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Asymptotic behavior for a class of the renewal nonlinear equation with diffusion

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In this paper, we consider nonlinear age-structured equation with diffusion under nonlocal boundary condition and non-negative initial data. More precisely, we prove that under some assumptions on the nonlinear term in a model of McKendrick–Von Foerster with diffusion in age, solutions exist and converge (long-time convergence) towards a stationary solution. In the first part, we use classical analysis tools to prove the existence, uniqueness, and the positivity of the solution. In the second part, using comparison principle, we prove the convergence of this solution towards the stationary solution. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

In the study of population of cells, animals, or humans, one of the most used model is the McKendrick–Von Foerster model said also renewal model, where the density of population $n(t,x)$ at time t and age x, is described by the master equation

$$\begin{cases} n_t(t,x) + n_x(t,x) + d(x)n(t,x) = 0, \quad t \geq 0, \quad x \geq 0 \\ n(t,0) = \int_0^\infty B(x)n(t,x)\mathrm{d}x, \quad \text{and} \quad n(0,x) = n_0(x), \end{cases} \tag{1.1}$$

where $B \geq 0$ is the birth rate, $d \geq 0$ is the death rate. It is well known that long-time asymptotic is described by the first eigenvalue λ and positive engenvector N of the stationary problem of (1.1). More precisely, for a long time, $n \sim CstNe^{\lambda t}$, see for instance [1–3] and [4] (using general relative entropy method) and [5, 6] (using semigroup method). This of course does not take in account the use of resources. Indeed, the population growth is not limited in time when the Malthusian growth rate (eigenvalue λ) is strictly positive. Nevertheless, resources are limited, and so, we expect that there is a limitation of the size of the population (an 'equilibrium' between the quantity of resources and their use). On the other hand, we notice that the population goes to extinction when the Malthusian growth rate is strictly negative. Nevertheless, in this case, there is no limitation of the resource, and the extinction cannot be explained by the lack of resources. In order to take in account the consumption of nutrient, we can, for instance, change the linear birth term in (1.1) by a nonlinear birth term (see for instance [7–10]):

$$\begin{cases} n_t(t,x) + n_x(t,x) + d(x)n(t,x) = 0, \quad t \geq 0, \quad x \geq 0 \\ n(t,0) = f\left(\int_0^\infty B(x)n(t,x)\mathrm{d}x\right), \quad \text{and} \quad n(0,x) = n_0(x). \end{cases}$$

In [11, 12], the authors proposed defining 'biological age' according to the DNA content and diffusion accounts for its variability. Therefore, considering that the variable x is a biological age and can vary according to certain proteins causing degradation or recovery caused by external factors, the density of the population satisfies the master equation

$$\begin{cases} n_t(t,x) - n_{xx}(t,x) + (g(x)n(t,x))_x + d(x)n(t,x) = 0 \quad t \geq 0, \quad x \geq 0 \\ g(0)n(t,0) - n_x(t,0) = f\left(\int_0^\infty B(x)n(t,x)\mathrm{d}x\right), \quad \text{and} \quad n(0,x) = n_0(x) \in L^1_+(\mathbb{R}^+), \end{cases} \tag{1.2}$$

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where $B \geq 0$ is the birth rate, $d \geq 0$ is the death rate, and the diffusion term modelizes the variability evolution of the 'biological age' x. We notice that problem (1.2) arises in many other applications, see for instance [13–16] and the references therein. Recently, we have treated the linear case of this renewal model with diffusion [17], we proved the existence, uniqueness, and positivity of the solution, and we showed that for a long time, $n \sim CstNe^{\lambda t}$ (with the decay estimate of the solution). Nevertheless, it does not take into account the consumption of resources. In the system (1.2), we overcome this and consider the consumption of the resources, such as nutrients, by introducing nonlinearity in the birth term. In general, this term may limit the possible extra growth of the population. In the system (1.2), we notice that the nonlinear form of the recruitment term only takes in account newborns. This means that, in a population modelized by system (1.2), 'giving birth' and 'newborns' need more resources and may limit growth of the population.

2. Assumptions and main results

In this section, we give the main assumptions of this work and state the main results of the paper.

We suppose that B and d are non-negative continuous functions and satisfy

$$0 < B_m \leq B(x) \leq B_M \text{ and } 0 < d_m \leq d(x) \leq d_M. \tag{2.1}$$

The nonlinear growth function f is smooth (for instance $C^1(]0, \infty[)$), is nondecreasing, and verifies the condition

$$f(x) \leq \alpha x + \gamma, \tag{2.2}$$

with γ positive and $\alpha \in [0, A)$ with a constant A to be chosen later.

$$\exists s_0 > 0 \;:\; \forall s' \in]0, s_0[\;\; f(s')/s' > d_M/B_m, \tag{2.3}$$

and

$$\exists s_1 > 0 \;:\; \forall s' \in]s_1, \infty[\;\; f(s')/s' < d_m/B_M. \tag{2.4}$$

The growth rate g is a C^1 positive function satisfying

$$0 < g_m \leq g(x) \leq g_M, \tag{2.5}$$

$$\int_0^\infty e^{-G(x)} \mathrm{d}x < \infty, \tag{2.6}$$

where $G(x) := \int_0^x g(s)\mathrm{d}s$, and there exists a positive constant ε such that

$$d(x) + g'(x) \geq \varepsilon \tag{2.7}$$

Remark 2.1
We notice that the condition (2.3) (resp. (2.4)) avoids the extinction (resp. the unlimited growth) of the population, which cannot be explained by the lack of resources. Indeed, under the existence of solutions to (1.2) (proved in Section 3), we have

$$\frac{d}{\mathrm{d}t} \int_0^\infty n(t,x)\mathrm{d}x \geq f\left(B_m \int_0^\infty n(t,x)\mathrm{d}x\right) - d_M \int_0^\infty n(t,x)\mathrm{d}x,$$

and the condition (2.3) implies that $\int_0^\infty n(t,x)\mathrm{d}x \geq \min(\int_0^\infty n_0(x)\mathrm{d}x, s_0)$ for all time $t \geq 0$. This means that the trivial solution is not stable. For instance, the condition (2.3) is satisfied for functions that behave like $f(t) \sim_{t\to 0} Ct^\alpha$ with $\alpha < 1$ and $C > 0$ or $\alpha = 1$ and $C > d_M/B_m$.

We will prove the following result, which concerns the existence and uniqueness as well as the positive solution of problem (1.2).

Theorem 2.2
Under assumptions (2.1)–(2.6) and for all positive initial data $n_0 \in L^1(\mathbb{R}^+, e^{-G(x)}\mathrm{d}x)$, problem (1.2) has a unique positive solution $n \in C(\mathbb{R}^+, L^1(\mathbb{R}^+, e^{-G(x)}\mathrm{d}x))$.

The following theorem concerns the asymptotic behavior of the solution n to problem (1.2). To be more precise, let N be solution of the following stationary problem

$$\begin{cases} -N''(x) + (gN)'(x) + d(x)N(x) = 0, \quad x \geq 0, \\ g(0)N(0) - N'(0) = f\left(\int_0^\infty B(x)N(x)\mathrm{d}x\right), \\ \int_0^\infty N(x)\mathrm{d}x < \infty, \quad \text{and} \quad N \geq 0. \end{cases} \tag{2.8}$$

Then, we have

Theorem 2.3
Assume that (2.1)–(2.7) hold. Then, for all positive initial data $n_0 \in L^1(\mathbb{R}^+, e^{-G(x)}\mathrm{d}x)$, with $n_0(x) \leq CN(x)$, the solution n to problem (1.2) satisfies

$$\underline{N}(x) \leq \liminf_{t\to\infty} n(t,x) \leq \limsup_{t\to\infty} n(t,x) \leq \bar{N}(x),$$

where $\bar{N}$ (resp. $\underline{N}$) is the maximal (resp. minimal) nontrivial solution to the stationary problem (2.8). In addition, if the stationary problem (2.8) has a unique nontrivial solution N, then we have the convergence of $n(t,x)$ to $N(x)$.

The paper is organized as follows. In the next section, we establish the existence, uniqueness, and positivity of the solution to problem (1.2). Section 4 is devoted to prove the convergence of solutions towards the solution of the stationary problem N. In Section 5, we prove a blow-up result and an extinction result under some assumptions on the growth function f. Finally, the paper is supplemented by the numerical simulation, and discussion on the model and the theoretical results.

3. Existence results

The aim of this section is to prove Theorem 2.3. In order to do it, we begin by stating the comparison principle lemma, which is very useful for the rest of the paper.

We consider the following problem

$$\begin{cases} n_t(t,x) - n_{xx}(t,x) + (g(x)n(t,x))_x + d(x)n(t,x) = 0, \quad t \geq 0, \quad x \geq 0, \\ g(0)n(t,0) - n_x(t,0) = \alpha \displaystyle\int_0^\infty B(x)n(t,x)\mathrm{d}x + \gamma, \quad \text{and} \quad n(0,x) = n_0(x) \in L^1(\mathbb{R}^+), \end{cases} \tag{3.1}$$

then we have,

Lemma 3.1
Let v and u be a non-negative supersolution and subsolution of problem (3.1), respectively. If $v(0,x) \geq u(0,x)$, then $v(t,x) \geq u(t,x)$.

Proof
Let $\phi \in L^2((0,T); W^{1,2}(\mathbb{R}^+))$ be solution to the following problem

$$\begin{cases} \phi_\tau(\tau,x) + \phi_{xx}(\tau,x) + g(x)\phi_x(\tau,x) - d(x)\phi(\tau,x) = 0, \quad 0 < \tau < t, \quad x \geq 0, \\ \phi_x(\tau,0) = 0, \quad \text{and} \quad \phi(t,x) = \psi(t,x), \end{cases} \tag{3.2}$$

where $\psi(t,x) = 1_{\{u-v>0\}}$. It is not difficult to see that problem (3.2) has a positive solution. By setting $w = u - v$, we have

$$\begin{cases} w_t(t,x) - w_{xx}(t,x) + (g(x)w(t,x))_x + d(x)w(t,x) \leq 0, \quad t \geq 0, \quad x \geq 0 \\ g(0)w(t,0) - w_x(t,0) \leq \displaystyle\int_0^\infty B(x)w^+(t,x)\mathrm{d}x, \quad \text{and} \quad w(0,x) \leq 0. \end{cases} \tag{3.3}$$

If we multiply the first inequality in (3.3) by ϕ and integrate over $(0,t) \times \mathbb{R}^+$, we get

$$\int_0^\infty w(t,x)\phi(t,x)\mathrm{d}x \leq \int_0^t \int_0^\infty \phi(\tau,0)B(x)w(\tau,x)\mathrm{d}x\mathrm{d}\tau. \tag{3.4}$$

Hence, from (3.4), we obtain

$$\int_0^\infty w^+(t,x)\mathrm{d}x \leq M \int_0^t \int_0^\infty w^+(\tau,x)\mathrm{d}x\mathrm{d}\tau.$$

Applying Gronwall's inequality, the conclusion of this lemma follows. □

The existence, uniqueness, and positiveness of the solution to problem (3.1) are stated in the following lemma; the proof is almost similar to the lemma in [17]; however, we present it for the reader's convenience.

Lemma 3.2
Let $n_0 \in L^1(\mathbb{R}^+) \bigcap L^2(\mathbb{R}^+)$, then problem (3.1) has a unique positive solution in $L^1((0,T) \times \mathbb{R}^+) \bigcap L^2((0,T); W^{1,2}(\mathbb{R}^+))$.

To prove this existence result, we will argue by approximation; namely, we consider the case of bounded domain $[0,R]$, and then, we pass to the limit in R. To be more precise, we first show the next lemma.

Lemma 3.3
The following problem

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + (g(x)v(t,x))_x + (d(x)+\mu)v(t,x) = 0, & t \geq 0, \quad x \in (0,R), \\ g(0)v(t,0) - v_x(t,0) = \alpha \displaystyle\int_0^R B(y)v(t,y)\mathrm{d}y + \gamma e^{-\mu t}, & \text{and} \quad v(t,R) = 0, \\ v(0,x) = n_0(x). \end{cases} \tag{3.5}$$

has a unique positive solution v_R, moreover if $R_1 \leq R_2$, then $v_{R_1} \leq v_{R_2}$.

Proof
Existence, uniqueness of the positive solution to problem (3.5), can be showed by using the Picard Banach fixed-point theorem in the Banach space $X_R = \mathcal{C}([0,T], L^1(0,R))$ with $||\phi||_{X_R} = \sup_{t\in[0,T]} \int_0^R |\phi(t,x)|\mathrm{d}x$ and $\mu > 0$ to be chosen later. More precisely, given $m \in X_R$, we define $v := S(m)$ as the solution to the problem

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + (g(x)v(t,x))_x + (d(x)+\mu)v(t,x) = 0, \text{ in } (0,T)\times(0,R), \\ g(0)v(t,0) - v_x(t,0) = \alpha \displaystyle\int_0^R B(x)m(t,x)\mathrm{d}x + \gamma e^{-\mu t}, \quad v(t,R) = 0, \\ v(0,x) = n_0(x), \quad x \in (0,R). \end{cases} \tag{3.6}$$

It is clear that $v \in L^2(0,T;W^{1,2}(0,R))) \cap X_R$. For $m_1, m_2 \in X_R$, we consider $v_1 := S(m_1)$, $v_2 := S(m_2)$. Then, with $v = v_1 - v_2$ and $m = m_1 - m_2$, solve

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + (g(x)v(t,x))_x + (d(x)+\mu)v(t,x) = 0, & \text{in } (0,T)\times(0,R), \\ g(0)v(t,0) - v_x(t,0) = \alpha \displaystyle\int_0^R B(x)m(t,x)\mathrm{d}x, \quad v(t,R) = 0, \\ v(0,x) = 0. \end{cases} \tag{3.7}$$

Multiplying Equation (3.7) by $sgn(v)$ and integrating in x,

$$\frac{d}{\mathrm{d}t}\int_0^R |v(t,x)|\mathrm{d}x + \int_0^R (d(x)+\mu)|v(t,x)|\mathrm{d}x \leq \alpha \int_0^R B(y)|m(t,y)|\mathrm{d}y, \tag{3.8}$$

after integration over $(0,T_1)$, $T_1 \leq T$,

$$\int_0^R |v(T_1,x)|\mathrm{d}x + \int_0^{T_1}\int_0^R (d(x)+\mu)|v(t,x)|\mathrm{d}x \leq \alpha B_M T ||m||_{X_R}$$

and then $||v||_{X_R} \leq \alpha B_M T ||m||_{X_R}$. Hence, choosing T such $\alpha T B_M < 1$, we obtain that the operator S is a strict contraction in the Banach space X_R, which proves the existence of a unique fixed point v_R.

As usual, we can iterate the operator on $[T,2T], [2T,3T], \ldots$ because the condition on T does not depend on the iteration. With this iteration process, we have built a solution in $\mathcal{C}(\mathbb{R}^+, L^1(0,R))$. The positivity of the solution is a simple consequence of the aforementioned comparison lemma. The strong maximum principle allows us to get the strict positivity of v_R. Let $R_1 \leq R_2$ and consider the corresponding solutions v_{R_1} and v_{R_2}. It is clear that v_{R_2} is a supersolution to the v_{R_1} problem. Hence, using the comparison principle, we obtain that $v_{R_1} \leq v_{R_2}$. Therefore, the result follows. □

We return now to prove the first existence lemma.

Proof of Lemma 3.2.
Let v_R be the solution to problem (3.5) obtained earlier. We define v_R for $(t,x) \in (0,T)\times(R,\infty)$ by setting $v_R(t,x) = 0$. Then, $v_R \in X \equiv L^1((0,T)\times\mathbb{R}^+) \cap L^2(0,T;W^{1,2}(\mathbb{R}^+))$. We know that

$$\frac{d}{\mathrm{d}t}\int_0^R v_R(t,x)\mathrm{d}x + \int_0^R (d(x)+\mu)v_R(t,x)\mathrm{d}x \leq \alpha B_M \int_0^R v_R(t,y)\mathrm{d}y + \gamma e^{-\mu t}.$$

Choosing $\mu > \alpha B_M$, it follows that

$$\frac{d}{\mathrm{d}t}\int_0^R v_R(t,x)\mathrm{d}x + \int_0^R (d(x)+\mu - B_M)v_R(t,x)\mathrm{d}x \leq \gamma e^{-\mu t}.$$

Hence, from Gronwall lemma, we obtain

$$\int_0^R v_R(t,x)\mathrm{d}x \leq C,$$

with $C > 0$. Thus, we conclude that v_R is uniformly bounded in $L^1((0,T)\times\mathbb{R}^+)$.

Therefore, using the monotonicity of the sequence $\{v_R\}$, we get the existence of $v \in L^1((0,T)\times\mathbb{R}^+)$ such that $v_R \uparrow v$ as $R \to \infty$. By taking v_R as a test function in (3.5), after integration, it results to

$$\frac{1}{2}\frac{d}{dt}\int_0^R v_R^2(t,x)\mathrm{d}x + \int_0^R \left(d(x)+\frac{1}{2}g'(x)+\mu\right)|v_R(t,x)|^2\mathrm{d}x + \int_0^R |(v_R)_x(t,x)|^2\mathrm{d}x \le C,$$

by integration in time,

$$\int_0^R |v_R(t,x)|^2\mathrm{d}x + \int_0^t\int_0^R \left(d(x)+\frac{1}{2}g'(x)+\mu\right)|v_R(t,x)|^2\mathrm{d}x\mathrm{d}t + \int_0^t\int_0^R |(v_R)_x(t,x)|^2\mathrm{d}x \le \frac{1}{2}||n_0||_{L^2}^2 + Ct,$$

where C is a positive constant. Thus, for μ so large, we have $||v_R||_{L^2(0,T;W^{1,2}(\mathbb{R}^+))} \le C$ and then $v_R \rightharpoonup v$ weakly in $L^2(0,T;W^{1,2}(\mathbb{R}^+))$.

Therefore, classical regularity result of parabolic equation allows us to pass to the limit in the boundary condition to conclude that v solves problem (3.1). To get the uniqueness result for problem (3.1), we suppose that v_1, $v_2 \in X$ are two solutions of (3.1), then $w = v_1 - v_2$, solves (1.1) with $w(0,x) = 0$. Hence, multiply the equation of w by $sgn(w)$; after integrating over $(0,\infty)$, we obtain

$$\int_0^\infty |w|_t(t,x)\mathrm{d}x + \int_0^\infty (d(x)+\mu)|w(t,x)|\mathrm{d}x \le \alpha B_M \int_0^\infty |w(t,x)|\mathrm{d}x,$$

thus,

$$\int_0^\infty |w|_t(t,x)\mathrm{d}x + C_1\int_0^\infty |w(t,x)|\mathrm{d}x \le 0,$$

and again, by the Gronwall lemma, we conclude that $w = 0$. The result follows. □

We can now state the first theorem, which concern the existence, uniqueness, and positivity of the solution to problem (1.2) in a regularized space.

Theorem 3.4

Under assumptions (2.1)–(2.6) and for all positive initial data $n_0 \in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$, there is a unique positive solution n to problem (1.2) that belongs to $\mathcal{C}([0,\infty);L^2(\mathbb{R}^+))\cap L^1((0,T)\times\mathbb{R}^+)\cap L^2(0,T;W^{1,2}(\mathbb{R}^+))$ for all $T>0$.

Proof

We consider the following approximated problem.

$$\begin{cases} n_t^k(t,x) - n_{xx}^k(t,x) + (g(x)n^k(t,x))_x + d(x)n^k(t,x) = 0, \quad t\ge 0, \quad x \ge 0 \\ g(0)n^k(t,0) - n_x^k(t,0) = f\left(\int_0^\infty B(x)n^{k-1}(t,x)\mathrm{d}x\right), \quad \text{and} \quad n^k(0,x) = n_0(x) \\ n^0(t,x) = v(t,x), \end{cases} \tag{3.9}$$

where v is solution to problem (3.1). We can prove that $0 \le n^k \le v$ (because 0 is a subsolution to (1.2)).

Now, multiplying the solution of (3.9) by n^k and integrating over $(0,\infty)$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^\infty |n^k|^2\mathrm{d}x + \int_0^\infty d(x)|n^k(t,x)|^2\mathrm{d}x + \int_0^\infty |(n^k)_x(t,x)|^2\mathrm{d}x &\le \frac{1}{2}f^2\left(\int_0^\infty B(x)n^{k-1}(t,x)\mathrm{d}x\right), \\ &\le \alpha^2\left(\int_0^\infty B(x)v(t,x)\mathrm{d}x\right)^2 + \gamma^2 + \frac{1}{2}\int_0^\infty |g'(x)|v^2(t,x)\mathrm{d}x. \end{aligned}$$

From this, we deduce

$$n^k \rightharpoonup n \ \text{ in } \ L^2((0,T);W^{1,2}(\mathbb{R}^+)),$$

and consequently, n is solution to problem (1.2). The uniqueness is proved by the aforementioned comparison principle. The theorem is established. □

We are now able to prove the main result of this section.

Proof of Theorem 2.3

Consider $n_0 \in L^1(\mathbb{R}^+, e^{-G(x)}\mathrm{d}x)$. By density, we can find a nondecreasing sequence $n_0^k \in L^1(\mathbb{R}^+)\bigcap L^2(\mathbb{R}^+)$ such that $n_0^k \to n_0$ in $L^1(\mathbb{R}^+, e^{-G(x)}\mathrm{d}x)$. We denote by $n^{(k)}(t,x)$ the corresponding solution to problem (1.2). Let $w = (n^{(k)} - n^{(p)})e^{-\mu t}$; it follows that

$$\begin{cases} e^{-G(x)}w_t(t,x) - (e^{-G(x)}w_x(t,x))_x + d(x)e^{-G(x)}w(t,x) = 0 \quad t\ge 0, \quad x\ge 0 \\ g(0)w(t,0) - w_x(t,0) = \left(f\left(\int_0^\infty B(x)n^{(k)}(t,x)\mathrm{d}x\right) - f\left(\int_0^\infty B(x)n^{(p)}(t,x)\mathrm{d}x\right)\right)e^{-\mu t}, \\ w(0,x) = n_0^{(k)}(x) - n_0^{(p)}(x), \end{cases} \tag{3.10}$$

setting the truncated function

$$T_1(w) = \begin{cases} w & \text{if} \quad |w| \le 1 \\ \dfrac{w}{|w|} & \text{if} \quad |w| > 1. \end{cases}$$

and $\Theta(s) = \int_0^s T_1(\sigma)d\sigma$. Multiplying the equation of problem (3.10) by $T_1(w)$ and integrating

$$\frac{d}{dt}\int_0^\infty e^{-G(x)}\Theta(w(t,x))dx + \int_0^\infty e^{-G(x)} w_x(t,x)(T_1(w(t,x)))_x dx \\ + \int_0^\infty e^{-G(x)}(d(x) + g'(x) + \mu)w(t,x)T_1(w(t,x))dx = -w_x(t,0)T_1(w(t,0)),$$

in view of this, we have

$$\frac{d}{dt}\int_0^\infty e^{-G(x)}\Theta(w(t,x))dx \le (f'(\theta(t))\int_0^\infty B(x)w(t,x)dx - w(t,0))T_1(w(t,0)),$$

where $\theta(t)$ is a value between $\int_0^\infty B(x)n^{(k)}(t,x)dx$ and $\int_0^\infty B(x)n^{(p)}(t,x)dx$, which is uniformly bounded. Now, we have by integrating over $(0,t)$,

$$\begin{aligned} \int_0^\infty e^{-G(x)}\Theta(w(t,x))dx &\le C\int_0^t\int_0^\infty |w(s,x)|dxds + \int_0^\infty e^{-G(x)}\Theta(w(0,x))dx, \\ &\le C\int_0^t\int_0^\infty |w(s,x)|dxds \\ &\quad + \int_{\{x\in\mathbb{R}^+,|w(0,x)|>1\}} e^{-G(x)}|w(0,x)|dx + \frac{1}{2}\int_{\{x\in\mathbb{R}^+,|w(0,x)|\le 1\}} e^{-G(x)}|w(0,x)|^2dx. \end{aligned}$$

On the other hand, integrate Equation (1.2), and again, by Gronwall lemma, we obtain

$$\int_0^\infty\int_0^\infty n^{(k)}(t,x)dx \le C.$$

Therefore, using the monotonicity of the sequence $n^{(k)}$, we get existence of $n \in L^1(\mathbb{R}^+ \times \mathbb{R}^+)$ such that $n^{(k)} \longrightarrow n$ as $k \longrightarrow \infty$ in $L^1(\mathbb{R}^+ \times \mathbb{R}^+)$. Henceforth, w is a Cauchy sequence in $L^1(\mathbb{R}^+ \times \mathbb{R}^+)$. Moreover,

$$\int_0^\infty e^{-G(x)}\Theta(w(t,x))dx = \int_{\{x\in\mathbb{R}^+,|w|>1\}} e^{-G(x)}|w|dx + \frac{1}{2}\int_{\{x\in\mathbb{R}^+,|w|\le 1\}} e^{-G(x)}|w|^2dx,$$

and, by Holder inequality, we find

$$\int_{\{x\in\mathbb{R}^+,|w|\le 1\}} e^{-G(x)}|w|dx \le \left(\int_{\{x\in\mathbb{R}^+,|w|\le 1\}} e^{-G(x)}|w|^2dx\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-G(x)}dx\right)^{\frac{1}{2}}.$$

Combining these aforementioned results and the hypothesis on the initial data, we prove that $n^{(k)}$ is a Cauchy sequence $C(\mathbb{R}^+, L^1(\mathbb{R}^+, e^{-G(x)}dx))$. Therefore, it converges to a function $n \in C(\mathbb{R}^+, L^1(\mathbb{R}^+, e^{-G(x)}dx))$. The theorem is proved. □

4. Convergence to a stationary solution

In this section, we prove that under some hypotheses on the growth of f and the initial data, we have the convergence towards a nontrivial stationary solution.

Noticing that the trivial solution is not stable (Section 1) for all positive (nontrivial) initial datum $n_0 \in L^1(\mathbb{R}^+, e^{-G(x)}dx)$, $n(t,x)$ solution to (1.2) is also solution to the problem

$$\begin{cases} n_t(t,x) - n_{xx}(t,x) + (g(x)n(t,x))_x + d(x)n(t,x) = 0 \quad t \ge 0, \quad x \ge 0 \\ g(0)n(t,0) - n_x(t,0) = \tilde{f}\left(\int_0^\infty B(x)n(t,x)dx\right), \quad \text{and} \quad n(0,x) = n_0(x) \in L^1_+(\mathbb{R}^+), \end{cases} \tag{4.1}$$

where $\tilde{f}(x) = f(\max(x, \min(s_0, \int_0^\infty n_0(y)dy))$ is strictly positive and $\tilde{f}(0) \ne 0$.

We consider the following problem, which play a fundamental role in the convergence towards a nontrivial stationary solution,

$$\begin{cases} -u''(x) - g(x)u'(x) + d(x)u(x) = B(x), \quad x \ge 0, \\ u'(0) = 0, \\ u \in W^{1,\infty}(\mathbb{R}^+). \end{cases} \tag{4.2}$$

The next lemma states the existence result and boundedness of the solution to problem (4.2).

Lemma 4.1
Under the hypotheses (2.1) and (2.5), problem (4.2) has a unique positive solution u. In addition, we have

$$0 < u(x) \le \frac{B_M}{d_m}, \quad \forall \quad x \ge 0. \tag{4.3}$$

Proof
In order to prove the existence of solution, we argue by approximation; namely, we consider the case of bounded domain $[0, R]$, and then, we pass to the limit in R. So let us define the following approximated problem

$$\begin{cases} -u_R''(x) - g(x)u_R'(x) + d(x)u_R(x) = B(x), \quad x \in (0, R), \\ u_R'(0) = 0, \\ u_R(R) = 0. \end{cases} \tag{4.4}$$

For the existence and uniqueness of the solution to the current problem we use, for instance Lax–Milgram theorem. The positivity of this solution is proved by multiplying the corresponding equation by the negative part of u_R.

A classical maximum principle allows us to prove that the solution u_R is strictly positive in $[0, R)$. Now, remarking that $\overline{u}_R = \frac{B_M}{d_m}$ is a supersolution of problem (4.4). Consequently, by classical comparison principle we prove the following inequalities

$$0 < u_R(x) \le \frac{B_M}{d_m}.$$

In addition, notice that (again by principle comparison) the sequence u_R is nondecreasing with respect to R; therefore, u_R (seen as 0 outside $(0, R)$) converges to a strictly positive function u, solution of problem (4.2) in $W^{1,2}(\mathbb{R}^+)$. Now, we suppose that v is another solution of problem (4.2) then v is a supersolution of problem (4.4) in $(0, R)$ $(v(R) > 0)$; thus, $u_R \le v$ and by passing to the limit, we can prove that u is the minimal solution of problem (4.2).

Moreover, rewriting Equation (4.2) as

$$-\left(u'e^{G(x)}\right)' + d(x)e^{G(x)}u(x) = B(x)e^{G(x)}, \tag{4.5}$$

we obtain

$$u'(x) = e^{-G(x)}\int_0^x \frac{d(s)}{g(s)}g(s)e^{G(s)}u(s)\mathrm{d}s - e^{-G(x)}\int_0^x \frac{B(s)}{g(s)}g(s)e^{G(s)}\mathrm{d}s,$$

in view of this, we deduce

$$|u'(x)| \le \left(\frac{d_M}{g_m} + \frac{B_M}{g_m}\right)(1 - e^{-G(x)}).$$

The existence of the minimal solution in $W^{1,\infty}(\mathbb{R}^+)$ is proved. Concerning the uniqueness, we set $w = v - u \ge 0$; thus, w satisfies the following problem:

$$\begin{cases} -\left(w'e^{G(x)}\right)' + d(x)e^{G(x)}w(x) = 0, \quad x \ge 0, \\ w'(0) = 0, \\ w \in W^{1,\infty}(\mathbb{R}^+). \end{cases} \tag{4.6}$$

Multiplying the equation of problem (4.6) by a test function $\phi(x) = e^{-\delta G(x)}$, with $\delta > 1$, is a constant to be chosen later, and integrating by part, we obtain

$$-w(0)\phi'(0) + \int_0^\infty \left(d(x) + \delta g'(x) - \delta(\delta - 1)g^2(x)\right)\phi(x)e^{G(x)}w(x)\mathrm{d}x = 0.$$

Now choosing δ such that

$$d(x) + \delta g'(x) - \delta(\delta - 1)g^2(x) \ge \delta_1 > 0,$$

and because $\phi'(0) < 0$, we have $w(x) = 0$. Hence, the lemma is proved. □

In order to analyze the asymptotic behavior of the solution to problem (1.2), we first need to show the existence, uniqueness, and positivity of the stationary solution to problem (2.8). Indeed, we have the following lemma.

Lemma 4.2
Problem (2.8) has a positive solution N with $N \in W^{2,2}(\mathbb{R}^+)$ if and only if there exists a positive constant Γ such that

$$f(\Gamma) = \frac{1}{u(0)}\Gamma, \tag{4.7}$$

where u is solution to problem (4.2). Moreover, $\Gamma = \int_0^\infty B(x)N(x)\mathrm{d}x$.

Proof
Multiplying the equation of problem (2.8) by u, solution to problem (4.2) and integrating over $(0, \infty)$, we have

$$\int_0^\infty B(x)N(x)\mathrm{d}x = f\left(\int_0^\infty B(x)N(x)\mathrm{d}x\right)u(0). \tag{4.8}$$

Now, let N be solution of the following problem

$$\begin{cases} -N''(x) + (gN)'(x) + d(x)N(x) = 0, \quad x \geq 0, \\ g(0)N(0) - N'(0) = f(\Gamma), \\ \displaystyle\int_0^\infty N(x)\mathrm{d}x < \infty, \quad \text{and} \quad N \geq 0. \end{cases} \tag{4.9}$$

Multiplying Equation (4.9) by u and integrating, we find

$$\int_0^\infty B(x)N(x)\mathrm{d}x = f(\Gamma)u(0). \tag{4.10}$$

From hypothesis (4.7), we have $\Gamma = \int_0^\infty B(x)N(x)\mathrm{d}x$. □

We consider now the following auxiliary problem, setting $\tilde{u}(x) := V(x) + CN(x)$,

$$\begin{cases} U_t(t,x) - U_{xx}(t,x) + (g(x)U(t,x))_x + d(x)U(t,x) = 0, \quad t \geq 0, \quad x \geq 0, \\ g(0)U(t,0) - U_x(t,0) = \tilde{f}\left(\displaystyle\int_0^\infty B(x)U(t,x)\mathrm{d}x\right), \quad \text{and} \quad U(0,x) = \tilde{u}, \end{cases} \tag{4.11}$$

with $C \geq 1$ to be chosen later. N is the solution of stationary problem (2.8), and V satisfies

$$\begin{cases} -V''(x) + (gV)'(x) + d(x)V(x) = 0, \quad x \geq 0 \\ g(0)V(0) - V'(0) = \alpha \displaystyle\int_0^\infty B(x)V(x)\mathrm{d}x + \gamma \quad \text{and} \quad \int_0^\infty V(x)\mathrm{d}x < \infty. \end{cases} \tag{4.12}$$

First of all, we begin by studying the aforementioned stationary problem (4.12). Indeed, we have the following lemma, which has the same proof as that of Lemma 4.2. However, we give a somehow different proof.

Lemma 4.3
Problem (4.12) has a unique positive solution if and only if $\alpha < \frac{1}{u(0)}$, and $\gamma > 0$.

Proof
By multiplying the equation of problem (4.12) by u, solution of problem (4.2) and integrating, we obtain

$$(1 - \alpha u(0))\int_0^\infty B(x)V(x)\mathrm{d}x = \gamma u(0). \tag{4.13}$$

On the other hand, we define the operator A from $L^1(\mathbb{R}^+, B(x)\mathrm{d}x)$ to $L^1(\mathbb{R}^+, B(x)\mathrm{d}x)$ such that for each function m, we set $A(m) = V$, with V satisfies

$$\begin{cases} -V''(x) + (gV)'(x) + d(x)V(x) = 0, \quad x \geq 0, \\ g(0)V(0) - V'(0) = \dfrac{1}{u(0)}\displaystyle\int_0^\infty B(x)m(x)\mathrm{d}x + \gamma. \end{cases} \tag{4.14}$$

Thus, by applying the Banach Picard fixed-point theorem, we can prove the existence and uniqueness of the solution to problem (4.12). □

Next, we will prove that the solution n of problem (1.2) is bounded.

Lemma 4.4
Assume that (2.3), (2.4) and $n_0(x) \leq CN(x)$. Let $\bar{U}(t,x)$ and $\underline{U}(t,x)$ be the solutions of problem (4.11), with initial conditions $\tilde{u}$ and 0, respectively. Suppose that $\alpha < \frac{1}{u(0)}$ and $\gamma > 0$. Then, we have the following inequalities.

$$0 \leq \underline{U}(t,x) \leq n(t,x) \leq \bar{U}(t,x) \leq \tilde{u}(x), \forall t \geq 0$$

Furthermore, $\bar{U}(t,x)$ resp($\underline{U}(t,x)$) is nonincreasing in t (is nondecreasing in t).

Proof
First of all, remarking that assumptions (2.3) and (2.4) give the existence of $\Gamma > 0$ solution to $f(\Gamma) = \frac{1}{u(0)}\Gamma$, put $w(t,x) = \bar{U}(t,x) - \tilde{u}(x)$; it follows that

$$\begin{cases} w_t(t,x) - w_{xx}(t,x) + (g(x)w(t,x))_x + d(x)w(t,x) = 0, \quad t \geq 0, \quad x \geq 0 \\ g(0)w(t,0) - w_x(t,0) \leq \alpha \displaystyle\int_0^\infty B(x)w(t,x)\mathrm{d}x \\ w(0,x) = 0. \end{cases} \tag{4.15}$$

As a simple consequence of comparison principle, we deduce that $w \leq 0$. Similarly, by setting $w(t,x) = n(t,x) - \bar{U}(t,x)$, we have

$$\begin{cases} w_t(t,x) - w_{xx}(t,x) + (g(x)w(t,x))_x + d(x)w(t,x) = 0, \quad t \geq 0, \quad x \geq 0 \\ g(0)w(t,0) - w_x(t,0) = \tilde{f}\left(\displaystyle\int_0^\infty B(x)n(t,x)\mathrm{d}x\right) - \tilde{f}\left(\displaystyle\int_0^\infty B(x)\bar{U}(t,x)\mathrm{d}x\right) = \tilde{f}'(\theta(t))\displaystyle\int_0^\infty B(x)w(t,x)\mathrm{d}x, \\ w(0,x) \leq 0, \end{cases} \tag{4.16}$$

where $\theta(t)$ is a value between $\int_0^\infty B(x)n(t,x)\mathrm{d}x$ and $\int_0^\infty B(x)\bar{U}(t,x)\mathrm{d}x$. By remarking that $\theta(t)$ is uniformly bounded ($\bar{U} \leq \tilde{u}$) and, so, by using the fact that f is nondecreasing, the result is

$$\begin{cases} w_t(t,x) - w_{xx}(t,x) + (g(x)w(t,x))_x + d(x)w(t,x) = 0, \quad t \geq 0, \quad x \geq 0 \\ g(0)w(t,0) - w_x(t,0) \leq c\displaystyle\int_0^\infty B(x)w^+(t,x)\mathrm{d}x, \\ w(0,x) \leq 0, \end{cases} \tag{4.17}$$

where the positive constant c satisfying $f'(\theta(t)) \leq c$. Again, by comparison principle, we obtain $w \leq 0$. Concerning the monotonicity of $\bar{U}$, we put $w = \bar{U}(t+t_1,x) - \bar{U}(t,x)$ for all t_1 positive; it follows that w satisfies problem (4.16) with $w(0,x) = U(t_1,x) - \tilde{u}(x) \leq 0$, and again by comparison principle, we conclude the monotonicity of $\bar{U}(t,x)$. □

Now, we are able to prove Theorem 2.3.

Proof of Theorem 2.3
First, we know that $\bar{U}(t,x)$ converges to a limit, so setting $\bar{U}(t,x) \to \bar{U}_s(x)$ as $t \to \infty$. Consider the boundary-value problem

$$\begin{cases} -v''(x) + (gv)'(x) + d(x)v(x) = 0, \quad x \geq 0 \\ g(0)v(0) - v'(0) = \tilde{f}\left(\displaystyle\int_0^\infty B(y)\bar{U}_s(y)\mathrm{d}y\right). \end{cases} \tag{4.18}$$

Now, setting $w(t,x) = \bar{U}(t,x) - v(x)$, and $W(x) = \tilde{u}(x) - v(x)$, we claim that $W \geq 0$. Indeed, W satisfies

$$\begin{cases} -W''(x) + (gW)'(x) + d(x)W(x) \geq 0, \quad x \geq 0, \\ g(0)W(0) - W'(0) \geq 0, \end{cases} \tag{4.19}$$

then by a simple comparison principle, we conclude the claim. Now, following the same arguments as aforementioned and the fact that f is nondecreasing, we can prove that $w(t,x) \geq 0$.

Multiplying the solution of problem (4.16) by w and integrating over $(t,t+1) \times (0,\infty)$, we have

$$\int_t^{t+1}\int_0^\infty \frac{\partial}{\partial s}w(s,x)w(s,x)\mathrm{d}x + \int_t^{t+1}\int_0^\infty \left(d(x) + \frac{1}{2}g'(x)\right)|w(s,x)|^2\mathrm{d}x\mathrm{d}s + \int_t^{t+1}\int_0^\infty |w_x(s,x)|^2\mathrm{d}x\mathrm{d}s$$
$$\leq \frac{1}{2}\int_t^{t+1}\left(\tilde{f}\left(\int_0^\infty B(x)\bar{U}(s,x)\mathrm{d}x\right) - \tilde{f}\left(\int_0^\infty B(x)\bar{U}_s(x)\mathrm{d}x\right)\right)^2\mathrm{d}s.$$

From this and using the fact that $w(t,.)$ is nonincreasing in t, we deduce

$$\frac{1}{2}\int_0^\infty \left(|w(t+1,x)|^2-|w(t,x)|^2\right)\mathrm{d}x+\int_0^\infty\left(d(x)+\frac{1}{2}g'(x)\right)|w(t+1,x)|^2\mathrm{d}x$$
$$\leq \frac{1}{2}\left(\tilde f\left(\int_0^\infty B(x)\bar U(t,x)\mathrm{d}x\right)-\tilde f\left(\int_0^\infty B(x)\bar U_s(x)\mathrm{d}x\right)\right)^2.$$

In addition, $\bar U(t,x)\to \bar U_s(x)$ as $t\to\infty$ in $L^1(\mathbb{R}^+)$, and the continuity of $\tilde f$ leads to

$$\tilde f\left(\int_0^\infty B(x)\bar U(t,x)\mathrm{d}x\right)\to \tilde f\left(\int_0^\infty B(x)\bar U_s(x)\mathrm{d}x\right)$$

as $t\to\infty$.

By passing to the limit as $t\longrightarrow\infty$ and combining these aforementioned results, we obtain $w(t,.)\to 0$ in $L^2(\mathbb{R}^+)$; thus, $v=\bar U_s$, and consequently, $\bar U_s = N$. In order to prove that $\bar U_s$ is the maximal solution of the stationary problem (2.8), we set $w=\bar U(t,x)-N(x)$, with N is any solution of problem (2.8). Applying Lemma 3.1, we can prove that $\bar U(t,x)\geq N(x)$. Now, by passing to the limit, we obtain the desired result. Employing an argument similar to that aforementioned, we can prove that $\underline{U}(t,x)\to \underline{N}(x)$ as $t\to\infty$. Hence, we conclude the proof of the theorem. □

The following corollary concerns the case where problem (2.8) has multiple nontrivial stationary solutions.

Corollary 4.5
Suppose that problem (2.8) admits n nontrivial steady states noted $N_i(x)$ for $1\leq i\leq n$. Assume that the initial condition n_0 is a supersolution, resp. (subsolution) of problem (2.8), and satisfies either $N_i(x)\lneqq n_0(x)\lneqq N_{i+1}(x)$ for $1\leq i\leq n-1$ or $N_n(x)\leq n_0(x)\leq CN_n(x)$, resp $(n_0(x)\leq N_1(x))$. Then, the solution of problem (1.2) converges to $N_i(x)$ resp$(N_{i+1}(x))$.

Proof
Because n_0 is supersolution of problem (2.8), then by comparison principle, we have $0<n(t,x)\leq n_0(x)$ and $n(t,x)$ is nonincreasing with respect to t. Consequently, it admits a limit. By following the proof of Theorem 2.3, we get the result. □

5. Blow up/extinction

In this section, we are concerned with a blow up (resp. extinction) of the solution to problem (1.2) under some assumptions on the growth of f.

Theorem 5.1
If there exist M_1 and M_2 such that

$$\alpha x+\gamma\leq f(x)\leq M_1x+M_2, \tag{5.1}$$

with $\alpha\geq\frac{1}{u(0)}$ and $\gamma>0$ or $\alpha>\frac{1}{u(0)}$ and $\gamma=0$, the solution n to problem (1.2) satisfies

$$\lim_{t\to\infty}\int_0^\infty n(t,x)\mathrm{d}x=+\infty. \tag{5.2}$$

Proof
Multiplying the Equation (1.2) by u and integrating, we have

$$\begin{aligned}\int_0^\infty n_t(t,x)u(x)\mathrm{d}x&=\int_0^\infty (n_{xx}-(gn)_x-d(x)n)u(x)\mathrm{d}x\\&=u(0)(n(t,0)-n_x(t,0))+\int_0^\infty n(t,x)(u''(x)+g(x)u'(x)-d(x)u(x))\mathrm{d}x,\\&=u(0)f\left(\int_0^\infty B(x)n(t,x)\mathrm{d}x\right)-\int_0^\infty B(x)n(t,x)\mathrm{d}x,\\&\geq u(0)\left(\left(\alpha-\frac{1}{u(0)}\right)\int_0^\infty B(x)n(t,x)\mathrm{d}x\right)+\gamma).\end{aligned}$$

Hence, using Lemma 4.1, we obtain the result. □

We have directly the following extinction result.

Theorem 5.2
If there exist M_1 such that

$$f(x)\leq M_1x, \tag{5.3}$$

with $M_1 < \frac{1}{u(0)}$, the solution n to problem (1.2) satisfies

$$\lim_{t\to\infty} \int_0^\infty n(t,x)\mathrm{d}x = 0. \tag{5.4}$$

6. Numerical simulation

We present in figure 1–3 some examples to illustrate the result of the previous section, namely the steady state solution and the convergencetowards this one. For almost all these examples, we suppose that the recruitment term $f(x) = \sqrt{x}$ and the transport term $g(x) = 1$, which satisfy the assumptions of Theorem 2.3. The initial conditions are assumed to be equal to 1. We compute numerical solution to the system (1.2) by using finite difference with Dirichlet condition at the right boundary.

The birth and death terms are given by $B(x) = 5e^{-x} + 10(1 - e^{-x})$ and $d(x) = 2e^{-x} + 4(1 - e^{-x})$.

The next figure illustrate Theorem 2.3, namely the convergence of the solution to problem (1.2) towards the nontrivial solution of problem (2.8).

7. Conclusion

We have proved the existence and uniqueness, and study the dynamics of solutions to the nonlinear partial differential Equation (1.2).

We notice that assumptions (2.3)-(2.4) on the growth of the nonlinear birth rate function f depends on the supremum and infimum of the birth rate B and death rate d. In the general case, the birth rate vanishes when the age is too small (immature) or too large (do not give birth when the individual is too old). In the same way, the biological age of individuals is 'limited', and so, the death rate can go to infinity when the age is too large. In Figure 4, we conjecture that we can obtain the same result, even if the birth term vanishes and the death term goes to infinity, but not too fast (behaves as x in infinity for example). Therefore, it will be interesting to find assumptions on f, which extend the convergence result when the birth rate and death rate are non-negative and not necessarily bounded.

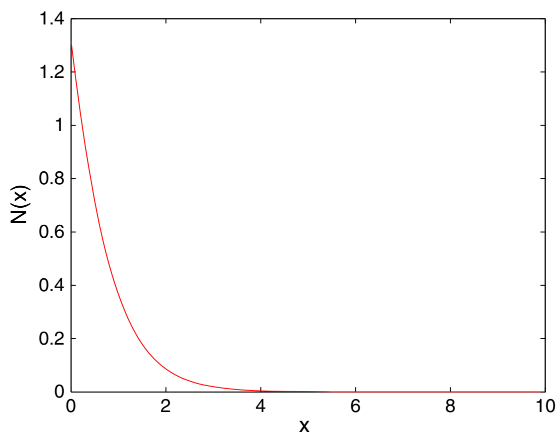

Figure 1. The numerical simulation of the steady state solution to problem (2.8).

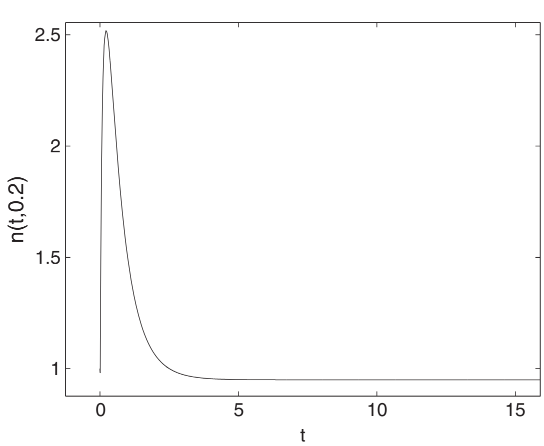

Figure 2. $x = 0.2$. The numerical simulation of the solution to problem (1.2). Here, we fix an age $x = 0.2$, we compute $n(t,x)$, and we observe the convergence in long-time asymptotic.

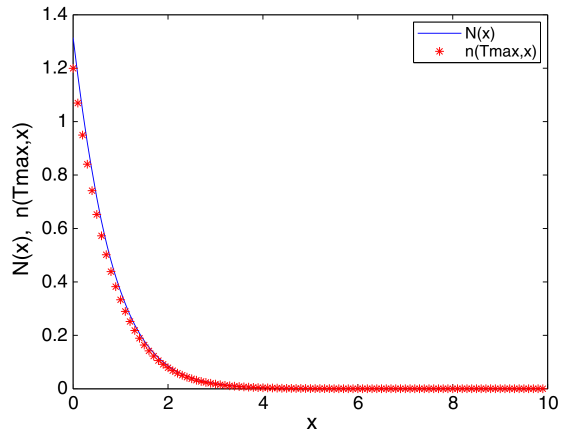

Figure 3. $Tmax = 500$. Convergence of the solution to problem (1.2) towards the nontrivial stationary solution of problem (2.8).

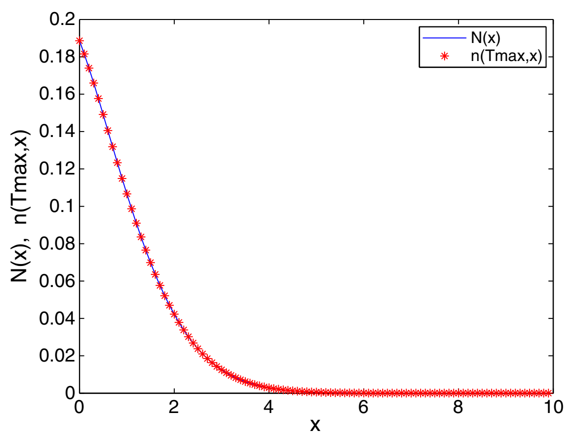

Figure 4. $Tmax = 500$, $B(x) = xe^{-x}$, and $d(x) = x$. Convergence of the solution to problem (1.2) towards the nontrivial stationary solution of problem (2.8).

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