \dots, N^H\},$$

$$\lambda_e^H = \sum_{i=1}^{E^h} P_{ie}^{\mathrm{edg}} \lambda_i^h, \ \forall e \in \{1, \dots, E^H\}.$$

- In order to use the same smoother as in the geometric case, the following relation will be kept at the coarse level:

$$\operatorname{grad} \phi_n^H = \sum_{e=1}^{E^H} G_{en}^H \lambda_e^H, \ \forall n \in \{1, \dots, N^H\}. \quad (1)$$

$(\phi_n^H)_{n\in\{1,\dots,N^H\}}$ is the coarse nodal basis and $(\lambda_e^H)_{e\in\{1,\dots,E^H\}}$ is the coarse edge basis.

Algebraic construction of prolongation operators

Relation (1) is implicitly satisfied in geometric multigrid.
This relation is wished to be kept for algebraic methods, in particular to define a coarse graph whose incident matrix G^H will be used in this relation.
By gathering the previous relations, the commutativity relation is obtained:

$$P^{\mathrm{edg}} G^H = G^h P^{\mathrm{nod}}.$$

G^h describes the fine graph, P^{nod} is the nodal prolongator computed by existing methods.

To define or to compute:

G^H and the edge prolongator P^{edg}.

Remark on the computation of P^{nod}

Many algebraic multilevel methods for grad div operator can provide P^{nod} operators satisfying the conditions required by our construction.
In particular, they build a decomposition of the domain Ω in overlapping subdomains $(\Omega_n^H)_{n=1,\dots,N^H}$ such that:

$$\operatorname{supp}\phi_n^H \subset \overline{\Omega_n^H}$$

This is equivalent to enforce zero coefficients in the n^{th} column of P^{nod}.
It is also required that P^{nod} satisfies a partition of unity property:

$$\forall x \in \overline{\Omega}, \sum_{n=1}^{N^H} \phi_n^H(x) = 1 \Leftrightarrow \sum_{n=1}^{N^H} P_{pn}^{\mathrm{nod}} = 1, \forall p \in \{1, \dots, N^h\}.$$

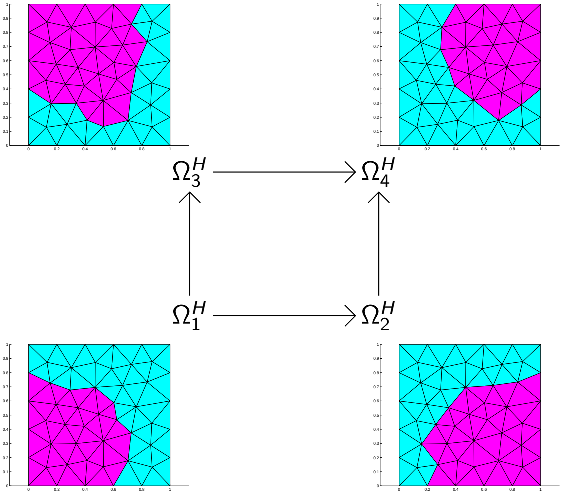
Coarse graph and support of the functions

After the computation of the nodal prolongator P^{nod}:

- Definition of a coarse graph or equivalently a coarse edge-node incidence matrix G^H.
  For a coase edge of extremities coarse nodes n and m, the condition $\Omega_n^H \cap \Omega_m^H \neq \emptyset$ has to be ensured.
- A coarse edge function λ_e^H is associated with each coarse edge e and $\operatorname{supp} \lambda_e^H \subset \overline{\Omega_n^H \cap \Omega_m^H}$.
  As for P^{nod}, it is equivalent to enforce zero coefficients in the e^{th} column of P^{edg}.

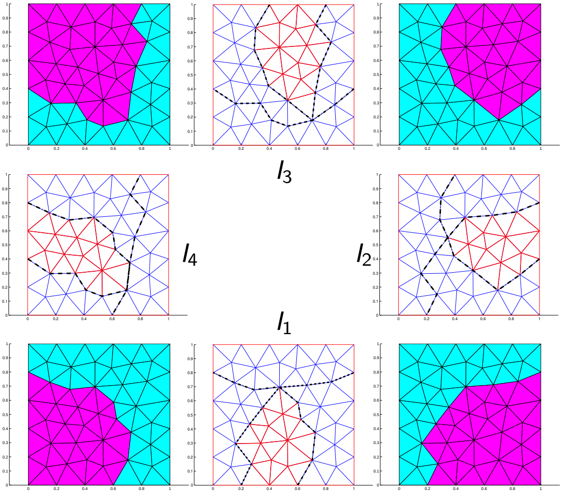
I_e: Set of the non-zero coefficients in the e^{th} column of P^{edg}.

Illustration



$$G^H = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

Illustration



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Final formulation

- Reitzinger and Schöberl[2] proposed a strategy to build nodal and edge prolongators satisfying the relation $P^{\mathrm{edg}} G^H = G^h P^{\mathrm{nod}}$.
  Their approach is attached to a particular choice for the nodal prolongator P^{nod} which leads to a unique matrix P^{edg} satisfying the commutation property.
- We wished to extend this approach to other choices of the nodal prolongator P^{nod}.
  Several matrices P^{edg} can then satisfy the commutativity relation and supplementary rules should be defined to make a judicious choice.

[2] S. Reitzinger and J. Schöberl. An algebraic multigrid method for finite element discretizations with edge elements. *Numer. Linear Algebra App.*, 9(3), 2002.

Final formulation

Ideas coming from nodal finite element strategy: constraints on the supports (to keep the sparsity or the matrices) + minimisation of an energy functional + approximation constraints.

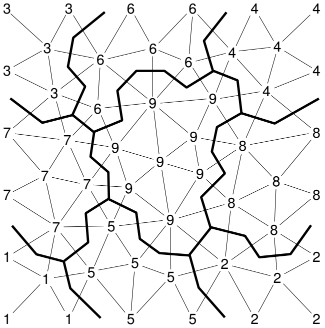
Optimisation problem

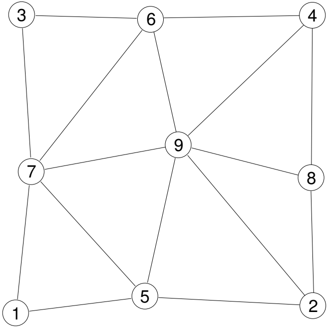
$$\begin{cases} \text{Find } P^{\mathrm{edg}} \text{ minimising } \displaystyle\sum_{e=1}^{E^H} \beta_e^t K_e \beta_e, \\ \text{under the constraint: } P^{\mathrm{edg}} G^H = G^h P^{\mathrm{nod}}. \end{cases}$$

- β_e: contains the non-zero coefficients of the e^{th} column of P^{edg}, *i.e.* these indexed by I_e.
- K_e: Local symmetric positive definite (SPD) matrix. The whole K_e matrices define the energy functional for the minimisation.

Subgraphs $\mathcal{S}^{H,i}$

$\mathcal{S}^{H,i}$: graph defined for each fine edge i such that one index e satisfies $i \in I_e$.
Subgraph of the coarse graph obtained by keeping only the coarse edges of index e such that $i \in I_e$.





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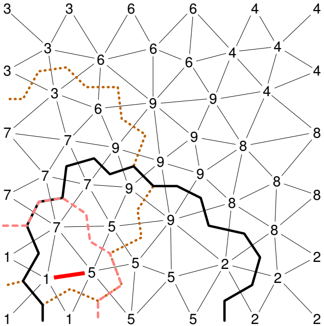

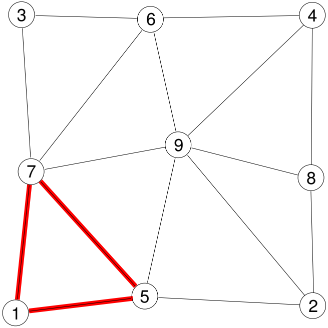

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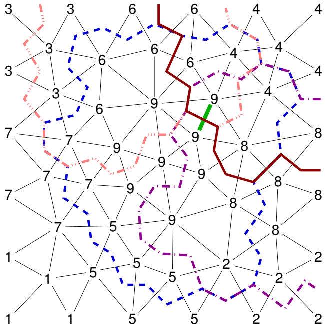
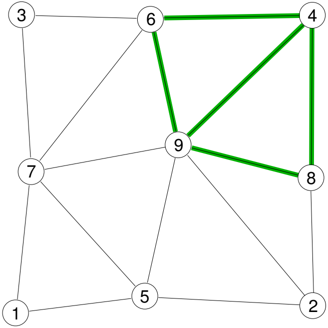




Flow problems

- We have shown that solving $P^{\mathrm{edg}} G^H = G^h P^{\mathrm{nod}}$ is equivalent to solve on each subgraph $\mathcal{S}^{H,i}$ a flow problem:

$$(G^{H,i})^t P_i^{\mathrm{edg}} = (G^h P^{\mathrm{nod}})_i.$$

- To determine: P_i^{edg}, vector containing the components of the i^{th} row of P^{edg} indexed by the indices of the edges of $\mathcal{S}^{H,i}$.
- Known: $(G^h P^{\mathrm{nod}})_i$, vector containing the components of the i^{th} row of $G^h P^{\mathrm{nod}}$ indexed by the indices of the nodes of $\mathcal{S}^{H,i}$.

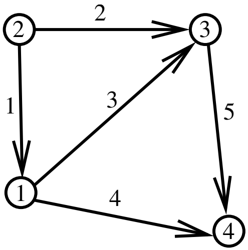
Resolution of flow problems

Solution of the flow problem:

$$P_i^{\mathrm{edg}} = (P_i^{\mathrm{edg}})' + (P_i^{\mathrm{edg}})''.$$

where:

- $(P_i^{\mathrm{edg}})'$ is a particular solution of the flow problem, determined thanks to a spanning tree of the subgraph,
- $(P_i^{\mathrm{edg}})''$ belongs to the kernel of $(G^{H,i})^t$ which is generated by the independent cycles of the subgraph.



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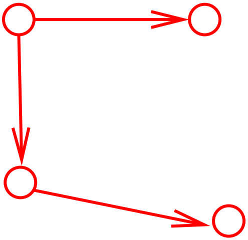

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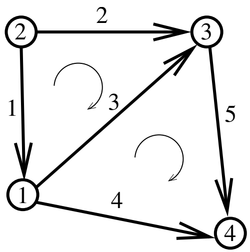


Linear system for the minimisation

After determining the particular solutions, the minimisation problem is equivalent to the resolution of a linear system of the form:

$$B^t D B \Gamma = -B^t D \widetilde{P^{\mathrm{edg}}},$$

where:

- Γ is the vector whose components give the coefficients of $((P_i^{\mathrm{edg}})'')^t$ in the bases of the kernel of $(G^{H,i})^t$.
- the matrix B gathers the basis vectors of these different kernels. It is a sparse full-rank matrix which is assembled during the resolution of flow problems.
- the matrix D is block-diagonal and its diagonal blocks are the matrices K_e involved in the energy functional.
- the vector $\widetilde{P}^{\mathrm{edg}}$ gathers the particular solutions $(P_i^{\mathrm{edg}})'$ from all flow problems.

Properties of the linear system

- The matrices K_e being SPD, $B^t DB$ is SPD and the system can be solved by the conjugate gradient method.
- Depending on the choice for K_e, the conditioning of D may be strongly dependent of the global dimension of the problem.
- However, if $D = \mathrm{Id}$, the conditioning of $B^t B$ is low and independent of the global dimension of the problem.

Moreover, a high accuracy is not required for the resolution of this system; this is only a setup phase before solving the initial system. The solution kept, whatever the accuracy is, always satisfies the commutativity relation.

Definition of Ω_n^H and of G^H

- Partition of the nodes of the graph (initialy of the mesh):

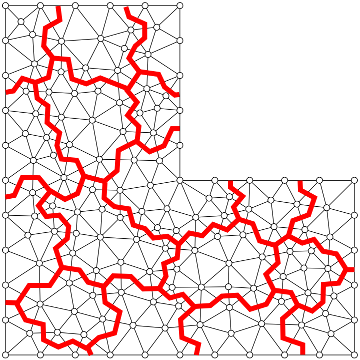
$$\{1,\ldots,N^h\} = \bigcup_{n=1}^{N^H} H_n.$$

Definition of subdomains $(\widetilde{\Omega}_n^H)_{n=1,\ldots,N^H}$:

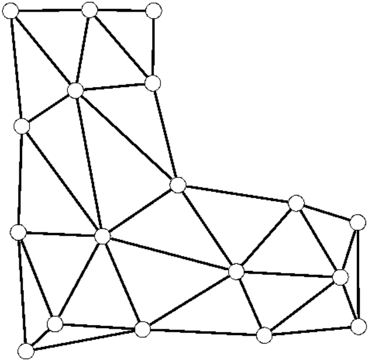
$$\widetilde{\Omega}_n^H = \bigcup_{p\in H_n} \mathrm{supp}(\phi_p^h).$$

- Creation of a coarse edge e of extremities n, m iff $\widetilde{\Omega}_n^H \cap \widetilde{\Omega}_m^H \neq \emptyset$.
- Subdomain Ω_n^H defined by extending $\widetilde{\Omega}_n^H$ to the nearest neighbours. Without this extension, no degree of freedom for the minimisation: this is the Reitzinger and Schöberl method (RS method).

Definition of Ω_n^H and of G^H

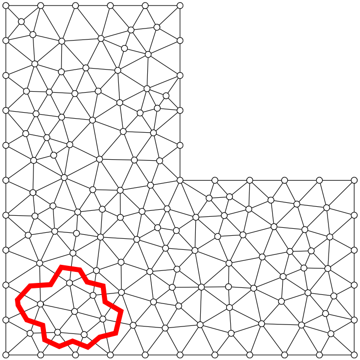


Initial graph.



Coarse graph representation.

Definition of Ω_n^H and of G^H



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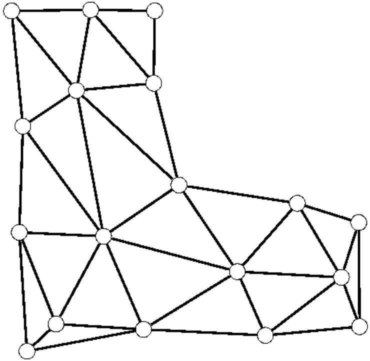

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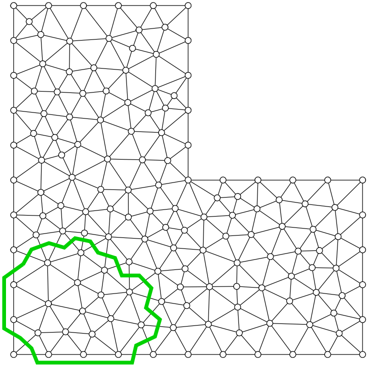

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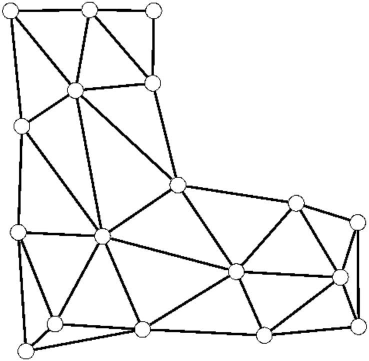


Coarse graph representation.

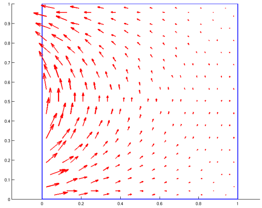
Choice for the matrices K_e

Different choices used for numerical experiments:

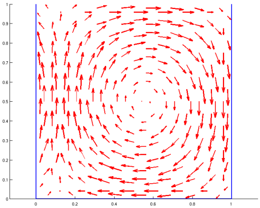
- Matrices extracted from the global matrix K of the problem; it is denoted by K,
- Matrices extracted from the matrix defined from $\int_\Omega \nu \operatorname{curl} E \cdot \operatorname{curl} E'$; it is denoted by S_ν,
- Matrices are all equal to the identity; it is denoted by Id.

2D problem

- $\Omega =]0;1[^2$.
- Tangential component of the field is enforced on the left side with $\mathbf{E}_y = \sin(2\pi y)$. On the remaining of the boundary: $(\text{curl}\,\mathbf{E}) \times \mathbf{n} = 0$.
- Constant coefficient in the subdomains with the following equations:
  - Basic test: $\text{curl}\,\text{curl}\,\mathbf{E} + \mathbf{E} = 0$.
  - Cavity in harmonic regime: $\text{curl}\,\text{curl}\,\mathbf{E} - \omega^2 \mathbf{E} = 0$ with $\omega = 1.5\pi$.

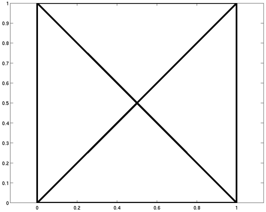


$\gamma = 1$

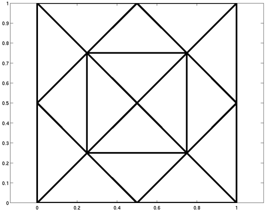


γ with $\omega = 1.5\pi$

Mesh used



Initial mesh τ_0^h



First refinement τ_1^h

For the mesh τ_i^h, i indicates the number of refinement.
NB: algebraic strategy works without knowledge of the grid structuration.

Resolution of the optimisation linear system

	$\mathcal{T}_2^h$	$\mathcal{T}_3^h$	$\mathcal{T}_4^h$	$\mathcal{T}_5^h$	$\mathcal{T}_6^h$	$\mathcal{T}_7^h$
optimisation system size	62	253	1016	4081	16355	65479
initial system size	100	392	1552	6176	24640	98432

Table: Number of unknows for the optimisation linear system and for the initial linear system — 2D, structured meshes.

The number of unknowns for the minimisation is roughly $2/3$ of the number of unknowns for the initial system.

Resolution of the optimisation linear system

Resolution by a conjugate gradient method without preconditioning.

	$\mathcal{T}_2^h$	$\mathcal{T}_3^h$	$\mathcal{T}_4^h$	$\mathcal{T}_5^h$	$\mathcal{T}_6^h$	$\mathcal{T}_7^h$
K	24	19	12	12	12	12
S_ν	14	12	12	12	12	12
Id	3	1	1	1	1	1

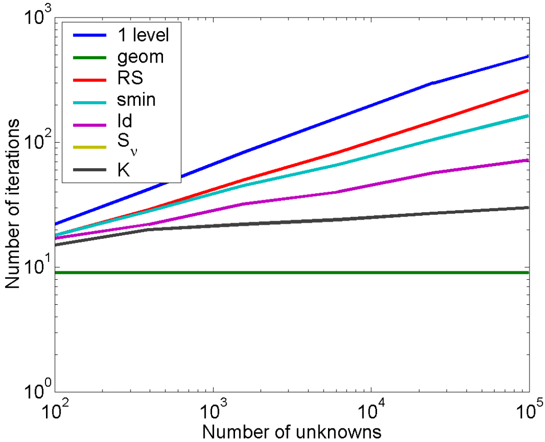
Table: Number of iterations for solving the optimisation linear system (division of the residual by 10^3) — 2D, structured meshes.

- Results given only for the finest level.
- Number of iterations independent of the dimension of the global system.

Resolution of the initial linear system - Case 1

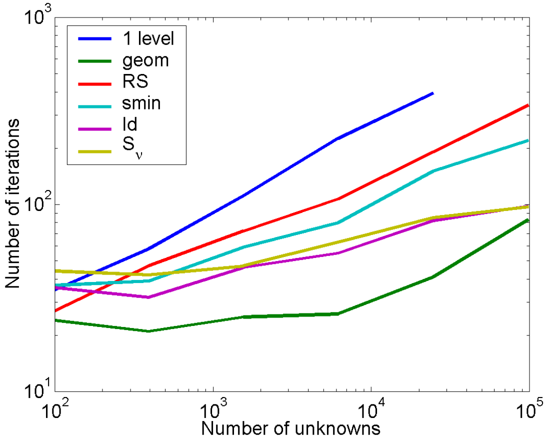
- Conjugate gradient preconditioned by a multilevel method using Hiptmair's smoother.
- Stopping criterion: division by 10^{10} of the residual norm.

geom: geometric multigrid,
1 level: use only one level,
smin: use only the particular solution of flow problems.



Resolution of the initial linear system - Case 2

- General deterioration of the results for $\gamma = -(1.5\pi)^2$.
- The hierarchy between the methods remains the same.



Final results and perspectives

Final results

- Description of the edge prolongators satisfying the commutativity relation. Generic algorithms to compute these prolongators.
- Addition of an optimisation process in order to optimise the choice of the edge prolongator.

Perspectives

- Introduction of other methods for the construction of the coarse graph.
- Optimisation of basis satisfying the commutativity relation but initially computed by another strategy.