# A singular asymptotic behavior of a transport equation

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Received *****; accepted after revision +++++

Presented by

**Abstract**

We consider a simple conservative transport equation where the speed is strictly decreasing. The monotonicity property of the speed rate leads to a singular asymptotic behavior and the concentration of the mass of the solution at a point. Thus, a model which contains a transport structure with monotone decay of the speed rate can be reduced by using the result of convergence to a Dirac mass. It is useful in the case where we have to simulate numerous nonlinear PDEs containing such a structure (as in [7]). Indeed, the concentration of the mass makes the variable in which the mass concentrate useless and thus we lose a dimension. The gain in time calculus is important when the number of equations is large.

*To cite this article: P. Michel, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

**Résumé**

**Etats asymptotiques singuliers pour certaines équations de transports.** On s'intéresse à des équations de transport dans lesquelles la vitesse de transport est strictement décroissante. Cette propriété de monotonie entraîne la concentration de la masse de la solution en un point et, par conséquent, un comportement asymptotique singulier. D'autre part, il est possible de réduire un modèle contenant une structure de transport avec la condition de monotonie (dans une direction) de la vitesse de transport. En utilisant la convergence de la solution en une masse de Dirac suivant la direction du transport, on rend cette direction inutile et on peut éliminer une variable d'espace. Cela réduit le coût numérique lorsqu'il faut simuler un certains nombres d'EDP contenant une telle structure (voir [7]). Le gain est d'autant plus important que le nombre d'équations est grand. *Pour citer cet article : P.Michel, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

## Version française abrégée

Ce papier est consacré à l'étude du comportement asymptotique de solutions d'équations de transport dans lesquelles la vitesse de transport est décroissante. Plus précisement, pour $n$ solution de

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$$\frac{\partial}{\partial t} n + \nabla(Vn) = 0, \tag{1}$$

avec $V$ régulière et strictement monotone,

$$\langle V(t,x) - V(t,y), x - y\rangle < -\epsilon \parallel x - y \parallel^2, \quad \forall t \geq 0, \quad \forall x \neq y \in \mathbb{R}^m, \tag{2}$$

avec $\epsilon > 0$. Toutes les solutions de (1) ont le mme type de comportement asymptotique décrit par les solutions "Dirac" le long des trajectoires

$$\frac{\partial}{\partial t} z = V(t,z). \tag{3}$$

**Proposition 0.1** *Supposons que (2) soit vérifié et soit $n$ solution de (1) avec $\int n(0,x)dx = 1$ alors la distance de Wasserstein entre $ndx$ et $\delta_{x=z(t)}$ tends vers* 0

$$\lim_{t\to\infty} d_{W_2}(n(t,x)dx, \delta_{x=z(t)}) = 0. \tag{4}$$

Cette proposition peut être étendue à des équations de renouvellement contenant une telle structure de transport comme par exemple

$$\begin{cases} \frac{\partial}{\partial t} n + \frac{\partial}{\partial a} n + \frac{\partial}{\partial x}(Vn) = 0, \\ \\ n(t,0,x) = 2n(t,1,x). \end{cases} \tag{5}$$

Dans ce cas, on a

**Proposition 0.2** *Si (2) est satisfaite et $n$ solution de (5) avec $\int n(0,a,x)2^a dx = 1$ alors on a*

$$\int n(t,a,x)2^{a-t} dadx = 1, \quad \forall t \geq 0,$$

$$\frac{d}{dt} \int n(t,a,x)2^{a-t} \mid x - z(t,a) \mid^2 dadx = 2 \int n2^{a-t} \langle V(t,x) - V(t,z), x - z\rangle dadx < 0, \tag{6}$$

*avec $z$ solution d'une équation de transport avec condition aux bords périodique.*

Cela implique qu'il y a concentration de la masse autour de $z$ et par conséquent, en temps long, on peut s'affranchir de la variable $x$. On peut considérer que $n \sim \delta_{x=z(t)} m(t,a)$ avec $m$ solution de l'équation de renouvellement

$$\begin{cases} \frac{\partial}{\partial t} m + \frac{\partial}{\partial a} m = 0, \\ \\ m(t,0) = 2m(t,1). \end{cases} \tag{7}$$

La perte d'une variable d'espace rend le système plus facile à implémenter et le gain en temps de calcul est appréciable lorsqu'il faut simuler un grand nombre d'EDP non linéaires couplées (voir [7]).

## 1. Introduction

In this paper, we are interested in the asymptotic behavior of transport equation when the transport speed is monotonic. More precisely, we consider $n$ solution to the conservative transport equation (see [1,4,5,6])

$$\frac{\partial}{\partial t} n + \nabla (V n) = 0, \tag{8}$$

where $V \in C^1([0,\infty] \times \mathbb{R}^m, \mathbb{R}^m)$ satisfies the monotonicity condition

$$\langle V(t,x) - V(t,y), x - y \rangle < -\epsilon \parallel x - y \parallel^2, \quad \forall t \geq 0, \quad \forall x \neq y \in \mathbb{R}^m, \tag{9}$$

where $\epsilon$ is strictly positive. We show that the limit of $n(t,.)$ when $t \to \infty$ is a Dirac measure supported by the trajectories of the characteristic eq. (independently of its initial data)

$$\frac{\partial}{\partial t} z = V(t,z). \tag{10}$$

To evaluate how $n(t,.)$ converges to this singular limit, we introduce the Wasserstein distance (see [15]) between measures

$$d_{W_p}(d\mu, d\nu) := \Big( \inf_{\eta \in \Gamma(\mu,\nu)} \Big\{ \int \int_{]-\infty,\infty[^2} \mid x - y \mid^p d\eta(x,y) \Big\} \Big)^{1/p}, \tag{11}$$

where $\Gamma(\mu,\nu)$ denotes the collection of all measures on $]-\infty,\infty[^2$ with marginals $\mu$ and $\nu$. We notice that

$$d_{W_p}(d\mu, d\nu) \leq \Big( \int \int_{]-\infty,\infty[^2} \mid x - y \mid^p d\mu(x) d\nu(y) \Big)^{1/p}. \tag{12}$$

Then we extend the convergence result to a singular limit, to a renewal equation of McKendrick-VonFoerster type. In this context, we use this result to reduce the dimension in a renewal model [7].

## 2. Transport equation : main result

We present here the main result of convergence of $n$ to a Dirac mass.

**Proposition 2.1** *Assume (9) and $n$ solution to (8) with $\int n(0,x)dx = 1$, then the Wasserstein distance between $ndx$ and $\delta_{x=z(t)}$ vanishes*

$$\lim_{t \to \infty} d_{W_2}(n(t,x)dx, \delta_{x=z(t)}) = 0. \tag{13}$$

**Proof.**

Under the assumption (9), we find directly that

$$\int n(t,x)dx = \int n(0,x)dx = 1, \tag{14}$$

$$\frac{d}{dt} \int n(t,x) \parallel x - z(t) \parallel^2 dx = 2 \int n(t,x) \langle V(t,x) - V(t,z), x - z \rangle dx < 0. \tag{15}$$

Since $V$ is strictly decreasing, there exists $\epsilon > 0$ such that,

$$\langle V(t,x) - V(t,z), x - z \rangle \leq -\epsilon \parallel x - z(t) \parallel^2 .$$

Therefore, we have

$$\frac{d}{dt} \int n(t,x) \parallel x - z(t) \parallel^2 dx \leq -2\epsilon \int n(t,x) \parallel x - z(t) \parallel^2 dx, \tag{16}$$

and using Gronwall inequality, we have exponential convergence to 0,

$$\lim_{t\to\infty}\int n(t,x) \parallel x - z(t) \parallel^2 dx = 0.$$

Moreover, let $w$ an another solution to (8) (with different initial data than $z$), then we have that

$$\frac{d}{dt} \parallel w(t) - z(t) \parallel^2 = 2\langle V(t,w) - V(t,z), w(t) - z(t)\rangle < 0. \tag{17}$$

Using (12) for $p = 2$ and (16) we prove that the Wasserstein distance between $ndx$ and $\delta_{x=z(t)}$ converges to 0 when $t \to \infty$. $\square$

We notice that, in dimension one ($m = 1$), we easily see the convergence of $n$ to a Dirac mass. Indeed, in this case, we have directly that

$$\frac{d}{dt}\int n(t,x)(x - z(t))_+^2 dx = 2\int n1_{x\geq z(t)}(V(t,x) - V(t,z(t)), x - z(t))dx < 0, \tag{18}$$

and thus the support of $n(t,.)$ must decrease. More precisely, under the assumption on the support of $n(0,.)$

$$Supp\ \ n(0,.) \subset [x_0, x_1], \tag{19}$$

we have the following Lemma.
**Lemma 2.1** *Assume (19) and let* $w$ *and* $z$ *solution to (10) with the initial condition* $w(0) = x_0$ *and* $z(0) = x_1$*. Then, under the monotonicity condition (9), we find that the support of* $n(t,.)$ *belongs to the interval* $[w(t), z(t)]$ *which length converges to* 0*,*

$$Supp\ \ n(t,.) \subset [w(t), z(t)], \tag{20}$$

*and* $\lim_{t\to\infty}(w(t) - z(t))^2 = 0$.

## 3. Application to renewal equations

We apply this result to a renewal equation (see [3,7,8,14,12])

$$\begin{cases} \frac{\partial}{\partial t}n + \frac{\partial}{\partial a}n + \frac{\partial}{\partial x}(Vn) = 0, \\ \\ n(t,0,x) = 2n(t,1,x). \end{cases} \tag{21}$$

We notice that the monotone decay assumption (9) is satisfied in [7] and the main equation (21) is similar to the Vlasov equation with friction [9,10]. In order to extend the result of Proposition 2.1, we define $z$ solution to the transport equation

$$\frac{\partial}{\partial t}z + \frac{\partial}{\partial a}z = V(t,z), \tag{22}$$

with periodic boundary condition

$$z(t,0) = z(t,1). \tag{23}$$

Here, two difficulties appear. The first one is that $n$ is no more conservative and according to ideas developed in [12,13,16] we have to use $\psi$ solution to the backward equation ($n\psi$ is conservative)

$$\begin{cases} \frac{\partial}{\partial t}\psi + \frac{\partial}{\partial a}\psi + V\frac{\partial}{\partial x}\psi = 0, \\ \\ \psi(t,0) = \frac{1}{2}\psi(t,1). \end{cases} \tag{24}$$

The second difficulty is that we can only conclude that $n\psi$ concentrate its mass in $x = z(t,a)$ and $n \sim m(t,a)\delta_{x=z(t,a)}$, thus we have to study $m(t,a)$ separately.

**Proposition 3.1** *Assume (9) and $n$ solution to (21) where $\int n(0,a,x)2^a dx = 1$ then we have*

$$\int n(t,a,x)2^{a-t}dadx = 1, \quad \forall t \geq 0,$$

$$\frac{d}{dt}\int n(t,a,x)2^{a-t} \parallel x - z(t,a) \parallel_2^2 dadx = 2\int n2^{a-t}\langle V(t,x) - V(t,z), x - z\rangle dadx < 0, \tag{25}$$

*and*

$$\begin{cases} \frac{\partial}{\partial t}\int ndx + \frac{\partial}{\partial a}\int ndx = 0, \\ \\ \int n(t,0,x)dx = 2\int n(t,1,x)dx. \end{cases} \tag{26}$$

**Proof.** It suffices to notice that $\psi(t,a) = 2^{a-t}$ is solution to (24). Indeed, in this case, we have

$$\begin{cases} \frac{\partial}{\partial t}n\psi + \frac{\partial}{\partial a}n\psi + \frac{\partial}{\partial x}Vn\psi = 0, \\ \\ n(t,0,x)\psi(t,0,x) = n(t,1,x)\psi(t,1,x), \end{cases} \tag{27}$$

and using (22) and (23), we find

$$\begin{cases} \frac{\partial}{\partial t}n\psi(x-z)^2 + \frac{\partial}{\partial a}n\psi(x-z)^2 + \frac{\partial}{\partial x}Vn(x-z)^2\psi = 2n(V(t,x) - V(t,z), x - z) \\ \\ n(t,0,x)\psi(t,0,x)(x - z(t,0))^2 = n(t,1,x)\psi(t,1)(x - z(t,1))^2. \end{cases} \tag{28}$$

Then, integrating (27) with respect to $a$ and $x$, we find that $\int n(t,a,x)\psi dadx$ is a constant. Now, integrating (28) with respect to $a$ and $x$, we find (25) and we conclude to the convergence of $n$ to a Dirac mass. □

Thus, we can reduce the PDE (21) to two equations (22) and (26) with less variables. Indeed, the numerical simulation of (21) needs to compute $P^2$ discretization points, while the numerical simulation of (22) and (26) need at most $2P$ points. An example of application is given by [7]. There a system of $N$ renewal PDEs (similar to (21)) is considered and the gain due to the reduction of the model in $2N$ equations with less variables is important. This system modelizes the growth of follicles structured in age and maturity with the maturity speed rate which satisfies the condition (9). The simplified model we obtain using the Proposition (3.1) is easier to compute numerically and can be compared to the classical model of Lacker [11].

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