Mathematical Models and Methods in Applied Sciences
© World Scientific Publishing Company

Large population and size scale limit of a stochastic particle model

Philippe MICHEL

Université de Lyon, École Centrale de Lyon, CNRS UMR 5208, Institut Camille Jordan, 36 avenue Guy de Collonge, F-69134 Ecully Cedex, France. *

philippe.michel@ec-lyon.fr

Received (Day Month Year)
Revised (Day Month Year)
Communicated by (xxxxxxxxxx)

The aim of this work is to study a stochastic individual-based model, structured with respect to age (progression within the cell cycle) and space (radial distance from the oocyte). We prove the existence of solutions and the convergence in large population and size scale limit to a solution of a partial differential equation.

Keywords: stochastic processes, partial differential equations, convergence.

AMS Subject Classification: 35L60, 35L65, 60G55

1. Introduction

The development of ovarian follicles is a unique instance of a morphogenesis process resulting from the interactions between somatic cells (granulosa cells) and germ cell (oocyte). In mammals, the initiation of follicular development from the pool of resting follicles is characterized by an increase in the oocyte size concomitant with the surrounding granulosa cells proliferating (see 19). In 5, the authors have introduced a multi-scale stochastic model, of the primordial follicle development, which take in accounts the molecular dialogue existing between the oocyte and granulosa cells. The population of granulosa (small cells) of diameter ϵ proliferate around the oocyte (large cell) of radius r_O (see fig. 1). There is a dialog (depending on the distance) between the large cell and the small cells (see fig. 1) which links growth and proliferation. Therefore, the model takes in account the location of small cells in space and their age in the cell cycle. The age of a cell is simply a positive real number and $Age = \mathbb{R}_+$.

In this model, the locations of small cells are given by their spherical coordinates (r, θ, ϕ) where $r \geq r_O(t)$ (the radius of the oocyte at time t). Therefore, there is a difficulty which appears since the space depends on time (and is probabilistic since

*This work was funded by INRIA - MYCENAE project team

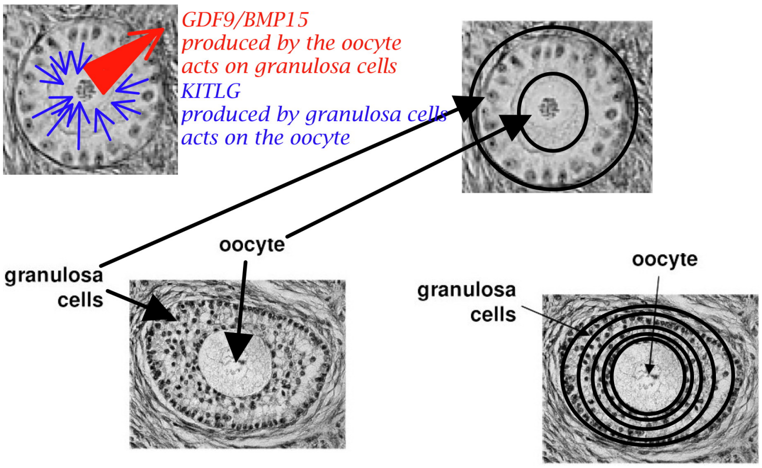

Fig. 1. Development of preantral follicles. Each follicle is comprised of a germ cell (the oocyte) and granulosa cells. The oocyte produces GDF9/BMP15 which makes granulosa cells proliferate around the oocyte and granulosa cells (small ones) produce KITLG that makes the oocyte grows. Therefore, the development of each follicle (in its basal follicular development), is coordinated by tight interactions existing between the oocytes and their surrounding granulosa cells.

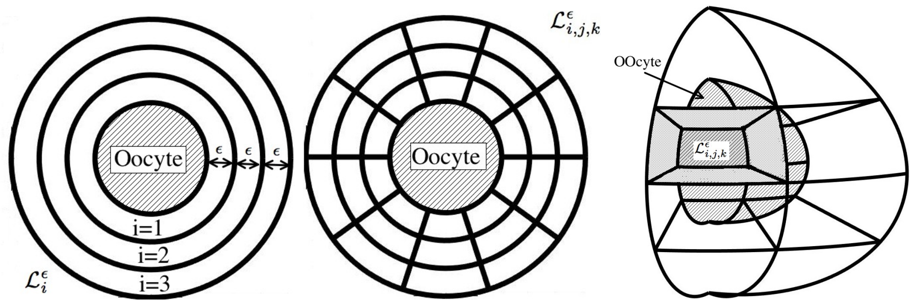

Fig. 2. Space layer decomposition. The physical space surround the ball (oocyte) in its center. We subdivide this space in layers $(\mathcal{L}_i^\epsilon)_i$ and mesh these layers uniformly $(\mathcal{L}_{i,j,k}^\epsilon)_{i,j,k}$ of volumes $(Vol_{i,j,k}^\epsilon)_{i,j,k}$. The figure at the center represents a two dimension subdivision and the right one is a tree dimension subdivision.

$(r_O(t))_t$ is a stochastic process). We choose an equivalent formulation of the small cells evolution model in space where the location of small cells do not depend of the radius of the oocyte, i.e., location in the simplified model (in spherical coordinate) (r, θ, ϕ) corresponds to $(r - 1 + r_O(t), \theta, \phi)$ in space. Indeed, with this formulation space remains independent of time and deterministic and we keep the informations of location and distance from small cells to the surface of the large cell. More precisely, it corresponds to have the location of the small cells in

$$Space = \mathbb{R}^3/B(0,1), \text{ where } B(0,1) = \{(r,\theta,\phi) \in Space \ : \ r < 1\}.$$

Now, the information of the distance to the surface of the oocyte of a small cell is given by the radius distance, therefore we discretize space in layer of thickness equal to the diameter of a small cell. Let $\epsilon > 0$ the diameter of a granulosa cell, then i^{th} layer is given by (see fig. 1)

$$\mathcal{L}_i^\epsilon = \{(r,\theta,\phi) \in Space \ : \ r \in [1+(i-1)\epsilon, 1+i\epsilon[\}.$$

In order to compute an overcrowding function, we discretize polar and azimuth angles with a precision depending of the small cell diameter, therefore we let $N = \mathbb{E}_{nt}(1/\epsilon) \in \mathbb{N}^*$ and we subdivide these layers (see fig. 2) in ($j \in [1,N]$ and $k \in [1,N]$)

$$\mathcal{L}_{i,j,k}^\epsilon = \{(r,\theta,\phi) \in Space \ : \ r \in [1+(i-1)\epsilon, 1+i\epsilon[, \\ \theta \in [\pi\frac{(j-1)}{N}, \pi\frac{j}{N}[, \quad \phi \in [-\pi + 2\frac{k-1}{N}\pi, -\pi + 2\frac{k}{N}\pi[\}.$$

A cell is in the subdivision $\mathcal{L}_{i,j,k}^\epsilon$ means that its location is (i,j,k) where i is the layer number, j the polar angle number and k the azimuth angle number. The last point arising by using the simplified formulation is the variation of the volume of each $\mathcal{L}_{i,j,k}^\epsilon$. Indeed, when the radius of the oocyte changes, the real volume (i.e. the real volume that cells can use) of the each layer changes

$$Vol_i^\epsilon = \frac{4\pi}{3}[(r_O+i\epsilon)^3 - (r_O+(i-1)\epsilon)^3] = \frac{4\pi}{3}[\epsilon^3(3i^2-3i+1) + 3\epsilon^2 r_O(2i-1) + 3\epsilon r_O^2],$$

and for each (i,j,k) the real volume of $\mathcal{L}_{i,j,k}^\epsilon$ is

$$Vol_{i,j,k}^\epsilon = Vol_i^\epsilon/N^2 \sim \epsilon^2\frac{4\pi}{3}[\epsilon^3(3i^2-3i+1) + 3\epsilon^2 r_O(2i-1) + 3\epsilon r_O^2].$$

For each time, a cell is defined by its age and its position (or similarly by a dirac mass on $Age \times Space$). For a given M (normalization parameter) and ϵ (cell size parameter), the whole population is then characterized by a punctual measure on this set, i.e., in $\mathcal{M}_P(Space \times Age)$. Let

$$Z_0^{M,\epsilon}(da,dp) = \frac{1}{M}\sum_{k=1}^{N_0^{M,\epsilon}} \delta_{(a_k^{M,\epsilon}, x_k^{M,\epsilon})} \in \mathcal{M}_P(Space \times Age), \qquad (1.1)$$

where $(a_k^{M,\epsilon}, x_k^{M,\epsilon})_k \subset Age \times Space$[a], the initial population sequence, such that

$$\sup_{M,\epsilon}(N_0^{M,\epsilon}/M) < \infty,$$

[a]which asymptoticly satisfy assumptions (2.5)-(2.6)

and $r_O^{M,\epsilon}(0) = r_0 > 0$ (a.e.). Let $Q(ds, Compt(dn), d\Theta, \Pi^\epsilon_{s-}(p, dp'))$ be a Poisson point measure on $\mathbb{R}_+ \times \varepsilon = \mathbb{R}_+ \times \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}^3$ with intensity

$$q(ds, Compt(dn), d\Theta, \Pi^\epsilon_{s-}(p, dp')) = ds \otimes Compt(dn) \otimes d\Theta \otimes \Pi^\epsilon_{s-}(p, dp'),$$
$$dp' = r^2 \sin(\theta) dr d\theta d\phi,$$

(where ds and $d\Theta$ are Lebesgue measures on $\mathbb{R}_+$, $Compt(dn)$ is the counting measure on $\mathbb{N}^*$ and $\Pi^\epsilon_{s-}(p, dp')$ the displacement distribution law (see (1.5))) and independent of $Z_0^{M,\epsilon}$ (see 4, 25, 17, 15, 3). Let us denote $X_k^{M,\epsilon}(t)$ and $A_k^{M,\epsilon}(t)$ the position and age of the k^{th} individual at time t (ranked in the lexicographic order on $\mathbb{R}^3 \times \mathbb{R}_+$, see 4, 25, 17, 15, 3 for details). Then the oocyte radius follows the equation

$$r_O^{M,\epsilon}(t) = r_O^{M,\epsilon}(0) + \sum_{i,j,k} \kappa((i-1)\epsilon) \int_0^t \left(r_O^{M,\epsilon}(s-)\right)^\alpha \langle \Psi^\epsilon_{i,j,k}, Z^{M,\epsilon}_{s-}\rangle ds, \tag{1.2}$$

where $\alpha < 0$ and $\kappa \in C^0_b(\mathbb{R}_+, \mathbb{R}_+)$ and $\Psi^\epsilon_{i,j,k}$ a regular approximation of the characteristic function $\chi_{\mathcal{L}^\epsilon_{i,j,k}}$ (see (7.5) in the annex : section 7.1). The population at time t, denoted by $Z_t^{M,\epsilon}$, is the set of all individuals alive at time t and follows the master equation

$$\begin{aligned} Z_t^{M,\epsilon}(da, dp) &= \frac{1}{M} \sum_{k=1}^{N_0^{M,\epsilon}} \delta_{(A_k^{M,\epsilon}(0)+t, X_k^{M,\epsilon}(0))} \\ &+ \frac{1}{M} \int_0^t \int_\varepsilon 1_{n < N^{M,\epsilon}_{s-}} \Big[(2\delta_{(t-s, X_n^{M,\epsilon}(s-))} - \delta_{(A_n^{M,\epsilon}(s-)+t-s, X_n^{M,\epsilon}(s-))}) 1_{0 \le \Theta < B^{M,\epsilon}_{n,s-}} \\ &+ (\delta_{(A_n^{M,\epsilon}(s-)+t-s, p')} - \delta_{(A_n^{M,\epsilon}(s-)+t-s, X_n^{M,\epsilon}(s-))}) 1_{0 \le \Theta - B^{M,\epsilon}_{n,s-} < P^\epsilon_{s-}(X_n^{M,\epsilon}(s-))} \Big] \\ & \qquad Q(ds, (dn), d\Theta, \Pi^\epsilon_{s-}(p, dp')), \end{aligned} \tag{1.3}$$

where the birth rate is

$$B^{M,\epsilon}_{n,s-} = B(A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-)), \text{ with } B(a,p) = 1 - e^{-a/\lambda(\|p\|_2)}, \ \lambda \in C^0(\mathbb{R}_+, \mathbb{R}_+), \tag{1.4}$$

the displacement rate is

$$P^\epsilon_{s-}(p) = C^\epsilon \sum_{i,j,k} \Psi^\epsilon_{i,j,k}(p) R(\langle \Psi^\epsilon_{i,j,k}, Z^{M,\epsilon}_{s-}\rangle \frac{Vol_G}{Vol^\epsilon_{i,j,k}}), \text{ with } R(x) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}}, \tag{1.5}$$

and the displacement distribution law is given by

$$\Pi^\epsilon_{s-}(p, dp') = \frac{\Psi^\epsilon(|p'-p|) G^\epsilon_{s-}(p') dp'}{\int \Psi^\epsilon(|q-p|) G^\epsilon_{s-}(q) dq}, \text{ with } G^\epsilon_{s-}(p') = 1 - \frac{P^\epsilon_{s-}(p')}{C^\epsilon}, \tag{1.6}$$

where $C^\epsilon = C/\epsilon^2 > 0$.

More precisely, at given time t :

(1) $\sum_{k=1}^{N_0^{M,\epsilon}} \delta_{(A_k^{M,\epsilon}(0)+t, X_k^{M,\epsilon}(0))}$ corresponds to the aging of the cell population from its initial state : $\sum_{k=1}^{N_0^{M,\epsilon}} \delta_{(A_k^{M,\epsilon}(0), X_k^{M,\epsilon}(0))}$.
(2) If a birth occured at time s, $0 \leq s < t$, and since the age is reset to 0 at birth, we add two new cells of age $(t-s)$ at time t : $2\delta_{(t-s, X_n^{M,\epsilon}(s-))}$ and we have to remove from the population the mother cell, whose age had just overcome $(t-s)$ at the mitosis time : $\delta_{(A_n^{M,\epsilon}(s-)+t-s, X_n^{M,\epsilon}(s-))}$.
(3) When a displacement occurs at time s, $0 \leq s < t$, we have to add a cell at the target location : $\delta_{(A_n^{M,\epsilon}(s-)+t-s, p')}$ and to remove one from the original location : $\delta_{(A_n^{M,\epsilon}(s-)+t-s, X_n^{M,\epsilon}(s-))}$.

The main objective of the paper is to prove the convergence of the stochastic processes $(Z_t^{M,\epsilon}(da, dp))_{\epsilon,M}$ as the population increases to infinity and the size of a cell converges to zero. The existence of solutions to the system (1.2)-(2.1) is quite classical (see 4, 25, 17, 15, 3)) and so be proved in annex 7. Difficulties arise in the study of convergence. Indeed, stochastic processes, their convergence or/and tightness (compactness) are powerful mathematical tools but are not well suited for the punctual (or regular) convergence. Therefore for a fixed initial population size (in an equivalent way, fixing M) we cannot pass to the limit as ϵ converges to zero. Thus, its not completely clear that $(Z_t^{M,\epsilon}(da, dp))_{\epsilon,M}$ converges for all sequences of $(\epsilon_k, M_k) \to (0, \infty)$. We show that the limit (weak-* limit 15, 4 for similar studies) of $(Z_t^{M,\epsilon}(da, dp))_{\epsilon,M}$ (for a fixed ϵ and $M \to \infty$) is well defined (unique) and is a strong solution to the following partial differential equation

$$\begin{cases} \text{Intermediate eq.} \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\rho^\epsilon - \int P_\infty^\epsilon(p')\frac{\Pi_\infty^\epsilon(p', dp)}{dp}\rho^\epsilon(t, a, p')dp' + P_\infty^\epsilon(p)\rho^\epsilon(t, a, p) = 0, \\ r_O^\epsilon(t) = r_O^\epsilon(0) + \sum_{i,j,k} \kappa((i-1)\epsilon) \int_0^t \left(r_O^\epsilon(s)\right)^\alpha \iint \Psi_{i,j,k}^\epsilon(p)\rho^\epsilon(t, a, p)dadpds \\ \rho^\epsilon(t, 0, p) = 2\int B(a, p)\rho^\epsilon(t, a, p)dadp, \ \rho^\epsilon(0, a, p') = \rho_0^\epsilon(a, p), \ r_O^\epsilon(0) = r_0. \end{cases} \tag{1.7}$$

with $M_\epsilon^1(t, p) := \int \rho^\epsilon(t, a, p)da$,

$$P_\infty^\epsilon(p') = \frac{C}{\epsilon^2}\sum_{i,j,k}\Psi_{i,j,k}^\epsilon(p')R(\frac{\int \Psi_{i,j,k}^\epsilon(p)M_\epsilon^1(t, p)dpVol_G}{Vol_{i,j,k}^\epsilon}), \text{with } R(x) = \frac{1}{1+e^{-\frac{x-\mu}{\sigma}}}, \tag{1.8}$$

$$\Pi_\infty^\epsilon(p', dp) = \frac{\Psi^\epsilon(|p'-p|)G_{s-}^\epsilon(p)dp}{\int \Psi^\epsilon(|q-p'|)G_{s-}^\epsilon(q)dq}, \\ \text{with } G_{s-}^\epsilon(.) = 1 - \sum_{i,j,k}\Psi_{i,j,k}^\epsilon(.)R(\frac{\int \Psi_{i,j,k}^\epsilon(p)M_\epsilon^1(t, p)dpVol_G}{Vol_{i,j,k}^\epsilon}). \tag{1.9}$$

Then, we prove that, the limit as ϵ converge to zero, is a weak solution (using Sobolev spaces and weak$-*$ convergence 2) to the following nonlinear partial differential equation (Transport 20, 22, Keller Segel type 12, 13)

$$Final\ eq. \begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\rho + div(C\rho R \nabla \Big(\log(1-R) \Big)) = 0, \quad t > 0,\ a > 0,\ r > 1, \\ \rho|_{a=0} = 2 \int B(a,p)\rho(t,a,p)da, \quad \rho_{t=0} = \rho_0, \\ \rho|_{r=1} = 0, \end{cases} \tag{1.10}$$

with $C > 0$, $R = R(\int_{\mathbb{R}_+} \rho(t,a,p)da \frac{Vol_G(3r^2+6r+3)}{3r^2+6rr_O(t)+3r_O(t)^2})$ and

$$r'_O(t) = \big(r_O(t)\big)^\alpha \iint_{Age \times Space} \kappa(r)\rho(t,a,p)r^2 \sin(\theta) da dr d\theta d\phi, \quad r_O(0) = r_0. \tag{1.11}$$

Therefore, the main difficulty is to mix both approach to prove the convergence. In section 2, we give main theorems of convergence and proves are done in sections 3, 4 (for stochastic tools) and 5 (for PDE tools).

2. Main results

Before proving the convergence of the stochastic process $Z_t^{M,\epsilon}$, we first show that we have existence for all $M > 0$ and $\epsilon > 0$ of $Z_t^{M,\epsilon}$ (proposition 2.1). Then, we adopt the following approach (see fig. 2) : we show, in theorem 2.1 I, that we can extract a subsequence $M_k \to \infty$ such that $(Z_t^{M_k,\epsilon})_{M_k,\epsilon}$ converges to a solution ρ^ϵ of an intermediate partial differential equation (1.7), then we prove, in theorem 2.1 II, that we can extract a subsequence $\epsilon_k \to \infty$ such that ρ^{ϵ_k} converges to ρ solution to (1.10). Finally, in theorem 2.1 III, we show that there exists a subsequence $(\epsilon_k, M_k) \to (0,\infty)$ such that $Z_t^{M,\epsilon}$ converges to ρ (see fig. 2).

Assumptions

I- Geometric : Assume that for all p, the matrix
$M_\Psi(p) := \frac{\int_q \Psi^\epsilon(|q-p|)(q-p)\ {}^t(q-p)dq}{\epsilon^2 \int \Psi^\epsilon(|q-p|)dq}$, satisfies

$$\begin{cases} M_\Psi(p) \text{ is definite positive} \\ \forall p\ \ \forall \epsilon > 0, \quad 0 < \inf_{\epsilon,p} \min\{\lambda \in Sp(M_\Psi(p))\} \\ \qquad\qquad\qquad\qquad \le \sup_{\epsilon,p} \max\{\lambda \in Sp(M_\Psi(p))\} < \infty \\ CM_\Psi(p) \to^{C^0(Space)} Ch(p). \end{cases} \tag{2.1}$$

Remark : the construction given in section 7.1 satisfies (2.1) and by symmetry of Ψ^ϵ *we have directly that* $Ch(p) = C.Id$ *for all* $p \in Space$. *Therefore, we consider that assumption (2.1) is satisfied in the whole paper. Moreover, we notice that*

$$Vol_G \int \Psi^\epsilon_{i,j,k} dp / Vol^\epsilon_{i,j,k} \to^{\epsilon \to 0} \frac{Vol_G(3r^2+6r+3)}{3r^2+6rr_O(t)+3r_O(t)^2}.$$

II- Uniform bounds on $Z_0^{M,\epsilon}$ and $\rho_0^\epsilon(a,p)$: assume there exists $m \geq 1$ and $w > 0$ s.t.

$$\sup_{(M,\epsilon)\in U_w} \mathbb{E}\Big((\iint (1+a^m+r^m) Z_0^{M,\epsilon}(da,dp))^2 + (\iint (1+a^{2m}+r^{2m}) Z_0^{M,\epsilon}(da,dp)) \Big) < \infty, \quad (2.2)$$

with

$$U_w := \{M\epsilon > w\}, \quad (2.3)$$

$$\sup_\epsilon \int (1+\|p\|)[\int \rho_0^\epsilon(a,p)da + (\int \rho_0^\epsilon(a,p)da)^2 + (\int |\frac{\partial}{\partial a}\rho_0^\epsilon(a,p)|da)^2]dp < \infty, \quad (2.4)$$

III- Convergence of $Z_0^{M,\epsilon}$ and $\rho_0^\epsilon(a,p)$:

$$Z_0^{M,\epsilon}(da,dp) \rightharpoonup_{M\to\infty} \rho_0^\epsilon(a,p)dadp, \quad with \quad \rho_0^\epsilon(a,p) \in C_0^1(\mathbb{R}^+\times\mathbb{R}^3), \quad (2.5)$$

$$\rho_0^\epsilon(a,p) \to_{\epsilon\to 0}^{L^2(\mathbb{R}^+\times)(Age\times Sapce)} \rho_0(a,p), \quad with \quad \rho_0(a,p) \in C_0^1(\mathbb{R}^+\times\mathbb{R}^3). \quad (2.6)$$

Proposition 2.1. *(Existence) Under assumptions (1.1)-(1.6) and (2.2) there exists a solution* $Z_t^{M,\epsilon} \in D(\mathbb{R}_+, \mathcal{M}_P)$ *and* $r_O \in C^0 \cap C_m^1(\mathbb{R}_+)$ *to (1.2)-(1.3) for all* $M, \epsilon > 0$*. Moreover, assuming that (2.2) is satisfied then we have that, for all* $T > 0$*,*

$$\sup_{M,\epsilon} \mathbb{E}\Big(\sup_{t\in[0,T\epsilon]} (\iint (1+a^m+r^m) Z_t^{M,\epsilon}(da,dp))^2 + (\iint (1+a^{2m}+r^{2m}) Z_t^{M,\epsilon}(da,dp)) \Big) < \infty. \quad (2.7)$$

The proof is given in section 7.2.

We have that.

Theorem 2.1.
I) Assume that (2.5) and (2.2) are satisfied. Then, for all $\epsilon > 0$*,* $(Z_t^{M,\epsilon}(da,dp))_M$ *is tight on* $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_F(Age \times Space), weak))$*. Its limit values* $\lim_{M_k\to\infty} Z_t^{M_k,\epsilon}(da,dp) = \rho^\epsilon(t,a,p)dadp$*, are continuous measure-valued process satisfying (1.7)-(1.9).*

II) Assume that (2.6), (2.4) and (2.1) are satisfied then $(\rho^\epsilon, r_O^\epsilon)$ *solution to (1.7) weakly converge to* (ρ, r_O) *weak solution to (1.10)-(1.11).*

III) Assume that (2.5), (2.6), (2.2) and (2.1) are satisfied. Then, for all $C > 0$*, there exists a subsequence* $(\epsilon_k, M_k) \subset U_C$ *s.t.* $M_k\epsilon_k \to \infty$ *and*

$$\lim_{k\to\infty} \langle Z_t^{M_k,\epsilon_k}(da,dp), \psi\rangle = \langle \rho(t,a,p)dadp, \psi\rangle, \quad \forall \psi \in C_0^1,$$

weak solution to (1.10)-(1.11).

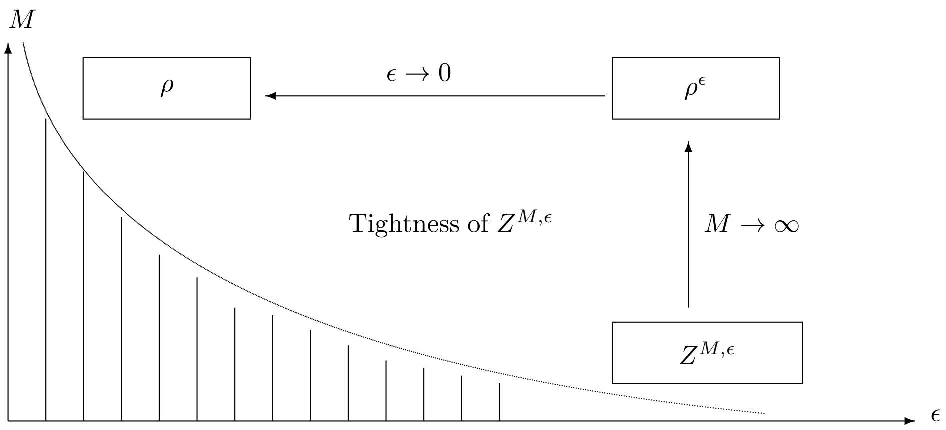

Fig. 3. Convergence proof. The main theorem 2.1 can be visualized as follows. Under assumptions given above, we prove the convergence as $M \to \infty$ then as $\epsilon \to 0$ and finally as $(M, \epsilon) \to (\infty, 0)$ (under condition on $\epsilon.M$)

3. Proof of the main theorem

The proof of the main theorem follows the scheme given in figure 3. The first and third part are stochastic processes results while the second point is a partial differential equation result.

3.1. *Proof of theorem 2.1 Part I : Convergence of the stochastic process $Z_t^{M,\epsilon}$ as $M \to \infty$*

To prove tightness of the sequence $Z_t^{M,\epsilon}(da, dp)$ (as probability measure on $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_F(Age \times Space), vague)))$, we use a Rolley criterium (see 24, 8, 9 for more details) which establishes that it suffices to prove that for all f of a dense subspace of $(C_0(Age \times Space, \mathbb{R}), \|.\|_\infty)$ (here $C_0^1(Age \times Space, \mathbb{R})$) the sequence $\langle f_t, Z_t^{M,\epsilon} \rangle$ is tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. In the section 4.2, we prove that

$$\langle f_t, Z_t^{M,\epsilon} \rangle = \underbrace{\mathcal{M}_t^{M,\epsilon}(f)}_{Martingale} + \underbrace{\mathcal{V}_t^{M,\epsilon}(f)}_{Finite\ Variation} .$$

Therefore, using a criterium of Aldous-Rebolledo 1, 9, 23, it suffices to prove that :
- for all $t \in \mathcal{T}$ (dense in $\mathbb{R}_+$), $\langle \mathcal{M}_t^{M,\epsilon}(f) \rangle$ and $\mathcal{V}_t^{M,\epsilon}(f)$ are tight on $\mathbb{R}$
- for all $T > 0$, $u > 0$, $\eta > 0$, there exists $\delta > 0$ and $N_0^{M,\epsilon} \in \mathbb{N}$ s.t.

$$\sup_{M \geq N_0^{M,\epsilon}} \mathbb{P}\Big(|\langle \mathcal{M}_{T_M}^{M,\epsilon}(f) \rangle - \langle \mathcal{M}_{S_M}^{M,\epsilon}(f) \rangle| \geq \eta, \quad T_M < S_M + \delta \Big) \leq u,$$

$$\sup_{M \geq N_0^{M,\epsilon}} \mathbb{P}\Big(|\mathcal{V}_{T_M}^{M,\epsilon}(f) - \mathcal{V}_{S_M}^{M,\epsilon}(f)| \geq \eta, \quad T_{M,\epsilon} < S_M + \delta \Big) \leq u,$$

for any sequences of stopping times (S_M, T_M) of the natural filtration $\mathcal{F}_M$, so that $S_M \leq T_M \leq T$. Both points are a direct consequence of the bounds (4.10)-(4.12) (see 26 for more details). Using Prohorov theorem, we can extract a subsequence $Z_t^{M_k,\epsilon}(da, dp)$ which vague converges to $Z_t^{\epsilon}(da, dp)$ and by construction, for all $f \in C^1(Age \times Space)$,

$$\sup_{t\in\mathbb{R}_+} |\langle f_t, Z_t^{M_k,\epsilon}\rangle - \langle f_t, Z_{t-}^{M_k,\epsilon}\rangle| \leq C\frac{\|f\|_{W^{1,\infty}}}{M_k}, \text{ with } C > 0,$$

the limit process is a.e. continuous. Finally, using a result of convergence 16[b], to prove the weak convergence, it suffices to add tightness of $\langle 1, Z_t^{M,\epsilon}\rangle$ (which came directly from bound (4.6)). Now using (4.10), we have that the martingale part of the process satisfies

$$\mathbb{E}(|\mathcal{M}_t^{M_k,\epsilon}(f)|)^2 \leq \mathbb{E}(|\mathcal{M}_t^{M_k,\epsilon}(f)|^2) = \mathbb{E}(\langle \mathcal{M}_t^{M_k,\epsilon}(f)\rangle) \leq C\frac{\|f\|_{W^{1,\infty}}}{M_k} \to_{k\to\infty} 0,$$

with $C > 0$. By passing to the limit in (1.8)-(1.9), we have that

$$P_{s-}^{\epsilon} \to_{M\to\infty} P_{\infty}^{\epsilon}(s-) = \frac{C}{\epsilon^2}\sum_{i,j,k}\Psi_{i,j,k}^{\epsilon}(.)R(\frac{\langle \Psi_{i,j,k}^{\epsilon}, Z_{s-}^{\epsilon}(da,dp)\rangle Vol_G}{Vol_{i,j,k}^{\epsilon}}),$$
$$\text{with } R(x) = \frac{1}{1+e^{-\frac{x-\mu}{\sigma}}},$$

and

$$\Pi_{s-}^{\epsilon}(., dq) \to_{M\to\infty} \Pi_{\infty} = \frac{\Psi^{\epsilon}(|p'-p|)G_{s-}^{\epsilon}(p')dp'}{\int \Psi^{\epsilon}(|q-p|)G_{s-}^{\epsilon}(q)dq},$$
$$\text{with } G_{s-}^{\epsilon}(p') = 1 - \sum_{i,j,k}\Psi_{i,j,k}^{\epsilon}(.)R(\frac{\langle \Psi_{i,j,k}^{\epsilon}, Z_t^{\epsilon}(da,dp)\rangle Vol_G}{Vol_{i,j,k}^{\epsilon}}).$$

Therefore, we find that for all $f \in W^{1,\infty}$ (see lemma 4.2 and the section 4.2),

$$\begin{aligned}0 = \langle f_t, Z_t^{M,\epsilon}\rangle - \langle f_0, Z_0^{M,\epsilon}\rangle - \int_0^t \langle (\frac{\partial}{\partial u} + \frac{\partial}{\partial a})f(u,a,p), Z_u^{M,\epsilon}(da,dp)\rangle du \\ - \int_0^t \langle (2f(s,0,p) - f(s,a,p))B(a,p) \\ + \int_{Space}(f(s,p,p') - f(s,a,p))P_{\infty}^{\epsilon}(s-)(p)\Pi_{\infty}(p,dp'), Z_s^{M,\epsilon}\rangle ds,\end{aligned}$$

is satisfied. The limit is a weak solution to the partial differential equation (1.7), which is unique (see the proof of II).

[b]Authors prove that the convergence in law in $\mathcal{D}([0,T]; (M_F, \text{vague topology}))$ of $(M_n)_n$ implies the convergence in law in $\mathcal{D}([0,T]; (M_F, \text{étroite topology}))$ of $(M_n)_n$ under two more conditions : regularity of the limit $(\mathcal{C}^0([0,T]; (M_F, \text{étroite})))$ and the convergence in law of $\langle M_n, 1\rangle$ in $\mathcal{D}([0,T]; \mathbb{R})$. With M_F finite measure space.

3.2. *Proof of theorem 2.1 Part II : Convergence of ρ_t^ϵ as $\epsilon \to 0$*

Using a fixed point theorem (contraction in Banach space), we prove (in lemma 5.1) the existence of solution to the master equation (1.7). To prove the convergence as $\epsilon \to 0$, we first prove, in lemmas 5.2-5.5, that under assumption (2.4), for any $T > 0$, $M_\epsilon^1(t,p) := \int \rho^\epsilon(t,a,p)da$ (resp. ρ^ϵ) belongs to a compact set of $L^2([0,T] \times Space)$ (resp. $L^2([0,T] \times Age \times Space)$)and $\nabla M_\epsilon^1(t,p)$ belongs to a weak$-*$ compact set of $L^2([0,T] \times Space)$. Assume (2.6) - (2.1), let $f \in C_c^1(Age \times Space)$, multiply (1.7) by f and integrate with respect to a : we find that

$$\frac{\partial}{\partial t}\int_p f(a,p)\rho^\epsilon(t,a,p)dadp - \iint B(a,p)f(a,p)\rho^\epsilon(t,a,p)dadp$$
$$= \int_{p,p'} \Big(f(a,p') - f(a,p)\Big)\Pi_{s-}^\epsilon(p,dp')P_{s-}^\epsilon(p)\rho^\epsilon(t,a,p)dadp.$$

Rewritting the second member as follows, we have that

$$\int_{a,p,p'} \Big(f(a,p') - f(a,p)\Big)\Pi_{s-}^\epsilon(p,dp')P_{s-}^\epsilon(p)\rho^\epsilon(t,a,p)dpda$$
$$= \int_{p,p'} C^\epsilon\Big(f(a,p') - f(a,p)\Big)\frac{\Psi^\epsilon(|p'-p|)G_{s-}^\epsilon(p')dp'}{\int \Psi^\epsilon(|q-p|)G_{s-}^\epsilon(q)dq}\frac{P_{s-}^\epsilon(p)}{C^\epsilon}\rho^\epsilon(t,a,p)dadp =$$
$$= \int_p^t \nabla f(a,p)\frac{C^\epsilon\int_q \Psi^\epsilon(|q-p|)(q-p)G_{s-}^\epsilon(q)dq}{\int \Psi^\epsilon(|q-p|)G_{s-}^\epsilon(q)dq}\frac{P_{s-}^\epsilon(p)}{C^\epsilon}\rho^\epsilon dadp + o(1)$$
$$= \int_p^t \nabla f(a,p)\frac{C^\epsilon\int_q \Psi^\epsilon(|q-p|)(q-p)\ {}^t(q-p)dq}{\int \Psi^\epsilon(|q-p|)dq}\frac{P_{s-}^\epsilon(p)}{C^\epsilon}\rho^\epsilon\frac{\nabla G_{s-}^\epsilon}{G_{s-}^\epsilon}(p)dadp + o(1)$$
$$= \int_p \nabla f(a,p)\rho^\epsilon Ch(p)R^t\nabla\Big(\log(1-R)\Big)dpda + o(1).$$

By passing to the limit $\nabla f(a,p)\rho^\epsilon(t,a,p)Ch(p)$ in L^2 and $\nabla\Big(\log(1-R)\Big)$ in weak$-*$ L^2 (see lemmas 5.3-5.4 and 2), we have that ρ is a solution to (1.10).

3.3. *Proof of theorem 2.1 Part III : Convergence of $Z_t^{M,\epsilon}$ as $M \to \infty$ and $\epsilon \to 0$*

Let $T > 0$. Changing the time scale $t \mapsto t\epsilon$. We first notice that for $(M,\epsilon) \in U_1$, bounds given in the proof of theorem 2.1 Part I are independent of ϵ, M. Indeed, the time scale appeared in lemma 4.5 that gives

$$\sup_{M,\epsilon}\mathbb{E}\Big(\sup_{t\in[0,T\epsilon]}\iint r^m Z_t^{M,\epsilon}(da,dp)\Big) < \infty, \quad \forall T > 0.$$

Then, changing the time scale, we find that all bounds given in Proof of theorem 2.1 Part I and in technical part 4.2 are in $O(T/\epsilon M)$ therefore uniform in U_1 (and so the tightness of $Z_t^{M,\epsilon}$ is assured in U_1). To obtain the convergence we need that $\epsilon M \to \infty$. Now, we construct a (ϵ_k, M_k) so that the limit is the one we are expecting

(i.e., ρ). Since C_0^1 is a separable set, there exists $(\psi_k)_k$ dense subset of C_0^1. Let any sequence $(\epsilon_k)_k$ converging to 0. Using theorem 2.1, there exists M^1 s.t.

$$\sup_{t\leq T} |\langle Z_t^{M^1,\epsilon_1}(da,dp),\psi_1\rangle - \langle \rho_{\epsilon_1}(t,a,p)dadp,\ \psi_1\rangle| \leq 1.$$

For the same reason, we can find M^2 s.t.

$$\sup_{t\leq T} |\langle Z_t^{M^2,\epsilon_2}(da,dp),\psi_i\rangle - \langle \rho_{\epsilon_2}(t,a,p)dadp,\psi_j\rangle| \leq 1/4,\ j=1,2.$$

and so on, there exists M^n s.t.

$$\sup_{t\leq T} |\langle Z_t^{M^n,\epsilon_n}(da,dp),\psi_j\rangle - \langle \rho_{\epsilon_n}(t,a,p)dadp,\psi_j\rangle| \leq 1/2^n,\ j=1..n.$$

Therefore, using the theorem 2.1 part I, we have that

$$\lim_{n\to\infty}\sup_{t\leq T} |\langle Z_t^{M^n,\epsilon_n}(da,dp),\psi\rangle - \langle \rho(t,a,p)dadp,\psi\rangle| = 0, \quad \forall \psi\in C_0^1.$$

4. Stochastic calculus and technical lemmas

In this part, we give the doob decomposition of the stochastic process $Z_s^{M,\epsilon}$ (issued from Ito calculus). Then, we give uniform bounds which are used to prove the tightness of $Z_s^{M,\epsilon}$.

4.1. *Ito calculus and first lemmas*

For convenience, we introduced stopping time $(\tau_N^{M,\epsilon})$

$$\tau_N^{M,\epsilon} = \inf\{s\geq 0, \quad \langle 1, Z_s^{M,\epsilon}\wedge r_O^{M,\epsilon}(s)\rangle \geq N\}. \tag{4.1}$$

We first give computational lemmas on the population evolution.

Lemma 4.1. *For all $f\in C^0(Space\times Age)$, $t>0$ and r_O, $Z_t^{M,\epsilon}$ solution to (1.2)-(1.3) we have*

$$\begin{aligned}\iint_{\mathbb{R}_+\times Space} f(t,a,p)Z_t^{M,\epsilon}(da,dp) &= \frac{1}{M}\sum_{k=1}^{N_0^{M,\epsilon}} f(t,A_n^{M,\epsilon}(0)+t,X_n^{M,\epsilon}(0))\\ +\frac{1}{M}\int_0^t\int_\varepsilon 1_{n<N_{s-}^{M,\epsilon}}\Big[&(2f(t,t-s,X_n^{M,\epsilon}(s^-))-f(t,A_n^{M,\epsilon}(s-)+t-s,X_n^{M,\epsilon}(s-)))1_{\Theta<B_{n,s-}^{M,\epsilon}}\\ +(f(t,A_n^{M,\epsilon}(s-)+t-s,p')-&f(t,A_n^{M,\epsilon}(s-)+t-s,X_n^{M,\epsilon}(s-)))1_{0\leq\Theta-B_{n,s-}^{M,\epsilon}<P_{s-}^{\epsilon}(X_n(s-))}\Big]\\ &Q(ds,(dn),d\Theta,\Pi_{s-}^{M,\epsilon}(p,dp'))\end{aligned} \tag{4.2}$$

Proof. Direct computation (see 4 for more details). □

Lemma 4.2. *Let $F\in C^1(\mathbb{R},\mathbb{R})$ and $f\in\mathcal{B}(\mathbb{R}_+\times\mathbb{R}_+\times Space,\mathbb{R})$ s.t. for all $p\in Space$, $(s,a)\in\mathbb{R}_+^2\mapsto f(s,a,p)\in C_b^{1,1}(\mathbb{R}_+\times Space,\mathbb{R})$ with uniform (in space)* *bounds of the partial derivation of f then, for all $t > 0$,*

$$\int_0^t F'(\langle f_u, Z_u^{M,\epsilon}\rangle)\iint_{\mathbb{R}_+\times Space}(\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,a,p)Z_u^{M,\epsilon}(da,dp)du =$$
$$F(\langle f_t, Z_t^{M,\epsilon}\rangle) - F(\langle f_0, Z_0^{M,\epsilon}\rangle) - \Big[\int_0^t\int_{\varepsilon} 1_{n<N_{s-}^{M,\epsilon}}\Big[(F(\langle f_s, Z_s\rangle + \frac{2}{M}f(s,0,X_n^{M,\epsilon}(s-))$$
$$-\frac{1}{M}f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-))) - F(\langle f_s, Z_s^{M,\epsilon}\rangle)1_{0\le\Theta<B_{n,s-}^{M,\epsilon}}$$
$$+(F(\langle f_s, Z_s\rangle + \frac{1}{M}f(s,A_n^{M,\epsilon}(s-),p') - \frac{1}{M}f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-))) - F(\langle f_s, Z_s\rangle))$$
$$1_{0\le\Theta-B_{n,s-}^{M,\epsilon}<P_{s-}^{\epsilon}(X_n(s-))}\Big]Q(ds,(dn),d\Theta,\Pi_{s-}^{M,\epsilon}(p,dp'))\Big] \quad (4.3)$$

with $\langle f_s, Z_s^{M,\epsilon}\rangle = \iint_{\mathbb{R}_+\times Space} f(s,a,p)Z_s^{M,\epsilon}(da,dp)$.

Proof. Using (4.2) of lemma 4.1, in the particular case $(\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,a,p)$, and integrating in time, we have that $Z_t^{M,\epsilon}$ satisfies (for all f given in assumptions)

$$\int_0^t\iint_{\mathbb{R}_+\times Space}(\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,a,p)Z_u^{M,\epsilon}(da,dp)du =$$
$$\frac{1}{M}\int_0^t\sum_{k=1}^{N_0^{M,\epsilon}}(\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,A_n^{M,\epsilon}(0)+u,X_n^{M,\epsilon}(0))du$$
$$+\frac{1}{M}\int_0^t\int_0^u\int_{\varepsilon}1_{n<N_{s-}^{M,\epsilon}}\Big[(2(\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,u-s,X_n^{M,\epsilon}(s-))$$
$$-(\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,A_n^{M,\epsilon}(s-)+u-s,X_n^{M,\epsilon}(s-)))1_{0\le\Theta<B_{n,s-}^{M,\epsilon}}$$
$$+((\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,A_n^{M,\epsilon}(s-)+u-s,p') - f(u,A_n^{M,\epsilon}(s-)+u-s,X_n^{M,\epsilon}(s-)))$$
$$1_{0\le\Theta-B_{n,s-}^{M,\epsilon}<P_{s-}^{\epsilon}(X_n(s-))}\Big]Q(ds,(dn),d\Theta,\Pi_{s-}^{M,\epsilon}(p,dp'))du. \quad (4.4)$$

Now, using Fubinni theorem $\int_{u=0}^t\int_{s=0}^u = \int_{s=0}^t\int_{u=s}^t$ and $((\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f)(u,.+u,.) =$

$\frac{d}{du}(f(u,.+u,.))$, we have that

$$\int_0^t \iint_{\mathbb{R}_+\times Space} (\frac{\partial}{\partial u}+\frac{\partial}{\partial a}) f(u,a,p) Z_u^{M,\epsilon}(da,dp)du =$$
$$\frac{1}{M}\sum_{k=1}^{N_0^{M,\epsilon}} [f(t, A_n^{M,\epsilon}(0)+t, X_n^{M,\epsilon}(0)) - f(0, A_n^{M,\epsilon}(0), X_n^{M,\epsilon}(0))]$$
$$+\frac{1}{M}\int_0^t \int_\varepsilon 1_{n<N_{s-}^{M,\epsilon}} \Big[(2(f(t,t-s,X_n^{M,\epsilon}(s-)) - f(s,0,X_n^{M,\epsilon}(s-)))$$
$$-(f(t,A_n^{M,\epsilon}(s-)+t-s, X_n^{M,\epsilon}(s-))) - f(s, A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))))1_{0\le\Theta<B_{n,s-}^{M,\epsilon}}$$
$$+(f(t,A_n^{M,\epsilon}(s-)+t-s,p') - f(s,A_n^{M,\epsilon}(s-),J))-$$
$$(f(t,A_n^{M,\epsilon}(s-)+t-s, X_n^{M,\epsilon}(s-)) - f(s,A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))))$$
$$1_{0\le\Theta-B_{n,s-}^{M,\epsilon}<P_{s-}^{\epsilon}(X_n(s-))}\Big] duQ(ds,(dn),d\Theta,\Pi_{s-}^{M,\epsilon}(p,dp')).$$

Using formula (4.2), we find finally that

$$\int_0^t \iint_{\mathbb{R}_+\times Space} (\frac{\partial}{\partial u}+\frac{\partial}{\partial a}) f(u,a,p) Z_u^{M,\epsilon}(da,dp)du = \langle f_t, Z_t^{M,\epsilon}\rangle - \langle f_0, Z_0^{M,\epsilon}\rangle$$
$$-\frac{1}{M}\Big[\int_0^t \int_\varepsilon 1_{n<N_{s-}^{M,\epsilon}} \Big((2f(s,0,X_n^{M,\epsilon}(s-)) - f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-)))1_{0\le\Theta<B_{n,s-}^{M,\epsilon}}$$
$$+(f(s,A_n^{M,\epsilon}(s-),p') - f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-)))1_{0\le\Theta-B_{n,s-}^{M,\epsilon}<P_{s-}^{\epsilon}(X_n(s-))}\Big)$$
$$Q(ds,(dn),d\Theta,\Pi_{s-}^{M,\epsilon}(p,dp'))\Big].$$

Now, using the Itô formula with jump processes, we find that (4.3) is satisfied (see 4, 25, 17, 15, 3 for more details). □

4.2. *Doob decomposition of $Z_t^{M,\epsilon}$*

Let $m \ge 1$ and assume that

$$\sup_{M,\epsilon} \mathbb{E}\Big(\langle 1+a^{2m}+r^{2m}, Z_0^{M,\epsilon}(da,dp)\rangle + \langle 1+a^m+r^m, Z_0^{M,\epsilon}(da,dp)\rangle^2) < \infty, \quad (4.5)$$

then we have that for all $T>0$ (see lemmas 4.3-4.7),

$$\sup_{M,\epsilon}\mathbb{E}\Big(\sup_{t\in[0,T\epsilon]}\langle 1+a^{2m}+r^{2m}, Z_t^{M,\epsilon}(da,dp)\rangle + \sup_{t\in[0,T\epsilon]}\langle 1+a^m+r^m, Z_t^{M,\epsilon}(da,dp)\rangle^2\Big) < \infty. \quad (4.6)$$

Moreover, for all f,

$$\langle f_t, Z_t^{M,\epsilon}\rangle = \mathcal{M}_t^{M,\epsilon}(f) + \mathcal{V}_t^{M,\epsilon}(f),$$

where

$$\mathcal{M}_t^{M,\epsilon}(f) = \langle f_t, Z_t^{M,\epsilon}\rangle - \langle f_0, Z_0^{M,\epsilon}\rangle - \int_0^t \langle (\frac{\partial}{\partial u} + \frac{\partial}{\partial a}) f(u,a,p), Z_u^{M,\epsilon}(da,dp)\rangle du$$
$$- \int_0^t \langle (2f(s,0,p) - f(s,a,p))B(a,p) +$$
$$\int_{Space} (f(s,p,p') - f(s,a,p)) P_{s-}^{\epsilon}(p)\Pi_{s-}^{\epsilon}(p,dp'), Z_s^{M,\epsilon}\rangle ds, \quad (4.7)$$

is an L^2 martingale càdlàg nul at $t=0$ of quadratic previsible increasing process

$$\langle \mathcal{M}_t^{M,\epsilon}(f)\rangle = \frac{1}{M}\int_0^t \langle (2f(s,0,p) - f(s,a,p))^2 B(a,p)$$
$$+ \int_{Space} (f(s,a,p') - f(s,a,p))^2 P_{s-}^{\epsilon}\Pi_{s-}^{\epsilon}(p,dp'), Z_s^{M,\epsilon}\rangle ds, \quad (4.8)$$

and

$$\mathcal{V}_t^{M,\epsilon}(f) = \langle f_0, Z_0^{M,\epsilon}\rangle + \int_0^t \langle (\frac{\partial}{\partial u} + \frac{\partial}{\partial a}) f(u,a,p), Z_u^{M,\epsilon}(da,dp)\rangle du$$
$$+ \int_0^t \langle (2f(s,0,p) - f(s,a,p))B(a,p)$$
$$+ \int_{Space} (f(s,p,p') - f(s,a,p)) P_{s-}^{\epsilon}(p)\Pi_{s-}^{\epsilon}(p,dp'), Z_s^{M,\epsilon}\rangle ds, \quad (4.9)$$

which satisfy in average (in $d\mathcal{P}(\omega)$) an uniform $W^{1,\infty}_{loc}(\mathbb{R}_+)$ bound. More precisely, for all stopping time T, S

$$\mathbb{E}\Big(\sup_{t\le T} |\langle \mathcal{M}_t^{M,\epsilon}(f)\rangle|\Big) \le \Big(9\|f\|^2_{L^\infty} + \|f\|^2_{W^{1,\infty}}(1+C)\Big)\mathbb{E}\Big(\frac{|T|}{M}\sup_{s\le T}\langle 1, Z_s^{M,\epsilon}\rangle\Big), \quad (4.10)$$

$$\mathbb{E}\Big(|\langle \mathcal{M}_T^{M,\epsilon}(f)\rangle - \langle \mathcal{M}_S^{M,\epsilon}(f)\rangle|\Big) \le$$
$$\Big(9\|f\|^2_{L^\infty} + \|f\|^2_{W^{1,\infty}}(1+C)\Big)\mathbb{E}\Big(\frac{|T-S|}{M}\sup_{s\le \max(T,S)}\langle 1, Z_s^{M,\epsilon}\rangle\Big), \quad (4.11)$$

$$\mathbb{E}\Big(\sup_{t\le T} |\langle \mathcal{V}_t^{M,\epsilon}(f)\rangle|\Big) \le \mathbb{E}\Big(|\langle f_0, Z_0^{M,\epsilon}|\Big)$$
$$+ \Big(3\|f\|_{L^\infty} + \|f\|_{W^{1,\infty}}(1+C)\Big)(1 + \mathbb{E}\Big(|T|\sup_{s\le T}\langle 1, Z_s^{M,\epsilon}\rangle\Big)), \quad (4.12)$$

and

$$\mathbb{E}\Big(|\langle \mathcal{V}_T^{M,\epsilon}(f)\rangle - \langle \mathcal{V}_S^{M,\epsilon}(f)\rangle|\Big) \le$$
$$\Big(3\|f\|_{L^\infty} + \|f\|_{W^{1,\infty}}(1+C)\Big)\mathbb{E}\Big(\frac{|T-S|}{M}(1 + \sup_{s\le \max(T,S)}\langle 1, Z_s^{M,\epsilon}\rangle)\Big). \quad (4.13)$$

Proof. To prove bound (4.6), we use technical lemmas 4.3-4.7 (proves are similar to 26, 17). Now, for all $f \in C^0(Space \times Age)$, $t > 0$ and r_O, $Z_t^{M,\epsilon}$ solution to (1.2)-(1.3) we have (see lemma 4.2)

$$\begin{aligned}
0 = \langle f_t, Z_t^{M,\epsilon}\rangle - \langle f_0, Z_0^{M,\epsilon}\rangle - \int_0^t \langle (\frac{\partial}{\partial u} + \frac{\partial}{\partial a}) f(u,a,p), Z_u^{M,\epsilon}(da,dp)\rangle du \\
- \frac{1}{M}\Big[\int_0^t \int_\varepsilon 1_{n<N_{s-}^{M,\epsilon}} \Big[(2f(s,0,X_n^{M,\epsilon}(s-)) - f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-))) \\
1_{0\le\Theta<B(A_n^M(s-),X_n^M(s-))} + (f(s,A_n^{M,\epsilon}(s-),p') - f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-))) \\
1_{B(A_n^M(s-),X_n^M(s-))\le\Theta<B(A_n^M(s-),X_n^M(s-))+P_{s-}^\epsilon(X_n^M(s-))}\Big] \\
Q(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp'))\Big]. \quad (4.14)
\end{aligned}$$

Using (4.2), we have that

$$\begin{aligned}
\mathcal{M}_t^{M,\epsilon}(f) = -\frac{1}{M}\int_0^t \int_\varepsilon 1_{n<N_{s-}^{M,\epsilon}} \Big[(2f(s,0,X_n^{M,\epsilon}(s-)) - f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-))) \\
1_{0\le\Theta<B(A_n^M(s-),X_n^M(s-))} \\
+ (f(s,A_n^{M,\epsilon}(s-),p') - f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-))) \\
1_{B(A_n^M(s-),X_n^M(s-))\le\Theta<B(A_n^M(s-),X_n^M(s-))+P_{s-}^\epsilon(X_n^M(s-))}\Big] \tilde{Q}(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp')), \\
(4.15)
\end{aligned}$$

with $\tilde{Q} = Q - ds \otimes Compt(dn) \otimes d\Theta \otimes \Pi_{s-}^{M,\epsilon}(p,dp')$ be the compensated Poisson process of Q introduced in the introduction. Therefore $\mathcal{M}_t^{M,\epsilon}(f)$ is a local martingale associated to the stopping times sequence $(\tau_N^{M,\epsilon})_N$ introduced in (4.1). Using Ito formula, we have that

$$\begin{aligned}
\mathcal{M}_t^{M,\epsilon}(f)^2 = -\int_0^t \int_\varepsilon 1_{n<N_{s-}^{M,\epsilon}} \\
\Big[((\langle f_s, Z_s^{M,\epsilon}\rangle + \frac{2}{M} f(s,0,X_n^{M,\epsilon}(s-)) - \frac{1}{M} f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-)))^2 - \langle f_s, Z_s^{M,\epsilon}\rangle^2) \\
1_{0\le\Theta<B(A_n^M(s-),X_n^M(s-))} \\
+((\langle f_s, Z_s^{M,\epsilon}\rangle + \frac{1}{M} f(s,A_n^{M,\epsilon}(s-),p') - \frac{1}{M} f(s,A_n^{M,\epsilon}(s-),X_n^{M,\epsilon}(s-)))^2 - \langle f_s, Z_s^{M,\epsilon}\rangle^2) \\
1_{B(A_n^M(s-),X_n^M(s-))\le\Theta<B(A_n^M(s-),X_n^M(s-))+P_{s-}^\epsilon(X_n^M(s-))}\Big] \\
\tilde{Q}(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp')). \quad (4.16)
\end{aligned}$$

Using lemma 4.2, we find that

$$\langle f_t, Z_t^{M,\epsilon}\rangle^2 - \langle f_0, Z_0^{M,\epsilon}\rangle^2 - 2\int_0^t \langle f_u, Z_u^{M,\epsilon}\rangle\langle(\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,a,p), Z_u^{M,\epsilon}(da,dp)\rangle du$$
$$-2\int_0^t \langle f_s, Z_s^{M,\epsilon}\rangle\langle(2f(s,0,p)-f(s,a,p))B(a,p)$$
$$+\int_{Space}(f(s,a,p')-f(s,a,p))P_{s-}^{\epsilon}(p)\Pi_{s-}^{\epsilon}(p,dp'), Z_s^{M,\epsilon}\rangle ds$$
$$-\frac{1}{M}\int_0^t \langle(2f(s,0,p)-f(s,a,p))^2 B(a,p)$$
$$+\int_{Space}(f(s,a,p')-f(s,a,p))^2 P_{s-}^{\epsilon}(p)\Pi_{s-}^{\epsilon}(p,dp'), Z_s^{M,\epsilon}\rangle ds = -\mathcal{M}_t^{M,\epsilon}(f)^2,$$

which means that

$$\mathcal{M}_t^{M,\epsilon}(f)^2 = Local_{Martingale} + \frac{1}{M}\int_0^t \langle(2f(s,0,p)-f(s,a,p))^2 B(a,p)$$
$$+\int_{Space}(f(s,a,p')-f(s,a,p))^2 P_{s-}^{\epsilon}(p)\Pi_{s-}^{\epsilon}(p,dp'), Z_s^{M,\epsilon}\rangle ds,$$

where

$$Local_{Martingale} = -\langle f_t, Z_t^{M,\epsilon}\rangle^2 + \langle f_0, Z_0^{M,\epsilon}\rangle^2$$
$$+2\int_0^t \langle f_u, Z_u^{M,\epsilon}\rangle\langle(\frac{\partial}{\partial u}+\frac{\partial}{\partial a})f(u,a,p), Z_u^{M,\epsilon}(da,dp)\rangle du$$
$$+2\int_0^t \langle f_s, Z_s^{M,\epsilon}\rangle\langle(2f(s,0,p)-f(s,a,p))B(a,p)$$
$$+\int_{Space}(f(s,a,p')-f(s,a,p))P_{s-}^{\epsilon}(p)\Pi_{s-}^{\epsilon}(p,dp'), Z_s^{M,\epsilon}\rangle ds.$$

Using uniqueness of Doob decomposition we find that (4.8) is satisfied and

$$\mathbb{E}(\langle\mathcal{M}_t^{M,\epsilon}(f)\rangle) \leq$$
$$\frac{T}{M}3(1+C)\|f\|_{L^\infty}^2\mathbb{E}\Big(\sup_{t\in[0,T]}\langle 1, Z_t^{M,\epsilon}(da,dp)\rangle + \sup_{t\in[0,T]}\langle 1, Z_t^{M,\epsilon}(da,dp)\rangle^2\Big) < \infty.$$

Therefore $\mathcal{M}_t^{M,\epsilon}(f)$ is a L^2 martingale of angle brackets process (4.8). We have directly (using Rolle's theorem) that $\mathcal{M}_t^{M,\epsilon}(f)$ and $\mathcal{V}_t^{M,\epsilon}(f)$ satisfy (4.10)-(4.12). □

4.3. *Uniform bounds on $Z_t^{M,\epsilon}$*

Lemma 4.3. *Let $q \geq 1$ and assume that $\sup_{M,\epsilon}\mathbb{E}\Big((\iint 1 Z_0^{M,\epsilon}(da,dp))^q\Big) < \infty$, then we have that*
$C_0^T = \sup_{M,\epsilon}\mathbb{E}\Big((\sup_{t\in[0,T]}\iint 1 Z_t^{M,\epsilon}(da,dp))^q\Big) < \infty, \quad \forall T > 0.$

Proof. Using equation (4.3) for $F : x \mapsto x^q$ and $f = 1$, we find that

$$0 = (\langle 1, Z_t^{M,\epsilon}\rangle)^q - (\langle 1, Z_0^{M,\epsilon}\rangle)^q - \int_0^t \int_\varepsilon 1_{n<N_{s-}^M} \Big[((\langle 1, Z_s^{M,\epsilon}\rangle + \frac{1}{M})^q - \langle 1, Z_s^{M,\epsilon}\rangle^q) 1_{0\le\Theta<B_{n,s-}^{M,\epsilon}} Q(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp'))\Big].$$

Since $(1+y)^q - y^q \le q2^{q-1}(1+y^{q-1})$, we have that

$$\langle 1, Z_t^{M,\epsilon}\rangle^q \le \langle 1, Z_0^{M,\epsilon}\rangle^q + q2^{q-1}\int_0^t \int_\varepsilon 1_{n<N_{s-}^M} \Big[(1+\langle 1, Z_s^{M,\epsilon}\rangle^{q-1}) 1_{0\le\Theta<B_{n,s-}^{M,\epsilon}} Q(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp'))\Big],$$

$$\sup_{u\le\min(T_N^M,t)} \langle 1, Z_u^{M,\epsilon}\rangle^q \le \langle 1, Z_0^{M,\epsilon}\rangle^q + q2^{q-1}\int_0^t \int_\varepsilon 1_{n<N_{s-}^M} \Big[(1+\langle 1, Z_s^{M,\epsilon}\rangle^{q-1}) 1_{0\le\Theta<B_{n,s-}^{M,\epsilon}} Q(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp'))\Big],$$

and

$$\mathbb{E}(\sup_{u\le\min(T_N^M,t)} \langle 1, Z_u^{M,\epsilon}\rangle^q) \le \mathbb{E}(\langle 1, Z_0^{M,\epsilon}\rangle^q) + q2^{q-1}\int_0^t \int_{Age\times Space} \mathbb{E}((1+\langle 1, Z_s^{M,\epsilon}\rangle^{q-1}) Z_s^{M,\epsilon}) ds.$$

Using the stopping time $\tau_N^{M,\epsilon}$ with $\bar{N}_t = E(\sup_{s\le\min(t,\tau_N^{M,\epsilon})} N_s)$ and noticing that $x^q + x \le 2(1+x)$ for all $x \ge 0$ and $q \ge 1$, we have that

$$\mathbb{E}(\sup_{u\le\min(T_N^M,t)} \langle 1, Z_u^{M,\epsilon}\rangle^q) \le [\mathbb{E}(\langle 1, Z_0^{M,\epsilon}\rangle^q) + tq2^q] + q2^q \int_0^t \mathbb{E}(\sup_{u\le\min(T_N^M,s)} \langle 1, Z_u^{M,\epsilon}\rangle^q) ds.$$

Now, using by Gronwall lemma, we find that

$$\mathbb{E}(\sup_{u\le\min(\tau_N^{M,\epsilon},t)} \langle 1, Z_u^{M,\epsilon}\rangle^q) \le [\mathbb{E}(\langle 1, Z_0^{M,\epsilon}\rangle^q) + tq2^q] e^{q2^q t}.$$

Thus, $\lim_{N\to\infty} \tau_N^{M,\epsilon} = \infty$ for all (M,ϵ) and for all $t \le T$

$$C_0^t = \sup_{M,\epsilon} \mathbb{E}((\sup_{u\le t} \langle 1, Z_u^{M,\epsilon}\rangle)^q) \le [\sup_{M,\epsilon} \mathbb{E}(\langle 1, Z_0^{M,\epsilon}\rangle^q) + tq2^q] e^{q2^q t}. \quad (4.17)$$

□

Lemma 4.4. *Let $m \ge 1$ and assume that (where $dp = r^2 \sin(\theta) dr d\theta d\phi$)*

$$\sup_{M,\epsilon} \mathbb{E}\Big(\iint a^m Z_0^{M,\epsilon}(da,dp))\Big) < \infty, \quad (4.18)$$

then we have that

$$C_m^T = \sup_{M,\epsilon} \mathbb{E}\Big(\sup_{t\in[0,T]} \iint a^m Z_t^{M,\epsilon}(da,dp)\Big) < \infty, \quad \forall T > 0. \quad (4.19)$$

Proof. We have that $Z_u^{M,\epsilon}$ satisfies (4.3), for $F : x \mapsto x$ and $f(a) = a^m$, we find that

$$\langle a^m, Z_t^{M,\epsilon}\rangle = \langle a^m, Z_0^{M,\epsilon}\rangle + m\int_0^t \iint_{\mathbb{R}_+\times Space} a^{m-1} Z_u^{M,\epsilon}(da,dp)du$$
$$- \frac{1}{M}\Big[\int_0^t \int_\varepsilon 1_{n<N_{s-}^M} A_n^{M,\epsilon}(s-)^m 1_{0\le\Theta<B_{n,s-}^{M,\epsilon}} Q(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp'))\Big],$$

$$\langle a^m, Z_t^{M,\epsilon}\rangle \le \langle a^m, Z_0^{M,\epsilon}\rangle + m\int_0^t \iint_{\mathbb{R}_+\times Space} a^{m-1} Z_u^{M,\epsilon}(da,dp)du,$$

and finally we find that

$$\mathbb{E}\Big(\langle a^m, Z_t^{M,\epsilon}\rangle\Big) \le \mathbb{E}\Big(\langle a^m, Z_0^{M,\epsilon}\rangle\Big) + m\int_0^t \mathbb{E}\Big(\langle a^{m-1} Z_u^{M,\epsilon}(da,dp)\rangle\Big)du.$$

Using that $a^{m-1} \le a^m + 1$ and (4.17), we have that

$$\mathbb{E}\Big(\sup_{u\le\min(\tau_N^{M,\epsilon},t)} \langle a^m, Z_u^{M,\epsilon}\rangle\Big) \le \mathbb{E}\Big(\langle a^m, Z_0^{M,\epsilon}\rangle\Big) + mt\Big[\sup_{M,\epsilon}\mathbb{E}(\langle 1, Z_0^{M,\epsilon}\rangle) + 2t]e^{2t}\Big]$$
$$+ m\int_0^t \mathbb{E}\Big(\langle a^m, Z_u^{M,\epsilon}(da,dp)\rangle\Big)du.$$

Using Gronwall lemma we find

$$\mathbb{E}\Big(\sup_{u\le\min(\tau_N^{M,\epsilon},t)} \langle a^m, Z_u^{M,\epsilon}\rangle\Big) \le \Big(\mathbb{E}\Big(\langle a^m, Z_0^{M,\epsilon}\rangle\Big) + mt[\sup_{M,\epsilon}\mathbb{E}(\langle 1, Z_0^{M,\epsilon}\rangle) + 2t]e^{2t}\Big)e^{mt}.$$

Therefore, under assumption (4.18), we prove that (4.19) is satisfied. □

Lemma 4.5. *Let $m \ge 1$ and assume that (where $dp = r^2\sin(\theta)drd\theta d\phi$)*

$$\sup_{M,\epsilon}\mathbb{E}\Big(\iint r^m Z_0^{M,\epsilon}(da,dp))\Big) < \infty, \quad \sup_{M,\epsilon}\mathbb{E}\Big(\iint 1 Z_0^{M,\epsilon}(da,dp))\Big) < \infty, \tag{4.20}$$

then we have that

$$D_m^T = \sup_{M,\epsilon}\mathbb{E}\Big(\sup_{t\in[0,T\epsilon]} \iint r^m Z_t^{M,\epsilon}(da,dp)\Big) < \infty, \quad \forall T > 0. \tag{4.21}$$

Proof. We have that $Z_u^{M,\epsilon}$ satisfies (4.3), for $F : x \mapsto x$ and $f(a) = r^m$, we find that

$$\sup_{u\le\min(\tau_N^{M,\epsilon},t)} \langle r^m, Z_u^{M,\epsilon}\rangle = \langle r^m, Z_0^{M,\epsilon}\rangle + \frac{1}{M}\Big[\int_0^t\int_\varepsilon 1_{n<N_{s-}^M}\Big[(X_n^{M,\epsilon}(s-))^m 1_{0\le\Theta<B_{n,s-}^{M,\epsilon}}$$
$$+ ((p')^m - (X_n^{M,\epsilon}(s-))^m) 1_{B_{n,s-}^{M,\epsilon}\le\Theta<B_{n,s-}^{M,\epsilon}+P_{s-}^\epsilon)}\Big] Q(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp'))\Big],$$

and finally we find that

$$\mathbb{E}\Big(\sup_{u\le\min(\tau_N^{M,\epsilon},t)}\langle r^m, Z_u^{M,\epsilon}\rangle\Big)\le\mathbb{E}\Big(\langle r^m, Z_0^{M,\epsilon}\rangle\Big)$$
$$+\int_0^t\mathbb{E}\Big(\langle r^m, Z_u^{M,\epsilon}\rangle\Big)+\|P_{s-}^\epsilon\|\epsilon m2^m(2\mathbb{E}\Big(\langle 1, Z_u^{M,\epsilon}\rangle\Big)+\mathbb{E}\Big(\langle r^m, Z_u^{M,\epsilon}\rangle\Big))du,$$

$$\mathbb{E}\Big(\sup_{u\le\min(\tau_N^{M,\epsilon},t)}\langle r^m, Z_u^{M,\epsilon}\rangle\Big)\le\mathbb{E}\Big(\langle r^m, Z_0^{M,\epsilon}\rangle\Big)$$
$$+\int_0^t\mathbb{E}\Big(\langle r^m, Z_u^{M,\epsilon}\rangle\Big)(1+\|P_{s-}^\epsilon\|\epsilon m2^m)du+\|P_{s-}^\epsilon\|\epsilon m2^{m+1}tC_0^T.$$

Using Gronwall lemma we find that for all $t<T$:

$$\mathbb{E}\Big(\sup_{u\le\min(\tau_N^{M,\epsilon},t\epsilon)}\langle r^m, Z_u^{M,\epsilon}\rangle\Big)\le$$
$$\Big(\mathbb{E}\Big(\langle r^m, Z_0^{M,\epsilon}\rangle+\|P_{s-}^\epsilon\|\epsilon^2m2^{m+1}tC_0^T\Big)e^{(1+\|P_{s-}^\epsilon\|\epsilon^2m2^m)t},$$

with and (4.21) holds. □

Lemma 4.6. *Let $m\ge1$ and assume that*

$$\sup_{M,\epsilon}\mathbb{E}\Big((\iint a^mZ_0(da,dp))^2\Big)<\infty,\quad \sup_{M,\epsilon}\mathbb{E}\Big((\iint a^{2m}Z_0(da,dp))\Big),\tag{4.22}$$

then we have that

$$C_{m,2}^T=\sup_{M,\epsilon}\mathbb{E}\Big((\sup_{t\in[0,T]}\iint a^mZ_t^{M,\epsilon}(da,dp))^2\Big)<\infty,\quad\forall T>0.\tag{4.23}$$

Proof. We have that (see lemma 4.2), for all $t\le T$,

$$(\langle a^m, Z_t^{M,\epsilon}\rangle)^2=(\langle a^m, Z_0^{M,\epsilon}\rangle)^2$$
$$+2m\int_0^t(\langle a^m, Z_u^{M,\epsilon}\rangle)\iint_{\mathbb{R}_+\times Space}a^{m-1}Z_u^{M,\epsilon}(da,dp)du$$
$$+\int_0^t\int_\varepsilon 1_{n<N_{s-}^M}\Big[(-\frac{2}{M}A_n^{M,\epsilon}(s-)^m(\langle a^m, Z_s^{M,\epsilon}\rangle)+\frac{1}{M^2}A_n^{M,\epsilon}(s-)^{2m})1_{0\le\Theta<B_{n,s-}^{M,\epsilon}}\Big]$$
$$Q(ds,(dn),d\Theta,\Pi_{s-}^\epsilon(p,dp')),$$

therefore, we find that,

$$\mathbb{E}\Big(\sup_{u\le\min(\tau_N^{M,\epsilon},t)}(\langle a^m, Z_u^{M,\epsilon}\rangle)^2\Big)\le\mathbb{E}\Big((\langle a^m, Z_0^{M,\epsilon}\rangle)^2\Big)$$
$$+2m\int_0^t\mathbb{E}\Big((\langle a^m, Z_u^{M,\epsilon}\rangle)\iint_{\mathbb{R}_+\times Space}a^{m-1}Z_u^{M,\epsilon}(da,dp)\Big)du+\int_0^t\frac{1}{M}\mathbb{E}\Big(\langle a^{2m}, Z_u^{M,\epsilon}\rangle\Big)du,$$

and using the lemma 4.4 and the Gronwall lemma (and noticing that $a^{m-1} \leq a^m + 1$), (4.23) holds. □

Lemma 4.7. *Let $m \geq 1$ and assume that*

$$\sup_{M,\epsilon} \mathbb{E}\Big((\iint r^m Z_0(da, dp))^2 \Big) < \infty, \quad \sup_{M,\epsilon} \mathbb{E}\Big((\iint r^{2m} Z_0(da, dp)) \Big),$$

and (2.3), then we have that

$$D^T_{m,2} = \sup_{M,\epsilon} \mathbb{E}\Big((\sup_{t\in[0,T\epsilon]} \iint r^m Z^{M,\epsilon}_t (da, dp))^2 \Big) < \infty, \quad \forall T > 0.$$

The proof of this lemma is similar to the previous ones.

5. Partial differential Equations calculus and lemmas

In this part, we prove the existence, regularity and compactness of solution to the intermediate equation.

5.1. *Proof of existence and uniqueness of solution to (1.7)*

Lemma 5.1. *(Existence/Uniqueness) Let $Z^\epsilon_0(a, p) = \rho^\epsilon_0(a, p)dadp$ satisfying (2.6) with $\rho^\epsilon_0 \in C^1$ and*

$$\rho^\epsilon_0(a = 0, p) = 2 \int B(a, p)\rho^\epsilon_0(a, p)da, \tag{5.1}$$

then $Z^\epsilon_t(a, p) = \lim_{M_k\to\infty} Z^{M_k,\epsilon}_t(da, dp)$ is equal to $\rho^\epsilon(t, a, p)dadp$ with ρ^ϵ C^1 solution to (1.7).

Proof. Let $\mathcal{T} : g \mapsto f$ solution to

$$\begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})f - K[g](t, p) = -K_T(t, p)f(t, a, p), \\ f(t, 0, p) = 2 \int B(a, p)f(t, a, p)da, \quad f(t = 0, ., .) = \rho^\epsilon_0(., .), \end{cases} \tag{5.2}$$

with $K[f](t, p) = \int \tilde{P}^\epsilon_\infty(p')\frac{\tilde{\Pi}^\epsilon_\infty(p',dp)}{dp} f(t, a, p')dp'$, $K_T(t, p) = \tilde{P}^\epsilon_\infty(p)$,

$$\tilde{P}^\epsilon_\infty(p') = \frac{C}{\epsilon^2} \sum_{i,j,k} \Psi^\epsilon_{i,j,k}(p')R(\frac{\int \Psi^\epsilon_{i,j,k}(p)gdadpVol_G}{Vol^\epsilon_{i,j,k}}),$$

and $\frac{\tilde{\Pi}^\epsilon_\infty(p',dp)}{dp} = \frac{\Psi^\epsilon(|p-p'|)\tilde{G}^\epsilon_{s-}(p)}{\int \Psi^\epsilon(|q-p'|)\tilde{G}^\epsilon_{s-}(q)dq}$, with $\tilde{G}^\epsilon_{s-}(.) = 1 - \sum_{i,j,k} \Psi^\epsilon_{i,j,k}(.)R(\frac{\int \Psi^\epsilon_{i,j,k}(p)gdadpVol_G}{Vol^\epsilon_{i,j,k}})$. Using Gronwall lemma and $\|B\|_\infty \leq 1$, we have that

$$\iint f(t, a, p)dadp \leq \iint \rho^\epsilon_0(a, p)dadpe^{2t}.$$

By computation, and using that $|R'| \leq \frac{1}{\sigma}$, we find that

$$|K[g] - K[\tilde{g}]|(t,p) \leq \frac{3Vol_G}{\sigma(1 - R(\iint \rho_0^\epsilon(a,p)dadpe^{2t}))Vol^\epsilon_{i,j,k}} \iint |g - \tilde{g}|dadp.$$

Therefore, using the Characteristics (in (a,t)) of the transport equation (5.2) (see 20), we have existence and uniqueness of the solution. Moreover, for $\phi : t \mapsto \int_t^\infty 2e^{-\frac{C}{\epsilon^2}(s-t)}ds$, and $f = \mathcal{T}(g)$, $\tilde{f} = \mathcal{T}(\tilde{g})$, we have that (multiplying (5.2) by ϕ and integrating in (a,p))

$$\frac{d}{dt}\iint |f - \tilde{f}|\phi(t)dadp \leq \frac{3Vol_G}{\sigma(1 - R(\iint \rho_0^\epsilon(a,p)dadpe^{2t}))Vol^\epsilon_{i,j,k}} \iint |g - \tilde{g}|\phi(t)dadp,$$

and so

$$\iint |f-\tilde{f}|\phi(t)dadp \leq \int_0^t \frac{3Vol_G}{\sigma(1 - R(\iint \rho_0^\epsilon(a,p)dadpe^{2s}))Vol^\epsilon_{i,j,k}} \iint |g-\tilde{g}|\phi(s)dadpds.$$

Therefore $\mathcal{T} : g \in E \mapsto f \in E$ with $E = C([0,T], L^1(\mathbb{R}_+ \times \mathbb{R}^3))$, is a contracting mapping for $T > 0$ small enough and there exists an unique solution to (1.7). Now, we construct a solution on $[T, 2T]$... and finally on $\mathbb{R}_+$. Since Ψ_ϵ and R are C^∞, the regularity of ρ^ϵ, solution to the transport equation (1.7) with the boundary condition 5.1, is given by those of ρ_0^ϵ (under the assumption (5.1), see 20, 18, 11, 27). Using uniqueness, we find that $Z_t^\epsilon(a,p) = \rho^\epsilon(t,a,p)dadp$. □

5.2. *Compactness of $M_\epsilon^1(t,p) := \int \rho^\epsilon(t,a,p)da$*

Lemma 5.2. *Let $T > 0$, ρ^ϵ solution to (1.7) and assume that (2.4) is satisfied then we can extract a convergent subsequence of $M_\epsilon^1(t,p) := \int \rho^\epsilon(t,a,p)da$ in $L^2([0,T] \times$ Space).*

Proof. Since $M_\epsilon^1(0,p) \in L^2(Space)$ by assumption (2.4), we have, by integrating equation (1.7) that $M_\epsilon^1(t,p) := \int \rho^\epsilon(t,a,p)da$ follows the equation

$$\begin{cases} \frac{\partial}{\partial t}M_\epsilon^1(t,p) - \int P_\infty^\epsilon(p')\frac{\Pi_\infty^\epsilon(p',dp)}{dp}M_\epsilon^1(t,p')dp' + P_\infty^\epsilon(p)M_\epsilon^1(t,p) = 2\int B(a,p)\rho^\epsilon(t,a,p)da, \\ M_\epsilon^1(t=0,p) = \int \rho_0^\epsilon(a,p)da. \end{cases} \tag{5.3}$$

First integrating (5.3) with respect to p and Gronwall lemma, we have that

$$\int M_\epsilon^1(0,p)dp \leq \int M_\epsilon^1(t,p)dp \leq \int M_\epsilon^1(0,p)dpe^{2T}, \quad \forall t \in [0,T],$$

therefore ρ^ϵ is uniformly bounded in $L^\infty([0,T], L^1(Age \times Space))$. Secondly, multiplying (5.3) by M_ϵ^1 and integrating with respect to p we find that

$$\frac{\partial}{\partial t}\frac{1}{2}\int (M_\epsilon^1(t,p))^2dp = \iint B(a,p)\rho^\epsilon(t,a,p)daM_\epsilon^1(t,p)dp$$
$$\iint [\frac{P_\infty^\epsilon(p')M_\epsilon^1(t,p')}{\int \Psi^\epsilon(|q-p'|)G^\epsilon_{s-}(q)dq}G^\epsilon_{s-}(p)](M_\epsilon^1(t,p) - M_\epsilon^1(t,p'))\Psi^\epsilon(|p'-p|)dpdp',$$

that can be rewritten as,

$$\frac{\partial}{\partial t}\frac{1}{2}\int (M^1_\epsilon(t,p))^2 dp = \iint B(a,p)\rho^\epsilon(t,a,p)da M^1_\epsilon(t,p)dp$$
$$+\frac{1}{2}\iint [\frac{P^\epsilon_\infty(p')M^1_\epsilon(t,p')}{\int \Psi^\epsilon(|q-p'|)G^\epsilon_{s-}(q)dq}G^\epsilon_{s-}(p) - \frac{P^\epsilon_\infty(p)M^1_\epsilon(t,p)}{\int \Psi^\epsilon(|q-p|)G^\epsilon_{s-}(q)dq}G^\epsilon_{s-}(p')]$$
$$(M^1_\epsilon(t,p) - M^1_\epsilon(t,p'))\Psi^\epsilon(|p'-p|)dpdp'.$$

Let $p_+ = (p+p')/2$ and $p_- = (p-p')/2$, then we have $p = p_+ + p_-$, $p' = p_+ - p_-$ and for all A, B C^1-functions we have that

$$\begin{aligned} A(p')B(p) - A(p)B(p') &= A(p_+ - p_-)B(p_+ + p_-) - B(p_+ - p_-)A(p_+ + p_-) \\ &= (A(p_+) - \nabla A(p_+)p_- + o(p_-))(B(p_+) + \nabla B(p_+)p_- + o(p_-)) \\ &\quad -(A(p_+) + \nabla A(p_+)p_- + o(p_-))(B(p_+) - \nabla B(p_+)p_- + o(p_-)) \\ &= 2(A(p_+)\nabla B(p_+)p_- - B(p_+)\nabla A(p_+)p_-) + o(p_-). \end{aligned}$$

Noticing that $M^1_\epsilon(t,p) - M^1_\epsilon(t,p') = 2\nabla M^1_\epsilon(t,p_+)p_- + o(p_-)$ and changing the variables in the integral, we find that

$$\frac{\partial}{\partial t}\frac{1}{2}\int (M^1_\epsilon(t,p))^2 dp = 4\iint [(\frac{P^\epsilon_\infty M^1_\epsilon(t,.)}{\int \Psi^\epsilon(|q-.|)G^\epsilon_{s-}(q)dq}(p_+)\nabla G^\epsilon_{s-}(p_+)2p_-$$
$$- G^\epsilon_{s-}(p_+)\nabla\frac{P^\epsilon_\infty M^1_\epsilon(t,.)}{\int \Psi^\epsilon(|q-.|)G^\epsilon_{s-}(q)dq}(p_+)2p_-]$$
$$(2\nabla M^1_\epsilon(t,p_+)p_-)\Psi^\epsilon(2|p_-|)dp_+dp_- + \iint B(a,p)\rho^\epsilon(t,a,p)da M^1_\epsilon(t,p)dp + o(1).$$

Using the definition of P^ϵ_∞ (1.8)-(1.9) we find that

$$\frac{\partial}{\partial t}\frac{1}{2}\int (M^1_\epsilon(t,p))^2 dp = 4C\iint [M^1_\epsilon(t,.)\frac{R}{1-R}(M^1_\epsilon(t,.))(1-R)'\nabla M^1_\epsilon(t,p_+)2\frac{p_-}{\epsilon}$$
$$-M^1_\epsilon(t,p_+)(1-R)(\frac{R}{1-R})'(M^1_\epsilon(t,p_+))\nabla M^1_\epsilon(p_+)2\frac{p_-}{\epsilon} - R(M^1_\epsilon(t,p_+))\nabla M^1_\epsilon(p_+)2\frac{p_-}{\epsilon}]$$
$$(2\nabla M^1_\epsilon(t,p_+)\frac{p_-}{\epsilon})\Psi^\epsilon(2|p_-|)dp_+dp_- + \iint B(a,p)\rho^\epsilon(t,a,p)da M^1_\epsilon(t,p)dp + o(1).$$

and so

$$\frac{\partial}{\partial t}\frac{1}{2}\int (M^1_\epsilon(t,p))^2 dp = -4C\int [1 + \frac{2M^1_\epsilon(t,p_+)}{\sigma}]R(M^1_\epsilon(t,p_+))(\nabla M^1_\epsilon(t,p_+)$$
$$\int 2\frac{p_-}{\epsilon}\Psi^\epsilon(2|p_-|)2\frac{{}^tp_-}{\epsilon}dp_- \,{}^t\nabla M^1_\epsilon(t,p_+))\Psi^\epsilon(2|p_-|)dp_+$$
$$+ \iint B(a,p)\rho^\epsilon(t,a,p)da M^1_\epsilon(t,p)dp + o(1).$$

Therefore, we have the following bound

$$\frac{\partial}{\partial t}\frac{1}{2}\int (M_\epsilon^1(t,p))^2 dp + 4CR(0)\lambda_m \iint [1+\frac{2M_\epsilon^1(t,p_+)}{\sigma}]\|\nabla M_\epsilon^1(t,.)\|^2(t,p_+)dp_- \leq \int (M_\epsilon^1(t,p))^2 dp,$$

with $\lambda_m > 0$ the infimum (with respect of ϵ) of the minimum of the eigenvalues of the definite positive matrix $\int 2\frac{p_-}{\epsilon}\Psi^\epsilon(2|p_-|)2\frac{{}^t p_-}{\epsilon}dp_-$. Finally, using Gronwall inequality we have that for all $T>0$

$$\int (M_\epsilon^1(t,p))^2 dp \leq \int (M_\epsilon^1(0,p))^2 dp e^{2T}, \quad \forall t \leq T,$$

and integrating the last inequality, we found that

$$\int_{[0,T]}\iint [1+\frac{2M_\epsilon^1(t,p_+)}{\sigma}]\|\nabla M_\epsilon^1(t,.)\|^2(t,p_+)dp_- \leq \frac{\int (M_\epsilon^1(0,p))^2 dp}{4CR(0)\lambda_m}e^{2T}(T+1).$$

We notice that, for all $T>0$, the same computation leads to (multiplying (5.3) by $(pM_\epsilon^1)^2$ and integrating with respect to p)

$$\int p(M_\epsilon^1(t,p))^2 dp \leq [\int p(M_\epsilon^1(0,p))^2 dp + T\|M_\epsilon^1\|_{C([0,T],H^1(Space))}\lambda_M]e^{2T}, \quad \forall t \leq T,$$

with $\lambda_M > 0$ the supremum (with respect of ϵ) of the maximum of the eigenvalues of the definite positive matrix $\int 2\frac{p_-}{\epsilon}\Psi^\epsilon(2|p_-|)2\frac{{}^t p_-}{\epsilon}dp_-$.Thus, we have that $(M_\epsilon^1)_\epsilon$ is uniformly bounded in $L^2([0,T],H^1(Space)\times L^2_{pdp}(Space))$. Moreover, we have directly that $|\frac{\partial}{\partial t}\frac{1}{2}\int (M_\epsilon^1(t,p))^2 dp|$ is uniformly bounded in $L^2([0,T])$. Therefore, by the (Lions-Aubin) result (see 14), we can extract a convergent subsequence of $(M_\epsilon^1)_\epsilon$ in $L^2([0,T]\times(Space))$. □

Lemma 5.3. *Let $T>0$. Assume that $(\nabla M_\epsilon^1(t,.))_\epsilon$ is $L^\infty([0,T];L^2(Space))$ and M_ϵ^1 converges $C^0([0,T];L^2(Space))$ to M^1 as ϵ to 0 then*

$$\sum_{i,j,k}\Psi_{i,j,k}^\epsilon(p')R(\frac{\int \Psi_{i,j,k}^\epsilon(p)M_\epsilon^1(t,.)dadpVol_G}{Vol_{i,j,k}^\epsilon}) \to_{C^0([0,T];L^2(Space))} R(\frac{Vol_G(3r^2+6r+3)}{3r^2+6rr_O(s)+3r_O(s)^2}M^1(t,p)).$$

Proof. Indeed, we have that

$$\int |\sum_{i,j,k} \Psi^\epsilon_{i,j,k}(p')R(\frac{\int \Psi^\epsilon_{i,j,k}(p)M^1_\epsilon(t,.)dadpVol_G}{Vol^\epsilon_{i,j,k}})$$
$$- R(\frac{Vol_G(3r^2+6r+3)}{3r^2+6rr_O(s)+3r_O(s)^2}M^1_\epsilon(t,p))|^2dp$$
$$\leq \|R'\|^2_\infty \int (\int_{u\leq 1} |M^1_\epsilon(p+\epsilon u) - M^1_\epsilon(p)|du)^2dp$$
$$\leq \epsilon^2 \int (\int_{u\leq 1}\int_{s\in[0,1]} |\nabla M^1_\epsilon(p+\epsilon su)|duds)^2dp$$
$$\leq \epsilon^2 \int_{u\leq 1}\int_{s\in[0,1]} (\int |\nabla M^1_\epsilon(p+\epsilon su)|^2dp)duds,$$

and so we find that

$$\int (R(\frac{Vol_G(3r^2+6r+3)}{3r^2+6rr_O(s)+3r_O(s)^2}M^1(t,p))$$
$$- R(\frac{Vol_G(3r^2+6r+3)}{3r^2+6rr_O(s)+3r_O(s)^2}M^1_\epsilon(t,p)))^2dp$$
$$\leq C\|R'\|^2_\infty \int (M^1(t,p) - M^1_\epsilon(t,p))^2dp,$$

with $C > 0$. Under the uniform boundness of $(\nabla M^1_\epsilon(t,.))_\epsilon$ and the $C^0([0,T];L^2(Space))$- convergence of M^1_ϵ we have that $\sum_{i,j,k} \Psi^\epsilon_{i,j,k}(p')R(\frac{\int \Psi^\epsilon_{i,j,k}(p)M^1_\epsilon(t,.)dadpVol_G}{Vol^\epsilon_{i,j,k}})$ converges (as $\epsilon \to 0$) to $R(M^1(t,p))$ in $C^0([0,T];L^2(Space))$. □

Lemma 5.4. *Let $T > 0$, ρ^ϵ solution to (1.7) and assume that (2.4) is satisfied then we can extract a convergent subsequence of $M^1_\epsilon(t,p) := \int \rho^\epsilon(t,a,p)da$ which limit is a weak solution to*

$$\frac{\partial}{\partial t}M^1(t,p) + div(M^1(t,p)Ch(p)R\nabla\Big(\log(1-R)\Big)) = H,$$

$$M^1(0,p) = \int \rho^\epsilon_0(a,p)da,$$

and $H \in C([0,T],H^1(Space))$ and R defined in (1.10).

Proof. Let $f \in C^1_c(\mathbb{R}_+ \times Space)$, multiplying (1.7) by f and integrating with respect to a we find that :

$$\frac{\partial}{\partial t}\int_p f(p)M^1_\epsilon(t,p)dp - \iint B(a,p)f(p)\rho^\epsilon(t,a,p)dadp$$
$$= \int_{p,p'} \Big(f(p') - f(p)\Big)\Pi^\epsilon_{s-}(p,dp')P^\epsilon_{s-}(p)M^1_\epsilon(t,p)dp.$$

Rewritting the second member as follows, we have that

$$\int_{p,p'} \Big(f(p') - f(p)\Big)\Pi^\epsilon_{s-}(p,dp')P^\epsilon_{s-}(p)M^1_\epsilon(t,p)dp$$
$$= \int_{p,p'} C^\epsilon\Big(f(p') - f(p)\Big)\frac{\Psi^\epsilon(|p'-p|)G^\epsilon_{s-}(p')dp'}{\int \Psi^\epsilon(|q-p|)G^\epsilon_{s-}(q)dq}\frac{P^\epsilon_{s-}(p)}{C^\epsilon}M^1_\epsilon(t,p)dp$$
$$= \int_p {}^t\nabla f(p)\frac{C^\epsilon \int_q \Psi^\epsilon(|q-p|)(q-p)G^\epsilon_{s-}(q)dq}{\int \Psi^\epsilon(|q-p|)G^\epsilon_{s-}(q)dq}\frac{P^\epsilon_{s-}(p)}{C^\epsilon}M^1_\epsilon(t,p)dp + o(1)$$
$$= \int_p {}^t\nabla f(p)\frac{C^\epsilon \int_q \Psi^\epsilon(|q-p|)(q-p)\,{}^t(q-p)dq}{\int \Psi^\epsilon(|q-p|)dq}\frac{P^\epsilon_{s-}(p)}{C^\epsilon}M^1_\epsilon(t,p)\frac{\nabla G^\epsilon_{s-}}{G^\epsilon_{s-}}(p)dp+o(1)$$
$$= \int_p \nabla f(p)M^1_\epsilon(t,p)Ch(p)R^t_\epsilon\nabla\Big(\log(1-R_\epsilon)\Big)dp + o(1).$$

Since, $M^1_\epsilon(t,p)$ is bounded $C([0,T],H^1(Space))$, we have that $\log(1-R_\epsilon) \to^{Distribution} \log(1-R)$ and by Banach Aologlu in L^2 (and identifying the limit)
$\nabla\log(1-R_\epsilon) \to^{*-weak\ L^2} \nabla\log(1-R)$. Now, by lemmas 5.2 and 5.3, we have that $\nabla f(p)M^1_\epsilon(t,p)Ch(p)R_\epsilon \to^{L^2} \nabla f(p)M^1(t,p)Ch(p)R$ and so (there exists $H \in C([0,T],L^2)$, the limit of $\iint B(a,p)f(p)\rho^\epsilon(t,a,p)dadp$ as $\epsilon\to 0$)

$$\frac{\partial}{\partial t}\int_p f(p)M^1(t,p)dp - \iint f(p)H(t,p)dp$$
$$= \int_p \nabla f(p)M^1(t,p)Ch(p)R^t\nabla\Big(\log(1-R)\Big)dp.$$

Therefore the limit is $C([0,T],H^1)$ and is a weak solution to

$$\frac{\partial}{\partial t}M^1(t,p) + div(M^1(t,p)Ch(p)R\nabla\Big(\log(1-R)\Big)) = H,$$

with $M^1(0,p) = \int \rho_0(a,p)da$. □

5.3. *Proof of existence and uniqueness of solution to (1.10)bis*

Lemma 5.5. *Let $T>0$, ρ^ϵ solution to (1.7) and assume that (2.4) is satisfied then we can extract a convergent subsequence of $(\rho^\epsilon)_\epsilon$ in $L^2([0,T]\times Space\times Age)$.*

Proof. Differentiate (1.7) with respect to a leads to

$$\begin{cases}(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\frac{\partial}{\partial a}\rho^\epsilon - \int P^\epsilon_\infty(p')\frac{\Pi^\epsilon_\infty(p',dp)}{dp}\frac{\partial}{\partial a}\rho^\epsilon(t,a,p')dp' + P^\epsilon_\infty(p)\frac{\partial}{\partial a}\rho^\epsilon(t,a,p) = 0,\\ \rho^\epsilon|_{a=0} = 2\int B(a,p)\rho^\epsilon(t,a,p)da, \quad \rho^\epsilon_{t=0} = \rho^\epsilon_0.\end{cases}$$

Therefore, we have that

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\int |\frac{\partial}{\partial a}\rho^\epsilon|dp \le 0,$$

and for all t, a and ϵ, we find that

$$\frac{\partial}{\partial t}\iint |\frac{\partial}{\partial a}\rho^\epsilon| dpda \le \int |\frac{\partial}{\partial a}\rho^\epsilon|(a=0)dp$$

Now, using the intermediate equation, we find that

$$\frac{\partial}{\partial a}\rho^\epsilon(a=0) = -2\int B(a,p)\frac{\partial}{\partial t}\rho^\epsilon(t,a,p)da$$
$$+2\int P^\epsilon_\infty(p')\frac{\Pi^\epsilon_\infty(p',dp)}{dp}\int B(a,p')\rho^\epsilon(t,a,p')dadp' - 2P^\epsilon_\infty(p)\int B(a,p)\rho^\epsilon(t,a,p)da,$$

and so, we have that

$$\frac{\partial}{\partial a}\rho^\epsilon(a=0) =$$
$$-2\int B(a,p)[-\frac{\partial}{\partial a}\rho^\epsilon + \int P^\epsilon_\infty(p')\frac{\Pi^\epsilon_\infty(p',dp)}{dp}\rho^\epsilon(t,a,p')dp' - P^\epsilon_\infty(p)\rho^\epsilon(t,a,p)]da$$
$$+2\int P^\epsilon_\infty(p')\frac{\Pi^\epsilon_\infty(p',dp)}{dp}\int B(a,p')\rho^\epsilon(t,a,p')dadp' - 2P^\epsilon_\infty(p)\int B(a,p)\rho^\epsilon(t,a,p)da,$$

$$\frac{\partial}{\partial a}\rho^\epsilon(a=0) = -2\int \rho^\epsilon\frac{\partial}{\partial a}B(a,p)da$$
$$-2\int B(a,p)[\int P^\epsilon_\infty(p')\frac{\Pi^\epsilon_\infty(p',dp)}{dp}\rho^\epsilon(t,a,p')dp' - P^\epsilon_\infty(p)\rho^\epsilon(t,a,p)]da$$
$$+2\int P^\epsilon_\infty(p')\frac{\Pi^\epsilon_\infty(p',dp)}{dp}\int B(a,p')\rho^\epsilon(t,a,p')dadp' - P^\epsilon_\infty(p)\int B(a,p)\rho^\epsilon(t,a,p)da,$$

and finally,

$$\frac{\partial}{\partial a}\rho^\epsilon(a=0) = -2\int \rho^\epsilon\frac{\partial}{\partial a}B(a,p)da$$
$$-2\int P^\epsilon_\infty(p')\frac{\Pi^\epsilon_\infty(p',dp)}{dp}\int (B(a,p) - B(a,p'))\rho^\epsilon(t,a,p')dadp'.$$

Therefore, using the same computation as lemma 5.2, we have that

$$\|\frac{\partial}{\partial a}\rho^\epsilon(a=0)\|_{L^2(Space)} \le 8\|B\|_{W^{1,\infty}}(1+\lambda_M^2)\sup_\epsilon \|M^1_\epsilon\|_{C([0,T],H^1(Space)}$$

with $\lambda_M > 0$ the sup (with respect of ϵ) of the maximum of the eigenvalues of the definite positive matrix $\int 2\frac{p_-}{\epsilon}\Psi^\epsilon(2|p_-|)2\frac{{}^t p_-}{\epsilon}dp_-$. And so, using the same computation as lemma 5.2, we have that

$$\rho^\epsilon \in C([0,T], H^1(Age \times Space).$$

Therefore, by the (Lions-Aubin) result (see 14), we can extract a convergent subsequence of $(\rho^\epsilon)_\epsilon$ in $L^2([0,T]\times(Space))$. □

6. Conclusion

In this paper, we prove the convergence of a stochastic process, which represent the evolution of a cell population, as its population size goes to infinity and its cell size converges to zero. A way to understand the result is to imagine that we observe the evolution of the cell population at a certain distance (depending of the number of cells). If the distance is fixed (i.e. a fixed cell size), then when the number of cells goes to infinity, the whole space is filled by cells (we are too close) and we can only observe $Z_t^{M,\epsilon} = Cst$ everywhere. If the distance is too far with respect to the size population, i.e. cells are too small, we observe a concentrated mass on the boundary of the oocyte (we are too far). To observe the cell population evolution we have to be, neither too close nor too far, which explains the balance between ϵ (cell size, or observation distance to the follicle) and M (cell population size). The proof is robust to the change of the birth rate B and cell displacement rate (function R), as long as there are smooth and bounded. The next step is to study the dynamics of the final equation (1.10) and make the link with the partial differential equation of the follicle evolution given in 19, 6, 7.

7. Annex

7.1. *Regular approximation of* $\chi_{\mathcal{L}^{\epsilon}_{i,j,k}}$

We introduce a regular approximation of the characteristic function $\chi_{[0,\epsilon]\times[0,\pi/N]\times[0,2\pi/N]}$ (see fig. 4). Let $\eta \in]0,1[$ and

$$\Psi^{\epsilon,\eta} : (r,\theta,\phi) \in \mathbb{R}^3 \mapsto \Psi^{\epsilon,\eta}_r(r)\Psi^{\epsilon,\eta}_\theta(\theta)\Psi^{\epsilon,\eta}_\phi(\phi), \tag{7.1}$$

a C^∞ positive function such that $\Psi^{\epsilon,\eta} \leq 1$, such that

$$(\Psi^{\epsilon}_r)|_{[\eta\epsilon,\epsilon(1-\eta)]} = 1, \quad \Psi^{\epsilon}_r|_{[-\eta\epsilon,\epsilon(1+\eta)]^c} = 0, \tag{7.2}$$

$$(\Psi^{\epsilon}_\theta)|_{[\frac{\eta}{N},\frac{\pi-\eta}{N}]} = 1,\ (\Psi^{\epsilon}_\theta)|_{[\frac{\pi+\eta}{N},\pi-\frac{\eta}{N}]} = 0,\ \Psi^{\epsilon}_\theta(\theta+\pi) = \Psi^{\epsilon}_\theta(\theta),\ \forall\theta, \tag{7.3}$$

$$(\Psi^{\epsilon}_\phi)|_{[-\pi+\frac{\eta}{N},-\pi+\frac{2\pi-\eta}{N}]} = 1,\ (\Psi^{\epsilon}_\phi)|_{[-\pi+\frac{2\pi+\eta}{N},\pi-\frac{\eta}{N}]} = 0,\ \Psi^{\epsilon}_\phi(\phi+2\pi) = \Psi^{\epsilon}_\phi(\phi),\ \forall\phi, \tag{7.4}$$

and satisfying the unity partition formulae

$$\sum_{i,j,k} \Psi^{\epsilon}_{i,j,k}(r,\theta,\phi) = 1, \quad \forall (r,\theta,\phi) \in [1+\eta\epsilon,\infty[\times[0,\pi]\times[-\pi,\pi].$$

with $\Psi^{\epsilon}_{i,j,k}$ a regular approximation of the characteristic function $\chi_{\mathcal{L}^{\epsilon}_{i,j,k}}$:

$$\Psi^{\epsilon}_{i,j,k}(r,\theta,\phi) = \Psi^{\epsilon}((r-1)-i\epsilon, \theta - \frac{j}{N}, \phi - \frac{k}{N}). \tag{7.5}$$

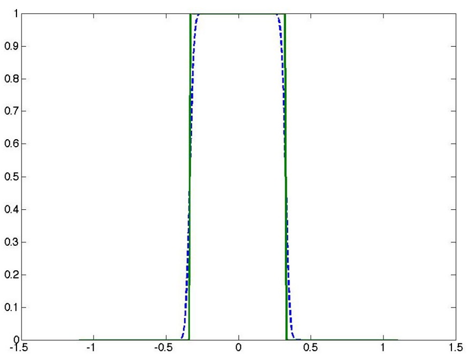

Fig. 4. Regularization of characteristic function χ. Using the convolution and Gaussian function it is easy to compute a regular approximation of χ which satisfies.

7.2. *Proof of proposition 2.1 : Existence of the stochastic process* $Z_t^{M,\epsilon}$

This process can be construct step by step (see 3, 15, 4, 25, 17), the only point is to prove global bounds

$$0 < E(\frac{N_0^{M,\epsilon}}{M}) \leq E(\sup_{s\leq t} N_s^{M,\epsilon}) \leq E(\frac{N_0^{M,\epsilon}}{M})e^{t/M} < \infty, \tag{7.6}$$

$$0 < r_O(0) \leq E(\sup_{s\leq t} r_O(s)) \leq r_O(0)e^{\frac{C}{\epsilon^2}E(\frac{N_0^{M,\epsilon}}{M})e^{t/M}t} < \infty, \tag{7.7}$$

where $N_s^{M,\epsilon} = \langle Z_s^{M,\epsilon}, 1\rangle$. Using (4.2), in the particular case $f = 1$, we have that (using (1.4), we have $B_{n,s-}^{M,\epsilon} \leq 1 < \infty$)

$$N_t^{M,\epsilon} = \iint_{\mathbb{R}_+\times Space} 1Z_t^{M,\epsilon}(da, dp) = \frac{1}{M}N_0^{M,\epsilon}$$
$$+ \frac{1}{M}\int_0^t \int_\varepsilon 1_{n<N_{s-}^{M,\epsilon}}\Big[11_{0\leq\Theta<B_{n,s-}^{M,\epsilon}}\Big]Q(ds,(dn), d\Theta, \Pi_{s-}^{M,\epsilon}(p, dp')).$$

Therefore, for $t \leq \tau_N^{M,\epsilon}$ (the stopping time $\tau_N^{M,\epsilon}$ is defined in (4.1)) and $N_t^{\bar{M},\epsilon} = E(\sup_{s\leq \min(t,\tau_N^{M,\epsilon})} N_s^{M,\epsilon})$, we have that

$$E(\frac{N_0^{M,\epsilon}}{M}) \leq N_t^{\bar{M},\epsilon} \leq E(\frac{N_0^{M,\epsilon}}{M}) + \int_0^t \frac{1}{M}N_s^{\bar{M},\epsilon}ds,$$

which implies (using Gronwall inequality on the right side) that

$$E(\frac{N_0^{M,\epsilon}}{M}) \leq \bar{N}_t \leq E(\frac{N_0^{M,\epsilon}}{M})e^{\frac{t}{M}}$$

and so, we find that

$$P(\inf_{M\geq N} \tau_N^{M,\epsilon} \geq t) = P(\tau_N^{M,\epsilon} \geq t) = P(\sup_{s\leq \min(t,\tau_N^{M,\epsilon})} N_s^{M,\epsilon} \geq N)$$
$$\leq E(\frac{N_0^{M,\epsilon}}{M})e^{\frac{t}{M}}/N,\ P(\lim_{N\to\infty} \tau_N^{M,\epsilon} = \infty) = 1.$$

Using Fatou lemma, we find that

$$E(\lim_{N\to\infty} \sup_{s\leq \min(t,\tau_N^{M,\epsilon})} N_s^{M,\epsilon}))$$
$$= E(\lim\inf_{N\to\infty} \sup_{s\leq \min(t,\tau_N^{M,\epsilon})} N_s^{M,\epsilon})) \leq \lim\inf_{N\to\infty} N_t^{\bar{M},\epsilon} \leq E(\frac{N_0^{M,\epsilon}}{M})e^{\frac{t}{M}},$$

and finally $E(\sup_{s\leq t} N_s^{M,\epsilon}) \leq E(\frac{N_0^{M,\epsilon}}{M})e^{\frac{t}{M}}$. Similarly, we have (7.7). Let $(T_k)_k$ the sequences of successive jumps of the $Z_s^{M,\epsilon}$ process, then, $\lim_k T_k = \infty$, *a.e.* Indeed, let $U_M \subset \Omega$, such that $\lim_k T_k(\omega) < M$ for $\omega \in U_M$, then necessarily $\lim_{k\to\infty} N_{T_k}^{M,\epsilon}(\omega) < \infty$ (otherwise $\tau_N^{M,\epsilon} \leq M/2$ for N large enough, which is with null probability) and so, for all $\omega \in U_M$, we can construct the time sequence $(T_k(\omega))_k$ as a subsequence of a Poisson point process of intensity $(1 + C/\epsilon^2)N_\infty$ (using (1.4) and (1.5) we have that $1 + C/\epsilon^2 < \infty$) where $N_\infty = \lim_{k\to\infty} N_{T_k}(\omega)$ which is a.e. unbounded. Moreover, we have for all $\epsilon > 0$, F and f belongs to $W^{1,\infty}$, the infinitesimal generator of the Markovian process $(Z_t^{M,\epsilon})_{t\geq 0}$ issued from $Z_0^{M,\epsilon}$ is given by

$$\mathcal{L}F_f(Z_0^{M,\epsilon}) = \frac{\partial}{\partial t}\mathbb{E}(F(\langle f, Z_t^{M,\epsilon}\rangle))|_{t=0} = F'(\langle f, Z_0^{M,\epsilon}\rangle)\iint_{\mathbb{R}_+\times Space}\frac{\partial}{\partial a}f(a,p)Z_0^{M,\epsilon}(da,dp)$$
$$+ (\iint_{\mathbb{R}_+\times Space}\Big[(F(\langle f, Z_0^{M,\epsilon}\rangle + \frac{2}{M}f(0,p) - \frac{1}{M}f(a,p)) - F(\langle f, Z_0^{M,\epsilon}\rangle)B(0,a)$$
$$+(F(\langle f, Z_0^{M,\epsilon}\rangle + \frac{1}{M}f(a,p') - \frac{1}{M}f(a,p)) - F(\langle f, Z_0^{M,\epsilon}\rangle))P_0^\epsilon \Pi_0^\epsilon(p,dp')\Big] Z_0^{M,\epsilon}(da,dp). \tag{7.8}$$

The infinitesimal generator of the Markovian process $(r_O^{M,\epsilon}(t))_{t\geq 0}$ issued from r_0 is given by

$$\mathcal{L}F(r_O^{M,\epsilon}(0)) = \frac{\partial}{\partial t}\mathbb{E}(F(r_O^{M,\epsilon}(t)))|_{t=0} = F'(r_0)\sum_{i,j,k}\kappa((i-1)\epsilon)r_0^\alpha\langle\Psi_{i,j,k}^\epsilon, Z_0^{M,\epsilon}\rangle. \tag{7.9}$$

Indeed, let $t \leq \tau_N^{M,\epsilon}$ with $\tau_N^{M,\epsilon} = \inf\{s \geq 0, \quad N_s^{M,\epsilon} \geq N\}$, then using lemma 4.2, we have that

$$\mathbb{E}((\int_0^{\min(t,\tau_N^{M,\epsilon})} F'(\langle f, Z_u^{M,\epsilon}\rangle)\iint_{\mathbb{R}_+\times Space}\frac{\partial}{\partial a}f(u,a,p)Z_u^{M,\epsilon}(da,dp)du) =$$
$$\mathbb{E}(F(\iint_{\mathbb{R}_+\times Space} f(a,p)Z_t^{M,\epsilon}(da,dp))) - \mathbb{E}(F(\iint_{\mathbb{R}_+\times Space} f(a,p)Z_0(da,dp)))$$
$$- \Big[\mathbb{E}(\int_0^{\min(t,\tau_N^{M,\epsilon})}\int_{\varepsilon} 1_{n<N_{s-}^{M,\epsilon}}\Big[(F(\langle f, Z_s\rangle + \frac{2}{M}f(0, X_n^{M,\epsilon}(s-))$$
$$- \frac{1}{M}f(A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) - F(\langle f_s, Z_s^{M,\epsilon}\rangle)1_{0\leq\Theta<B_{n,s-}^{M,\epsilon}}$$
$$+ (F(\langle f, Z_s\rangle + \frac{1}{M}f(A_n^{M,\epsilon}(s-), p') - \frac{1}{M}f(A_n^{M,\epsilon}(s-), X_n^{M,\epsilon}(s-))) - F(\langle f, Z_s\rangle))$$
$$1_{0\leq\Theta - B_{n,s-}^{M,\epsilon} < P_{s-}^\epsilon(X_n(s-))}\Big]Q(ds,(dn),d\Theta,\Pi_{s-}^{M,\epsilon}(p,dp'))))\Big],$$

which can be rewritten (for simplicity) as

$$\mathbb{E}(F(\iint_{\mathbb{R}_+\times Space} f(a,p)Z_{\min(t,\tau_N^{M,\epsilon})}(da,dp)))$$
$$= \mathbb{E}(F(\iint_{\mathbb{R}_+\times Space} f(a,p)Z_0(da,dp))) + \mathbb{E}(\Psi(\min(t,\tau_N^{M,\epsilon}), Z)),$$

with

$$\Psi(t,Z) = \int_0^t F'(\langle f, Z_u^{M,\epsilon}\rangle) \iint_{\mathbb{R}_+\times Space} \frac{\partial}{\partial a} f(a,p) Z_u^{M,\epsilon}(da,dp)du$$

$$+\int_0^t \iint_{\mathbb{R}_+\times Space} 1_{n<N_{s-}^{M,\epsilon}} \Big[(F(\langle f, Z_s\rangle + \frac{2}{M} f(s,0,p) - \frac{1}{M} f(s,a,p)) - F(\langle f, Z_s\rangle) B(p,a)$$

$$+(F(\langle f, Z_s\rangle + \frac{f(s,a,p') - f(s,a,p)}{M}) - F(\langle f, Z_s\rangle)) P_{s-}^{\epsilon}(p)\Pi_{s-}^{\epsilon}(p,dp') \Big] Z_s^{M,\epsilon}(da,dp),$$

with $B(p,a) = 1 - e^{-a/\lambda(\|p\|_2)}$. We have the following bound

$$|\Psi(\min(t,\tau_N^{M,\epsilon}), Z)| \leq TN\|F\|_{W^{1,\infty}}(1+\|f\|_{W^{1,\infty}}(1+\sup_p \int |p'-p| P^{\epsilon}(p)\Pi_{s-}(p,dp'))).$$

Since $Z_t^{M,\epsilon}$ is càd and $\tau_N^{M,\epsilon} > 0$, we have that

$$\frac{\partial}{\partial t}\Psi(\min(t,\tau_N^{M,\epsilon}), Z)|_{t=0} = F'(\langle f_0, Z_0^{M,\epsilon}\rangle) \iint_{\mathbb{R}_+\times Space} \frac{\partial}{\partial a} f(a,p) Z_0^{M,\epsilon}(da,dp)$$

$$+ (\iint_{\mathbb{R}_+\times Space} \Big[(F(\langle f, Z_0\rangle + \frac{2}{M} f(0,0,p) - \frac{1}{M} f(0,a,p)) - F(\langle f, Z_0^{M,\epsilon}\rangle) B(0,a)$$

$$+(F(\langle f, Z_0\rangle + \frac{f(0,a,p') - f(0,a,p)}{M}) - F(\langle f, Z_0^{M,\epsilon}\rangle)) P_0^{\epsilon}(p)\Pi_0^{\epsilon}(p,dp') \Big] Z_0^{M,\epsilon}(da,dp),$$

which is dominated by $T\frac{N_0^{M,\epsilon}}{M}\|F\|_{W^{1,\infty}}(1+\|f\|_{W^{1,\infty}}(1+C/\epsilon)) < \infty$. Therefore by derivation under domination, we find (7.8).

References

1. D. Aldous, Stopping times and tightness, *Ann. Probab.*, **6** (1978), 335-340.
2. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Science & Business Media, 2010.
3. N. Champagnat, R. Ferrière and S. Méléard, Individual-based probabilistic models of adaptive evolution and various scaling approximations, *In Proceedings of the 5th seminar on Stochastic Analysis, Random Fields and Applications, Probability in Progress Series*, Ascona, Suisse. Birkhauser.
4. R. Ferrière and V.C. Tran, Stochastic and deterministic models for age-structured populations with genetically variable traits, *ESAIM: Proceedings*, **27** (2009), 289-310.
5. F. Clément, P. Michel, D. Monniaux and T. Stiehl, *Coupled somatic cell kinetics and germ cell growth: multiscale model-based insight on ovarian follicular development*, SIAM : Multiscale Model. Simul., **11(3)** (2013), 719-746.
6. F. Clément, D. Monniaux. *Multiscale modelling of ovarian follicle selection.* Progress in Biophysics and Molecular Biology, **113(3)** (2013), 398-408.
7. N. Echenim, D. Monniaux, M. Sorine, F. Clément. *Multi-scale modeling of the follicle selection process in the ovary.* Mathematical Biosciences, **198** (2005), 57-79.
8. A. Etheridge, *An Introduction to Superprocesses*, *University Lecture Series, Amer. Math. Soc., Providence, RI*, **20**, 2000.
9. S. N. Ethier and T.G. Kurtz, Markov Processes: Characterization and Convergence, *WILEY*, ISBN: 978-0-471-76986-6, 2005.

10. H.J. Hwang, K. Kang and A. Stevens, *Global Solutions of Nonlinear Transport Equations for Chemosensitive Movement*, *SIAM J. Math. Analysis*, **36(4)** (2005), 1177-1199.
11. M. Iannelli, Mathematical Theory of Age-Structured Population Dynamics. *Applied Math. Monographs, CNR, Giardini Editori e Stampatori in Pisa*, 1995.
12. E. F. Keller and L. A. Segel, Initiation of slide mold aggregation viewed as an instability, *J. Theoret. Biol.*, **26** (1970), 399-415.
13. E. F. Keller and L. A. Segel, Model for chemotaxis, *J. Theoret. Biol.*, **30** (1971), 225-234.
14. J.-L. Lions, Equations differentielles operationnelles et problemes aux limites, *Die Grundlehren der mathematischen Wissenschaften, Bd. 111, Springer-Verlag, Berlin*, 1961.
15. S. Méléard, Convergence of the fluctuations for interacting diffusions with jumps associated with boltzmann equations, *Stochastics and Stochastics Reports*, **63** (1998), 195225.
16. S. Méléard and Roelly S., Sur les convergences étroite ou vague de processus valeurs mesures / On the vague and weak convergence of measure-valued processes, *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, **317(8)** (1993), 785-788.
17. S. Méléard and V.C. Tran, Trait substitution sequence process and canonical equation for age-structured populations, *Journal of Mathematical Biology*, **58(6)** (2009), 881-921.
18. J.A.J. Metz and O. Diekmann, The dynamics of physiologically structured populations. *L.N. in biomathematics, Springer*, **68**, 1986.
19. D. Monniaux, P. Michel, M. Postel, F. Clément, Multiscale modeling of ovarian follicular development: from follicular morphogenesis to selection for ovulation. *Biology of the Cell*, ehead of pub.
20. B. Perthame, Transport Equations in Biology, *Birkhuser, Frontiers in Mathematics*, 2007.
21. P.-L. Lions, B. Perthame, and E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, *J. Amer. Math. Soc.*, **7 (1)** (1994), 169-191.
22. B. Perthame, Kinetic formulation of conservation laws, *Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford*, **21**, 2002.
23. Rebolledo R., Sur l'existence de solutions certains problemes de semimartingales, *C. R. Acad. Sci. Paris AB*, **290(18)** (1980), A843 A846.
24. S. Roelly, A criterion of convergence of measure-valued processes: Application to measure branching processes. *Stochastics*, **17**(1986), 43-65.
25. V.C. Tran, Large population limit and time behaviour of a stochastic particle model describing an age-structured population, *ESAIM: PS*, **12** (2008), 345-386.
26. V.C. Tran, Modèles particulaires stochastiques pour des problèmes d'évolution adaptative et pour l'approximation de solutions statistiques. *Ph.D. thesis*, Université Paris X - Nanterre.
27. Webb G.F., Theory of nonlinear age-dependent population dynamics. *Marcel Dekker, New York*, 1985.