Mathematical Models and Methods in Applied Sciences

Existence of a solution to the cell division eigenproblem

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Received (Day Month Year)
Revised (Day Month Year)
Communicated by (xxxxxxxxxx)

We consider the cell division equation which describes the continuous growth of cells and their division in two pieces. Growth conserves the total number of cells while division conserves the total mass of the system but increases the number of cells. We give general assumptions on the coefficient so that we can prove the existence of a solution (λ, N, ϕ) to the related eigenproblem. We also prove that the solution can be obtained as the sum of an explicit series. Our motivation, besides its applications to the biology and fragmentation, is that the eigenelements allow to prove a priori estimates and long time asymptotics through the General Relative Entropy.[16]

Keywords: Relative entropy; fragmentation equations; cell division; long time asymptotic; steady state; eigenproblem; dual.

AMS Subject Classification: 22E46, 53C35, 57S20

1. Introduction

A classical subject in biology, see Ref. 2, 7, 8, 11, 12, 13, 20, 21, is the evolution of a density describing the number of individuals in a structured population, in particular, tumor cell population. Several models are written as Population Balance Equation (PBE) obtained combining several conservation laws (growth in size, number...). Here we focus on the cell division equation at the cellular scale. [1,3,20] Let $n(t,y)$ the density of population at time t and size y, then the time evolution of $n(t,y)$ is described by the master equation

$$\begin{cases} \frac{\partial n}{\partial t}(t,y) + \frac{\partial n}{\partial y}(t,y) + B(y)n(t,y) = \int_y^\infty b(y,y')n(t,y')dy', \quad y \geq 0, \\ \\ n(t,0) = 0, \quad t \geq 0, \end{cases} \tag{1.1}$$

where $B(y)$ denotes the rate of division of cells of size y, $b(y,y')/2B(y')$ is the probability that the mother cell of size y' gives rise to a cell of size y. A Similar model also arises to describe fragmentation in physics, see Ref. 14 and the growth term $\partial_y n$ arises after rescalling.[4,5,9] An interesting feature of this problem is the asymptotic behaviour of $n(t,y)$ that gives the invasive capacity of the population and thus a fitness measure of populations under different rates and probabilities in (1.1). It appears that the long time behaviour of n is directly linked with the existence of a solution (λ, N, ϕ), in L^1 sense, of the associated eigenproblem,[16,18,19,23]

$$\begin{cases} \dfrac{\partial}{\partial y} N(y) + [\lambda + B(y)]N(y) = \displaystyle\int_y^\infty b(y,y')N(y')dy', \quad y \geq 0, \\[2ex] N(y=0) = 0, \quad N \geq 0, \quad \displaystyle\int N(y)dy = 1, \\[2ex] -\dfrac{\partial}{\partial y}\phi(y) + [\lambda + B(y)]\phi(y) = \displaystyle\int_0^y b(y',y)\phi(y')dy', \quad y \geq 0, \\[2ex] \phi \geq 0, \quad \displaystyle\int N(y)\phi(y)dy = 1. \end{cases} \tag{1.2}$$

More precisely one can prove that $n(t,y)e^{-\lambda t}$ converges to $N(y)$. Thus the asymptotic behaviour is given by the Maltusian parameter λ and the asymptotic shape of n given by N. These are proved in the case where $b(y,y') = B(y')\delta_{y'=2y}$ (equal mitosis), i.e., a cell division in which a cell of size y gives birth to two cells of size $y/2$.[19] The methods we use here are closely related to those used in, see Ref. 9, 18, 19, we prove the existence and compactness of approximated solution to this eigenproblem, then we pass to the limit on the sequence of approximated solutions. Using the following results, we show, see Ref. 15, that the Maltusian parameter λ depends on the way of a cell divides. Moreover the symmetrical division is not necessarily the best fitted division, it depends on the birth rate B and an adaptive strategy would be to have a large spectrum of way to divide to be able to fit in the case of a change of the environment. Gene mutations of the tumor cells that alter the cell cycle, see Ref. 10, may alter the symmetry of the division and thus, by selection, optimize the Maltusian parameter.

This article is organized as follows : in section 3 we prove the existence of (λ, N, ϕ), theorems 2.1, 2.2 introduced in section 2, using a priori bounds (propositions 3.1, 3.2) proved in section 4 and the existence of a regularized solution using theorem 6.2 and lemma 3.1, 3.3. In section 5 we extend the proof of existence to the cell division equation with some non constant growth rate and in the section 6 we focus on the properties of these solutions under some more restrictive assumptions and prove theorem 6.2.

2. Existence of solution to the Eigenproblems for the cell division eq.

The purpose of this section is to prove the existence of a first eigenvalue λ and positive eigenvectors (N, ϕ) solution to (1.2). Throughout the paper, our assumptions on b and B are the following structural properties; for some $\kappa > 1$, $0 < \eta < 1$,

$$\int_0^{y'} b(y, y')dy = \kappa B(y'), \tag{2.1}$$

$$\int_0^{y'} yb(y, y')dy = y'B(y'), \tag{2.2}$$

$$\int_{(1-\eta)y'}^{y'} b(y, y')dy \leq B(y'). \tag{2.3}$$

We notice that (2.2) implies directly

$$\int_0^{y'} y^k b(y, y')dy \leq y'^k B(y'), \quad \forall k \geq 1. \tag{2.4}$$

The identity (2.1) with $\kappa = 2$ expresses that mother cells divide in two daughter cells. We use a more general parameter κ to cover also the fragmentation equation.[9] The identity (2.2) expresses that total size is conserved during the division process. The inequality (2.3) expresses that the division of the mother cell of size y' does not give more than one cell of size larger than $(1 - \eta)y'$ (where η is independent of y').

We prove the existence of a solution to this eigenproblem under general conditions on $b \geq 0$ and $B \geq 0$. Namely

$$\text{Supp}\, B \ \text{ is an interval}, \tag{2.5}$$

$$B \in L^\infty_{loc}(]0, \infty[) \bigcap L^1_{loc}([0, \infty[), \tag{2.6}$$

$$\overline{\lambda} := \sup_{y'} \int_0^{y'} b(y, y')e^{-\int_y^{y'} B(s)ds} dy < \infty, \tag{2.7}$$

$$\text{Supp}\, B \subset [0, A], \ \text{ and } \ \underline{\lambda} := \frac{\kappa - 1}{Ae^{\int_0^A(\overline{\lambda}+B(s))ds}} \Big(\int_{A(1-\eta)}^{A} B(y)dy - \frac{1}{\kappa - 1} \Big) > 0, \tag{2.8}$$

or

$$\lim_{y\to\infty} yB(y) = \infty. \tag{2.9}$$

Assumptions (2.8) or (2.9) mean that there is enough birth to avoid extinction, i.e., to avoid $\lambda < 0$. The inequality (2.7) gives a bound to avoid explosion, we control here the way to divide with respect to the birth rate. The assumption (2.5) means that all the cells which divide don't extinct, it could be replace by some more complicated assumptions implying a connexion between cells (fragments). These assumptions cover several cases where the probability of dividing a cell of size y' into two cells of size y and $y' - y$ only depends on $\frac{y}{y'}$, $b(y, y') = \wp(\frac{y}{y'})B(y')/y'$.

1) $B(y') = y'^{\gamma}$ the assumptions can be written in a simpler way,

$$\gamma \geq 0, \ \wp \in M^1([0,1]), \tag{2.10}$$

$$\overline{\lambda'} = \int_0^1 \frac{\wp(\theta)}{(1-\theta^{\gamma+1})^{\gamma/(\gamma+1)}} d\theta < \infty, \tag{2.11}$$

where $M^1([0,1])$ is the set of probability measure on $(0,1)$ (see the proof in the section 6, Theorem 6.1).

2) Equal mitosis, i.e., $b(y,y') = 2\delta_{(y=y'/2)} B(y')$ with *Supp* B connex. Then the assumptions are satisfied with $\kappa = 2$, $\eta = 1/2$, B is bounded and then $\overline{\lambda} \leq \sup B$ and either B has a compact support in $[0,A]$ and $\int_{A/2}^A B(y')dy' > 1$ and then $\underline{\lambda} = \frac{\int_{A/2}^A B(y)dy - 1}{Ae^{\int_0^A (\overline{\lambda} + B(s))ds}}$, either $\lim_{y\to\infty} B(y)y = \infty$.

3) Age structured,[22] in the particular case where the probability of dividing of a cell of size y' into cells of size y is given by $b(y,y') = B(y')(\delta_{(y=y')} + \delta_{(y=0)})$ we get a classical age structured eigenproblem. Indeed, the equation reads in distribution sense,

$$\begin{cases} \frac{\partial}{\partial y} N(y) + \lambda N(y) = \delta_{(y=0)} \int B(y')N(y')dy', \quad y \geq 0, \\ \\ N(0) = 0, \end{cases}$$

i.e., for all $\xi \in C_0^\infty([0,\infty[)$,

$$-\int_0^\infty N(y) \frac{\partial}{\partial y} \xi(y) dy + \lambda \int_0^\infty N(y)\xi(y)dy = \xi(0) \int B(y')N(y')dy', \quad y \geq 0,$$

which means that N is discontinuous at $y = 0$. Then in distribution sense, it is equivalent to

$$\begin{cases} \frac{\partial}{\partial y} \widetilde{N}(y) + \lambda \widetilde{N}(y) = 0, \quad y > 0, \\ \\ \widetilde{N}(y=0) = \int B(y')\widetilde{N}(y')dy', \end{cases}$$

with $N(y) = \widetilde{N}(y)$ for $y > 0$, $\widetilde{N}(0) = \lim_{y\to 0} N(y)$. Then our assumptions become $\kappa = 2$, $B \in L^\infty([0,\infty])$ and then $\overline{\lambda} \leq \sup B$ and either B has a compact support in $[0,A]$ and $\int_{A(1-\eta)}^A B(y)dy > 1$ for some $\eta < 1$, i.e., $\int_0^A B(y)dy > 1$ and then

$$\underline{\lambda} = \frac{1}{Ae^{\int_0^A (\overline{\lambda} + B(s))ds}} \Big(\int_0^A B(y)dy - 1 \Big),$$

either $\lim_{y\to\infty} B(y)y = \infty$.

Our purpose is to prove the following theorems of existence when B has a compact support,

Theorem 2.1.
Assume (2.1)-(2.7) and (2.8), then there exists a solution to (1.2), and we have

$$\underline{\lambda} \leq \lambda \leq \overline{\lambda} \tag{2.12}$$

$$\forall \epsilon > 0 \ \exists C_\epsilon > 0 \ \ : \ \ \int_0^\infty N(y) e^{\lambda(1-\epsilon)y} dy \leq C_\epsilon,$$
$$N(y) \leq C_\epsilon e^{-\lambda(1-\epsilon)y}(1 + \int_y^\infty B(y')e^{\lambda(1-\epsilon)y'} N(y')dy'), \tag{2.13}$$

$$\exists C > 0 \ \ : \ \ \phi(y) \leq C(1+y), \tag{2.14}$$

$$N \in BV, \ \ \phi \in BV_{loc}, \tag{2.15}$$

$$\phi(y) = 0, \quad y \geq A. \tag{2.16}$$

And when B has not a compact support, we have,

Theorem 2.2.
Assume (2.1)-(2.7) and (2.9), then there exists a solution to (1.2) in L^1 sense, and we have

$$0 \leq \lambda \leq \overline{\lambda}, \tag{2.17}$$

$$\forall \epsilon > 0 \ \exists C_\epsilon > 0 \ \ : \ \ \int_0^\infty N(y) e^{\lambda(1-\epsilon)y} dy \leq C_\epsilon,$$
$$N(y) \leq C_\epsilon e^{-\lambda(1-\epsilon)y}(1 + \int_y^\infty B(y')e^{\lambda(1-\epsilon)y'} N(y')dy'), \tag{2.18}$$

$$\exists C > 0 \ \ : \ \ \phi(y) \leq C(1+y), \tag{2.19}$$

$$N \in BV, \ \ \phi \in BV_{loc}. \tag{2.20}$$

3. Proof of the main theorems

3.1. *Proof of theorem 2.1*

First we prove there exists an approximation scheme of (1.2) and a sequence $(N^\epsilon, \lambda^\epsilon, \phi^\epsilon)$ of solutions to these approximated problems, then we prove the compactness of this sequence and finally pass to the limit to find a solution of (1.2).

Step1. Existence of an approximation scheme

We have the existence of solutions satisfying a sequence of regularized problem, indeed, consider

$$\begin{cases} \dfrac{\partial}{\partial y} N^\epsilon(y) + [\lambda^\epsilon + B^\epsilon(y)] N^\epsilon(y) = \displaystyle\int_y^\infty b^\epsilon(y, y') N^\epsilon(y') dy', \\ N^\epsilon(y=0) = 0, \quad N^\epsilon \geq 0, \quad \displaystyle\int_0^A N^\epsilon(y) dy = 1, \\ -\dfrac{\partial}{\partial y} \phi^\epsilon(y) + [\lambda^\epsilon + B^\epsilon(y)] \phi^\epsilon(y) = \displaystyle\int_0^y b^\epsilon(y', y) \phi^\epsilon(y') dy', \\ \phi^\epsilon \geq 0, \quad \displaystyle\int_0^A N^\epsilon(y) \phi^\epsilon(y) dy = 1, \end{cases} \tag{3.1}$$

where $\epsilon > 0$,

$$b^\epsilon(y, y') = \frac{1}{\epsilon^2} \sum_{k=0}^\infty \sum_{j=0}^\infty 1_{(y,y') \in [k\epsilon, (k+1)\epsilon] \times [j\epsilon, (j+1)\epsilon]} \int_{k\epsilon}^{(k+1)\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} b(z, z') dz' dz,$$

and

$$B^\epsilon(y) = \int_0^y \frac{y'}{y} b_\epsilon(y, y') dy'.$$

Lemma 3.1. *Assume (2.1)-(2.3), (2.6), (2.7) and (2.8) with* $A = \sup_y \{y \in SuppB\}$ *then for all* $\epsilon > 0$ *there exists a solution* $(N^\epsilon, \lambda^\epsilon, \phi^\epsilon)$ *to (3.1) satisfying*

$$N^\epsilon(y) e^{\int_0^y (\lambda^\epsilon + B^\epsilon(s)) ds} \text{ increases and belongs to } L^\infty([0, \infty[), \tag{3.2}$$

$$\phi^\epsilon(y) = 0 \text{ as } y \geq A, \tag{3.3}$$

Proof. We use the construction method given in section 6.3 (see theorem 6.2) to prove the existence of $(N^\epsilon, \lambda^\epsilon, \phi^\epsilon)$. We only have to notice that b^ϵ, B^ϵ satisfy (2.2), (2.3), (2.6), (6.10), (6.11), (6.12). Then, (3.2) is obtained by differentiation of $N^\epsilon(y) e^{\int_0^y (\lambda^\epsilon + B^\epsilon(s)) ds}$, indeed,

$$\partial_y (N^\epsilon(y) e^{\int_0^y (\lambda^\epsilon + B^\epsilon(s)) ds}) = \int_y^\infty N^\epsilon(y') e^{-\int_y^{y'} (\lambda^\epsilon + B^\epsilon(s)) ds} b(y, y') dy' \geq 0,$$

and $\partial_y (N^\epsilon(y) e^{\int_0^y (\lambda^\epsilon + B^\epsilon(s)) ds}) = 0$ if $y \geq A$. □

Step2. Compactness of the approximation scheme

Now, we have that a solution (N, λ, ϕ) to (1.2) satisfying (3.2), (3.3) belongs to a compact set of $L^1([0, \infty[) \times] - \infty, \infty[\times L^1_{loc}([0, \infty[, N(y) dy)$,

Proposition 3.1. *(Supp B is compact)*
Assume (2.1)-(2.8) are satisfied. If (λ, N, ϕ) *is solution to (1.2) and*

$$\lambda > 0, \tag{3.4}$$

then for some $C_0(\underline{\lambda}) < \infty$, *we have*

$$\lambda \in [\underline{\lambda}, \overline{\lambda}] \subset]0, \infty[, \tag{3.5}$$

$$\int_0^\infty N(y) e^{\underline{\lambda} y/2} dy \leq 1, \tag{3.6}$$

$$\int_0^\infty \mid \partial_y N(y) \mid dy \leq \frac{2\kappa}{\kappa - 1} \overline{\lambda}, \tag{3.7}$$

$$\phi(y) \leq C_0(\underline{\lambda})(1+y), \tag{3.8}$$

$$\mid \partial_y \phi(y) \mid \leq C_0(\underline{\lambda})(1+y)(\overline{\lambda} + \frac{2\kappa}{\kappa - 1} B(y)). \tag{3.9}$$

Proof. We refer to the lemma 4.1, 4.2 and 4.4 to prove this proposition. □

Thus the sequence $(N^\epsilon, \lambda^\epsilon, \phi^\epsilon)$ introduced in lemma 3.1 satisfies
- λ^ϵ belongs to the compact set $\Lambda^0 = [\inf_{\epsilon'} \underline{\lambda}^{\epsilon'}, \sup_{\epsilon'} \overline{\lambda}^{\epsilon'}]$,
- N^ϵ belongs to the compact set S^0_{dens} of $L^1([0,\infty[)$ given by

$$S^0_{dens} = \{f \in L^1 : \int_0^\infty f(y) dy = 1, \int_0^\infty f(y) e^{\inf_{\epsilon'} \underline{\lambda}^{\epsilon'} y/2} dy \leq 1,$$
$$\int_0^\infty \mid \partial_y f(y) \mid dy \leq \frac{2\kappa}{\kappa - 1} \sup_{\epsilon'} \overline{\lambda}^{\epsilon'} \}, \tag{3.10}$$

- ϕ^ϵ belongs to the compact set S^0_{dual} of $L^1([0,\infty[, N(y)dy)$ given by

$$S^0_{dual} = \{g \in L^1 : \int_0^\infty g(y) \inf_{\epsilon'} N^{\epsilon'}(y) dy \geq 1, \;\; g(y) \leq C_0(\inf_{\epsilon'} \underline{\lambda}^{\epsilon'})(1+y),$$
$$\mid \partial_y g(y) \mid \leq C_0(\inf_{\epsilon'} \underline{\lambda}^{\epsilon'})(1+y)(\sup_{\epsilon'} \overline{\lambda}^{\epsilon'} + \frac{2\kappa}{\kappa - 1} \sup_{\epsilon'} B^{\epsilon'}(y))\}. \tag{3.11}$$

Then we notice that $(N^\epsilon, \lambda^\epsilon, \phi^\epsilon) \in \Lambda^0 \times S^0_{dens} \times S^0_{dual}$ is a compact family and we can extract a convergent subsequence that gives a solution (N, λ, ϕ) to the limit problem (in the case when B has a compact support). Therefore Theorem 2.1 is proved in the case of (2.8).

Step3. Uniqueness of solution to (2.1)
Now, we prove the uniqueness of solution to this eigenproblem,

Lemma 3.2. *Assume there exists* (N, λ, ϕ) *solution to (1.2) and* (N_1, λ_1) *solution to (1.2) such that* $0 < \int_0^\infty N_1(y)\phi(y)dy < \infty$ *and* $N_1/N \in L^1([0,\infty[, N(y)\phi(y)dy)$ *then*

$$\lambda_1 = \lambda,$$

and there exists $C > 0$ *such that* $N_1 = CN$.

Proof. By integrating from 0 to A, $A = \sup_y \{y \in SuppB\}$,

$$\phi(y)\partial_y N_1(y) + (\lambda_1 + B(y))N_1(y)\phi(y) = \int_y^\infty b(y,y')N_1(y')dy'\phi(y),$$

and

$$-N_1(y)\partial_y\phi(y) + (\lambda + B(y))N_1(y)\phi(y) = \int_y^\infty b(y',y)\phi(y')dy'N_1(y),$$

with $N_1(0) = \phi(A) = 0$, we find,

$$\lambda_1 = \lambda.$$

We conclude using GRE method, see Ref. 16, 17, for all $C > 0$,

$$\partial_t \int_0^\infty \mid \frac{N_1}{N} - C \mid N(y)\phi(y)dy = \int_0^\infty [-\partial_y N_1(y) - (\lambda + B(y))N_1(y) + \int_y^\infty b(y,y')N_1(y')dy'] \operatorname{sgn}(\frac{N_1}{N}(y) - C)\phi(y)dy, \quad (3.12)$$

with

$$-\int_0^\infty \partial_y N_1(y) \operatorname{sgn}(\frac{N_1}{N}(y) - C)\phi(y)dy = -\int_0^\infty \partial_y[\mid \frac{N_1}{N}(y) - C \mid N(y)\phi(y)]dy - \int_0^\infty \frac{\partial_y N(y)}{N(y)}\frac{N_1}{N}(y) \operatorname{sgn}(\frac{N_1}{N}(y) - C)N(y)\phi(y)dy + \int_0^\infty \mid \frac{N_1}{N}(y) - C \mid \partial_y(N\phi)(y)dy,$$

thus using boundary condition,

$$-\int_0^\infty \partial_y N_1(y) \operatorname{sgn}(\frac{N_1}{N}(y) - C)\phi(y)dy = + \int_0^\infty \mid \frac{N_1}{N}(y) - C \mid \Big[\int b(y,y')N(y')\phi(y)dy' - \int b(y',y)\phi(y')N(y)dy'\Big]dy - \int_0^\infty\int_0^\infty b(y,y')N(y')dy'\frac{N_1}{N}(y) \operatorname{sgn}(\frac{N_1}{N}(y) - C)\phi(y)dy + \int_0^\infty (\lambda + B(y))\frac{N_1}{N}(y) \operatorname{sgn}(\frac{N_1}{N}(y) - C)N(y)\phi(y)dy, \quad (3.13)$$

$$-\int_0^\infty \partial_y N_1(y) \operatorname{sgn}(\frac{N_1}{N}(y) - C)\phi(y)dy = + \int_0^\infty\int_0^\infty b(y,y')(\mid \frac{N_1}{N}(y) - C \mid - \mid \frac{N_1}{N}(y') - C \mid)N(y')\phi(y)dy'dy - \int_0^\infty\int_0^\infty b(y,y')N(y')dy'\frac{N_1}{N}(y) \operatorname{sgn}(\frac{N_1}{N}(y) - C)\phi(y)dy + \int_0^\infty (\lambda + B(y))\frac{N_1}{N}(y)(\operatorname{sgn}\frac{N_1}{N}(y) - C)N(y)\phi(y)dy, \quad (3.14)$$

therefore we have,

$$\partial_t \int_0^\infty \mid \frac{N_1}{N} - C \mid N(y)\phi(y)dy =$$
$$+ \int_0^\infty \int_0^\infty b(y,y')(\mid \frac{N_1}{N}(y) - C \mid - \mid \frac{N_1}{N}(y') - C \mid) N(y')\phi(y)dy'dy$$
$$- \int_0^\infty \int_0^\infty b(y,y')N(y')dy' \frac{N_1}{N}(y)(\frac{N_1}{N}(y) - C)\phi(y)dy$$
$$+ \int_0^\infty \int_0^\infty b(y,y')\frac{N_1}{N}(y') \operatorname{sgn}(\frac{N_1}{N}(y) - C)\phi(y)N(y')dy'dy, \quad (3.15)$$

and finally,

$$\partial_t \int_0^\infty \mid \frac{N_1}{N} - C \mid N(y)\phi(y)dy = -\int_0^\infty \int_0^\infty b(y,y')N(y') \mid \frac{N_1}{N}(y') - C \mid$$
$$[\operatorname{sgn}(\frac{N_1}{N}(y) - C) \operatorname{sgn}(\frac{N_1}{N}(y') - C) - 1]\phi(y)dy'dy. \quad (3.16)$$

But $\int_0^\infty \mid \frac{N_1}{N} - C \mid N(y)\phi(y)dy$ is independent of t thus $[\operatorname{sgn}(\frac{N_1}{N}(y) - C) \operatorname{sgn}(\frac{N_1}{N}(y') - C) - 1] = 0$ on the support of $b(y,y')$ for all C thus $\frac{N_1}{N}(y) = \frac{N_1}{N}(y')$ on the support of $b(y,y')$ and

$$\partial_y \frac{N_1}{N}(y) = \int_0^\infty b(y,y')(\frac{N_1}{N}(y') - \frac{N_1}{N}(y))\frac{N(y')}{N(y)}dy' = 0,$$

i.e., N_1/N is a constant. □

3.2. *Proof of theorem 2.2*

First we prove there exists an approximation scheme of (1.2) and a sequence (N_A, λ_A, ϕ_A) of solutions to these approximated problems, then we prove the compactness of this sequence and finally pass to the limit to find a solution to (1.2).

Step1. Existence of an approximation scheme
We have the existence of solutions satisfying a sequence of regularized problem, indeed, consider,

$$\begin{cases} \frac{\partial}{\partial y} N_A^\epsilon(y) + [\lambda_A^\epsilon + B_A^\epsilon(y)]N_A^\epsilon(y) = \int_y^\infty b_A^\epsilon(y,y')N_A^\epsilon(y')dy', \\ N_A^\epsilon(y=0) = 0, \quad N_A^\epsilon \geq 0, \quad \int_0^A N_A^\epsilon(y)dy = 1, \\ -\frac{\partial}{\partial y}\phi_A^\epsilon(y) + [\lambda_A^\epsilon + B_A^\epsilon(y)]\phi_A^\epsilon(y) = \int_0^y b_A^\epsilon(y',y)\phi_A^\epsilon(y')dy', \\ \phi_A^\epsilon \geq 0, \quad \int_0^A N_A^\epsilon(y)\phi_A^\epsilon(y)dy = 1, \end{cases} \quad (3.17)$$

where $\epsilon > 0$, $b_A^\epsilon(y,y') = 1_{(y,y')\in[0,A]\times[0,A]} b^\epsilon(y,y')$, (see lemma 3.1 for the definition of b^ϵ) and $B_A^\epsilon(y) = \int_0^y \frac{y'}{y} b_A^\epsilon(y,y')dy'$.

Lemma 3.3. *Assume (2.1)-(2.6), (2.7) and (2.9), then for all $A > 0$ (large enough), $\epsilon > 0$ (ϵ small enough) there exists a solution $(N_A^\epsilon, \lambda_A^\epsilon, \phi_A^\epsilon)$ to (3.17) and it satisfies that*

$$N_A^\epsilon(y) e^{\int_0^y (\lambda_A^\epsilon + B_A^\epsilon(s))ds} \text{ increases and belongs to } L^\infty([0,\infty[), \tag{3.18}$$

$$\phi_A^\epsilon(y) = 0 \text{ as } y \geq A, \tag{3.19}$$

Proof. Indeed, using (2.9),

$$\int_{A(1-\eta)}^A B_A^\epsilon(y)dy \geq \ln(1/(1-\eta)) \inf_{y\in(A(1-\eta),A)} yB_A^\epsilon(y) \to \infty \text{ as } A \to \infty,\ \epsilon \to 0,$$

and thus for A large enough, ϵ small enough, B_A^ϵ satisfies (2.8). We use the construction method given in 6.3 (see theorem 6.2) to prove the existence of $(N_A^\epsilon, \lambda_A^\epsilon, \phi_A^\epsilon)$. We only have to notice that b_A^ϵ, B_A^ϵ satisfy (2.2), (2.3), (2.6), (6.10)-(6.12). □

Remark 3.1. We notice that the property

$$\phi_A^\epsilon(y) \to 0, \quad as\ y \to \infty, \tag{3.20}$$

is satisfied by the solution of these regularized problems and used to prove the a priori bounds on ϕ but is not satisfied, in general, by the solution ϕ obtained by passing to the limit in A.

Step2. Compactness of the approximation scheme

Now, we have that a solution (N, λ, ϕ) to (1.2) satisfying (3.18), (3.19) belongs to a compact set of $L^1([0,\infty[)\times]-\infty,\infty[\times L^1_{loc}([0,\infty[, N(y)dy)$,

Proposition 3.2. *($B > 0$ at ∞)*
Assume (2.1)-(2.7), (2.9), (3.4) then for some $C_0 \in L^\infty_{loc}(]0,\infty[)$, we have

$$\lambda \in]0, \overline{\lambda}] \subset [0,\infty[, \tag{3.21}$$

$$\int_0^\infty N(y)\sqrt{yB(y)}dy \leq 1/\sqrt{\kappa - 1}, \tag{3.22}$$

$$\phi(y) \leq C_0(\lambda)(1 + y/\lambda), \tag{3.23}$$

$$\int_0^\infty |\ \partial_y N(y)\ |\ dy \leq \frac{2\kappa}{\kappa - 1}\overline{\lambda}, \tag{3.24}$$

$$\forall y > 0,\ |\ \partial_y \phi(y)\ | \leq C_0(\lambda)(1+y)(\lambda + \frac{2\kappa}{\kappa-1}B(y)). \tag{3.25}$$

Proof. We refer to the lemma 4.1 and 4.4 to prove this proposition. □

Thus the sequence $(N^\epsilon_A, \lambda^\epsilon_A, \phi^\epsilon_A)$ introduced in lemma 3.3 satisfies
- λ^ϵ_A belongs to the compact set $\Lambda^1 = [0, \sup_{\epsilon,A} \overline{\lambda}^\epsilon_A]$,
- N^ϵ_A belongs to the compact set S^1_{dens} of $L^1([0,\infty[)$ given by

$$S^1_{dens} = \{f \in L^1 : \int f(y)dy = 1, \int f(y) \sup_{\epsilon',A'} \sqrt{yB^{\epsilon'}_{A'}(y)}dy \leq 1/\sqrt{\kappa - 1},$$
$$\int_0^\infty \mid \partial_y f(y) \mid dy \leq \frac{2\kappa}{\kappa - 1} \sup_{\epsilon',A'} \overline{\lambda}^{\epsilon'}_{A'}\}, \quad (3.26)$$

at this stage we do not know that λ^ϵ_A is uniformly positive. But for $\lambda^\epsilon_A > 0$
- ϕ^ϵ_A belongs to $S^1_{dual}(\lambda^\epsilon_A)$ compact set of $L^1([0,\infty[, N(y)dy)$ given by

$$S^1_{dual}(\lambda^\epsilon_A, N^\epsilon_A) = \{g \in L^1([0,\infty[, N(y)dy) : \int g(y)N^\epsilon_A(y)dy = 1,$$
$$g(y) \leq C_0(\lambda^\epsilon_A)(1+y), \mid \partial_y g(y) \mid \leq C_0(\lambda^\epsilon_A)(1+y)(\lambda + \frac{2\kappa}{\kappa - 1}B^\epsilon_A(y))\}, \quad (3.27)$$

where $C_0(\mu)$ is a positive function on $]0,\infty[$. Thus the sequence $(\lambda^\epsilon_A, N^\epsilon_A, \phi^\epsilon_{A,})$ satisfies (3.21), (3.19), and belongs to the set $(\Lambda^1, S^1_{dens}, S^1_{dual})$. Moreover $(\lambda^\epsilon_A, N^\epsilon_A)$ belongs to a compact set of $L^1([0,\infty[) \times [0,\infty[$ and we can extract a convergent subsequence giving a solution to the limit problem. But the limit is a function of $L^1([0,\infty[)$, satisfying $N(y)e^{\int_0^y(\lambda+B(s))ds}$ increases and $\sqrt{yB(y)} > 0$ for y large enough (see (2.9)), thus the limit of λ^ϵ_A, λ_∞ satisfies $\lambda_\infty = \int B(y)N(y)dy > 0$. Therefore we can extract a subsequence such that $\lambda^\epsilon_A > \lambda_\infty/2$ and for this sequence $S^1_{dual}(\lambda^\epsilon_A, N^\epsilon_A) \subset S^1_{dual}(\lambda_\infty/2, N^\epsilon_A)$ which is a compact set of $L^1([0,\infty[, N(y)dy)$ and we can extract a convergent subsequence of ϕ^ϵ_A. Therefore Theorem 2.2 is proved in the case of (2.9).

Step3. Uniqueness of solution to (2.2)
Now, we prove the uniqueness of solution to this eigenproblem,

Lemma 3.4. *Assume there exists (N, λ, ϕ) solution to (1.2) and (N_1, λ_1) solution to (1.2) such that $0 < \int_0^\infty N_1(y)\phi(y)dy < \infty$ and $N_1/N \in L^2([0,\infty[, N(y)\phi(y)dy)$ then*

$$\lambda_1 = \lambda,$$

and there exists $C > 0$ such that $N_1 = CN$.

Proof. By integrating from 0 to ∞,

$$\phi(y)\partial_y N_1(y) + (\lambda_1 + B(y))N_1(y)\phi(y) = \int_y^\infty b(y,y')N_1(y')dy'\phi(y),$$

and

$$-N_1(y)\partial_y\phi(y) + (\lambda + B(y))N_1(y)\phi(y) = \int_y^\infty b(y',y)\phi(y')dy'N_1(y),$$

with $(N_1\phi)(0) = (N_1\phi)(\infty) = 0$ (see uniform bounds on $N,\ \phi$), we find,

$$\lambda_1 = \lambda.$$

We conclude using GRE method (see lemma 3.2),

$$\partial_t \int_0^\infty \mid \frac{N_1}{N} - C \mid N(y)\phi(y)dy = -\int_0^\infty \int_0^\infty b(y,y')N(y') \mid \frac{N_1}{N}(y') - C \mid$$
$$[\ \mathrm{sgn}(\frac{N_1}{N}(y) - C)\ \mathrm{sgn}(\frac{N_1}{N}(y') - C) - 1]\phi(y)dy'dy. \quad (3.28)$$

But $\int_0^\infty \mid \frac{N_1}{N} - C \mid N(y)\phi(y)dy$ is independent of t thus $[\ \mathrm{sgn}(\frac{N_1}{N}(y) - C)\ \mathrm{sgn}(\frac{N_1}{N}(y') - C) - 1] = 0$ on the support of $b(y,y')$ for all C thus $\frac{N_1}{N}(y) = \frac{N_1}{N}(y')$ on the support of $b(y,y')$ and

$$\partial_y \frac{N_1}{N}(y) = \int_0^\infty b(y,y')(\frac{N_1}{N}(y') - \frac{N_1}{N}(y))\frac{N(y')}{N(y)}dy' = 0,$$

i.e., N_1/N is a constant. □

4. Proof of *Proposition* 3.1, 3.2

Let (λ, N, ϕ) solution to (1.2) under assumptions (2.1)-(2.7), (3.4) and either (2.8) or (2.9). We prove some a priori bound on (λ, N, ϕ), i.e., if we assume there exists a solution then this solution satisfies the following property $\lambda \in \Lambda,\ \ N \in S_{dens},\ \ \phi \in S_{dual}$ where S_{dens} denotes $S_{dens} = S^0_{dens}$ or S^1_{dens} depending on which case under consideration (proposition 3.1 or 3.2), the same for S_{dual} and Λ.

4.1. *First step. bounds on* $\boldsymbol{N, \lambda}$

Lemma 4.1. *Assume (2.1), (2.2), (2.6), (2.7), (3.4) and (3.2) then*

$$0 \le \lambda \le \overline{\lambda}, \quad (4.1)$$

$$\int_0^\infty \frac{(\lambda y)^k}{k!} N(y)dy \le 1, \quad (4.2)$$

$$\int_0^\infty e^{\lambda y(1-\eta)} N(y)dy \le \frac{1}{\eta}, \quad \forall \eta \in (0,1), \quad (4.3)$$

$$\int_0^\infty \mid N(z+h) - N(z) \mid dz \le \frac{2\kappa}{\kappa - 1} h\overline{\lambda}. \quad (4.4)$$

Also, assume (2.1), (2.2), (2.6), (2.7), (2.9) and (3.4) then

$$\int_0^\infty N(y')\sqrt{y'B(y')}dy' \le 1/\sqrt{\kappa - 1}. \quad (4.5)$$

Proof. We first prove the bound (4.1) then the bounds on N.

Upper bound (4.1) Consider the problem equivalent to (1.2),

$$\begin{cases} \frac{\partial}{\partial y}G(y) + \lambda G(y) = \int_y^\infty e^{-\int_y^{y'} B(s)ds} b(y,y')G(y')dy', \quad y \geq 0, \\ \\ G(y=0) = 0, \end{cases} \tag{4.6}$$

where $G(y) = N(y)e^{\int_0^y B(s)ds}$ thus $G \geq 0$. From (3.4), (3.2) we deduce that $G \in L^1([0,\infty[)$ and $\lim_{y\to\infty} G(y) = 0$. Next we can integrate this equation between 0 and ∞ and we find, using (2.7),

$$\lambda \int_0^\infty G(y)dy = \int_0^\infty \int_y^\infty e^{-\int_y^{y'} B(s)ds'} b(y,y')G(y')dy'dy,$$

$$\lambda \int_0^\infty G(y)dy = \int_0^\infty \int_0^\infty 1_{y'\geq y}\, e^{-\int_y^{y'} B(s)ds'} b(y,y')G(y')dy'dy,$$

$$\lambda \int_0^\infty G(y)dy = \int_0^\infty \Big[\int_0^{y'} e^{-\int_y^{y'} B(s)ds'} b(y,y')dy\Big] G(y')dy',$$

and finally we arrive at the upper bound

$$\lambda \leq \overline{\lambda} < \infty. \tag{4.7}$$

Step1. Upper bound (4.3) Now we prove the bound

$$\int_0^\infty yN(y)dy \leq \frac{1}{\lambda}. \tag{4.8}$$

Multiply (1.2) by y

$$y\frac{\partial}{\partial y}N(y) + y[\lambda + B(y)]N(y) = y\int_y^\infty b(y,y')N(y')dy', \quad y \geq 0, \tag{4.9}$$

and integrate (4.9) from 0 to ∞ using (2.2), we obtain

$$\int_0^\infty (1-\lambda y)N(y)dy \geq 0, \tag{4.10}$$

thus proving (4.8). To prove the bound (4.3), we multiply (1.2) by y^k (with $k > 0$) and integrate the equation from 0 to ∞, now we recall that (b, B) satisfies (2.4), then we obtain

$$\int_0^\infty \frac{(\lambda y)^k}{k!} N(y)dy \leq \int_0^\infty N(y)dy,$$

thus

$$\int_0^\infty e^{\lambda y(1-\eta)} N(y)dy = \sum_{k=0}^\infty (1-\eta)^k \int_0^\infty \frac{(\lambda y)^k}{k!} N(y)dy \leq \sum_{k=0}^\infty (1-\eta)^k = \frac{1}{\eta},$$

and (4.3) follows. Instead of integrating from 0 to ∞ we integrate (1.2) by y^k (with $k>0$) from 0 to Y then for all $k \geq 1$,

$$Y^k N(Y) - k\int_0^Y y^{k-1}N(y)dy + \lambda \int_0^Y N(y)dy = \int_0^Y y^k \int_y^\infty b(y,y')N(y')dy'dy - \int_0^Y y^k B(y)N(y)dy,$$

$$Y^k N(Y) - k\int_0^Y y^{k-1}N(y)dy + \lambda \int_0^Y N(y)dy = \int_0^\infty \int_0^{\min(y',Y)} y^k b(y,y')dyN(y')dy' - \int_0^Y y^k B(y)N(y)dy,$$

$$Y^k N(Y) \leq k\int_0^Y y^{k-1}N(y)dy + \int_Y^\infty y'^k B(y')N(y')dy',$$

$$\frac{\lambda^k(1-\epsilon)^k Y^k N(Y)}{k!} \leq \frac{(1-\epsilon)^k\lambda^k}{(k-1)!}\int_0^Y y^{k-1}N(y)dy + \int_Y^\infty \frac{\lambda^k}{k!}(1-\epsilon)^k y'^k B(y')N(y')dy',$$

using (4.2) and summing for all k we find

$$N(y) \leq C_\epsilon e^{-\lambda(1-\epsilon)y}(1 + \int_y^\infty B(y')e^{\lambda(1-\epsilon)y'}N(y')dy'),$$

with $C_\epsilon = 1/\epsilon$.

Step2. Bound (4.4) We integrate (1.2) from z to $z+h$

$$N(z+h) - N(z) = -\int_z^{z+h}(\lambda + B(y))N(y)dy + \int_z^{z+h}\int_y^\infty b(y,y')N(y')dy'dy,$$

and integrate $\mid N(z+h) - N(z) \mid$ from 0 to ∞ then using (2.1)

$$\int_0^\infty \mid N(z+h) - N(z)\mid dz \leq h\int_0^\infty (\lambda + B(y))N(y)dy + h\int_0^\infty\int_y^\infty b(y,y')N(y')dy'dy \leq \frac{2\kappa}{\kappa-1}h\overline{\lambda},$$

because

$$\int_0^\infty (\lambda + B(y))N(y)dy = \lambda\int_0^\infty N(y)dy + \int_0^\infty B(y)N(y)dy = \lambda + \int_0^\infty B(y)N(y)dy = \lambda + \frac{\lambda}{\kappa - 1},$$

then using (2.1)

$$\int_0^\infty \int_y^\infty b(y,y')N(y')dy'dy = \int_0^\infty \int_0^{y'} b(y,y')dyN(y')dy'$$
$$= \kappa \int_0^\infty B(y')N(y')dy' = \frac{\kappa\lambda}{\kappa - 1}.$$

Step3. Bound (4.5) Now, we assume (2.9) and integrate (1.2) from 0 to ∞ then

$$\lambda = \int_0^\infty (\int_0^{y'} b(y,y')dy - B(y'))N(y')dy',$$

but we have (4.8) thus, combining with (2.1)

$$\int_0^\infty y'N(y')dy' \int_0^\infty B(y')N(y')dy' \leq 1/(\kappa - 1),$$

and using Cauchy-Schwarz inequality we finally obtain

$$\int_0^\infty N(y')\sqrt{y'B(y')}dy' \leq 1/\sqrt{\kappa - 1}.$$ □

Lemma 4.2. *Assume (2.3), (2.6), (2.7), (2.8) (with* $A = \sup_y \{y \in SuppB\}$*)*

$$\int_0^{y'} b(y,y')dy \geq \kappa B(y'), \tag{4.11}$$

and (λ, N, ϕ) *is the solution to (1.2) then*

$$\lambda \geq \underline{\lambda}.$$

Proof. Assume $\lambda < \underline{\lambda}$, (2.6), (2.7), then there exists a solution to

$$\partial_y M + (\underline{\lambda} + B)M = \int_y^A b(y,y')M(y')dy', \tag{4.12}$$

such that $M(A) = N(A)$. We notice that

$$\overline{M}(y) := M(A-y), \quad \overline{N}(y) := N(A-y),$$

satisfy

$$\partial_y \overline{M} - (\underline{\lambda} + \overline{B})\overline{M} + \int_y^A b(A-y, A-y')\overline{M}(y')dy' = 0,$$

$$\partial_y \overline{N} - (\lambda + \overline{B})\overline{N} + \int_y^A b(A-y, A-y')\overline{N}(y')dy' = 0,$$

thus

$$\partial_y \overline{M} - (\lambda + \overline{B})\overline{M} + \int_y^A b(A-y, A-y')\overline{M}(y')dy' > 0,$$

and $M(A-y)$ is a supersolution to the equation satisfied by $N(A-y)$. Now, integrate (4.12) from 0 to A then

$$M(A) - M(0) = \int_0^A \Big[\int_0^{y'} b(y,y')dy - B(y')\Big] M(y')dy' - \underline{\lambda} \int_0^A M(y')dy'.$$

thus using (4.11)

$$-M(0) \geq \int_0^A \Big[(\kappa-1)B(y') - \underline{\lambda}\Big] M(y')dy' - M(A).$$

But we have $M(x) \geq M(A)$ on $[(1-\eta)A, A]$, indeed

$$M(A) - M(x) = \int_x^A \Big[\int_x^{y'} b(y,y')dy - B(y')\Big] M(y')dy' - \underline{\lambda} \int_x^A M(y')dy',$$

and for all $(1-\eta)A \leq x \leq y' \leq A$ we have $(1-\eta)y' \leq (1-\eta)A \leq x \leq y' \leq A$ and

$$x \geq (1-\eta)y',$$

thus using (2.3) we have on $[A(1-\eta), A]$

$$F(x) := \sup_{y' \geq x} \Big[\int_x^{y'} b(y,y')dy - B(y')\Big] \leq \sup_{y' \geq x \geq (1-\eta)y'} \Big[\int_{(1-\eta)y'}^{y'} b(y,y')dy - B(y')\Big] \leq 0.$$

Therefore,

$$-M(0) \geq M(A)(\kappa-1) \int_{A(1-\eta)}^A B(y')dy' - \underline{\lambda} \int_0^A M(y')dy' - M(A),$$

using (2.3), the growth of $M(y)e^{\int_0^y (\lambda + B(s))ds}$ and (2.8), we obtain

$$-M(0) \geq M(A)\Big(-\underline{\lambda} \int_0^A e^{\int_y^A (\lambda + B(s))ds} dy + (\kappa-1)\int_{A(1-\eta)}^A B(y')dy' - 1\Big) \geq 0.$$

Thus $M(0) \leq 0$ and $M(A-y)$ is a supersolution to $N(A-y)$, thus $N(0)$ must be non positive but $N(0) = 0$ therefore $\lambda < \underline{\lambda}$ is absurd and we have $\lambda \geq \underline{\lambda} = \frac{\kappa-1}{Ae^{\int_0^A (\overline{\lambda} + B(s))ds}} (\int_{A(1-\eta)}^A B(y')dy' - 1)$. □

4.2. *Second step. Bounds on ϕ*

First we prove that N is strictly positive on the support of B,

Lemma 4.3. *Under assumptions (2.1), (2.3), (2.5), (2.6), (2.7), if (N, λ) is solution to (1.2) and let $\underline{a} = \inf$ Supp $B \geq 0$, $\overline{a} = \sup$ Supp $B \leq \infty$ then*

$$N(y) > 0, \ \ \forall y > \underline{a}, \ \textit{i.e.,} \ \ \textit{Supp } N \subset \textit{Supp } B.$$

Proof. Let $a = \inf\{y : \ N(y) > 0\}$ then $a < \overline{a}$ otherwise $N(y) = 0$ for all $y \leq \overline{a}$ and $\partial_y N + \lambda N = 0$ for all $y \geq \overline{a}$ and $N = 0$. Moreover if we assume that $a > \underline{a}$ then

$$\int_a^\infty b(y,y')N(y')dy' = 0, \ \ \forall y \leq a,$$

and $b(y, y') = 0$ for all $y \le a$, $y' \ge a$, thus $\int_0^a b(y, y')dy = 0$ for all $y' \ge a$. But

$$\int_{(1-\eta)y'}^{y'} b(y, y')dy \le B(y'),$$

thus

$$\int_0^{(1-\eta)y'} b(y, y')dy \ge (\kappa - 1)B(y'),$$

and $B(y') = 0$ for all $a \le y' \le a/(1-\eta)$, which is absurd and contradict that $a > \underline{a}$. Indeed using $Supp(B)$ is a connex set (2.5) and $B(y') = 0$ on $(a, a/(1-\eta))$ we have $B(y') = 0$ on $(\underline{a}, a/(1-\eta))$ and $\underline{a} < \inf\ Supp\ B$. □

Lemma 4.4. *Under assumptions (2.1)-(2.6), (3.3) there exists $C_0(\lambda)$ strictly positive and defined on $]0, \infty]$, such that*

$$\phi(y) \le C_0(\lambda)(y + 1).$$

Proof. We refer to Ref. 19 to the following construction of a supersolution of ϕ.

Step1. Bound on $(0, 1/\lambda)$ We integrate the equation of ϕ from Y to x_0 where x_0 is chosen such that $\int_0^{x_0}\int_0^y b(y', y)dy'dy = \kappa \int_0^{x_0} B(y)dy = \alpha \in]0, 1/2]$ (using (2.1), (2.6)) then we have

$$\phi(Y) \le \phi(x_0) + \int_Y^{x_0}\int_0^y b(y', y)\phi(y')dy'dy,$$

and

$$\phi(Y) \le \phi(x_0) + \sup_{z\in(0,x_0)} \phi(z) \int_Y^{x_0}\int_0^y b(y', y)dy'dy \le \phi(x_0) + \alpha \sup_{z\in(0,x_0)} \phi(z),$$

$$\sup_{Y\in(0,x_0)} \phi(Y) \le \frac{1}{1-\alpha}\phi(x_0).$$

Using the decay of $\phi(y)e^{-\int_0^y(\lambda+B(s))ds}$, there exists $C(\lambda)$ such that

$$\sup_{(0,1/\lambda)} \phi \le C(\lambda)\phi(x_0).$$

Noticing that $\int_0^\infty N(y)\phi(y)dy = 1$, we have

$$1 \ge \int_0^{x_0} N(y)\phi(y)dy \ge \phi(x_0)\int_0^{x_0} e^{-\int_y^{x_0}(\lambda+B(s))ds}N(y)dy$$

moreover using lemma 4.3 we have $\int_0^{x_0} N(y)dy > 0$ and

$$\sup_{(0,1/\lambda)} \phi \le C_0(\lambda).$$

Step2. Bound on $(1/\lambda, \infty)$ Now, let $\underline{\phi}(y) = \phi(\chi(y))$, $\underline{B}(y) = B(\chi(y))$, then

$$\partial_y \underline{\phi} - \chi'(y)(\lambda + \underline{B}(y))\underline{\phi} = -\chi'(y) \int_0^{\chi(y)} b(y', \chi(y))\underline{\phi}(\chi^{-1}(y'))dy'.$$

Using (2.2), (3.3), we have $\underline{v}(y) = C\chi(y)$ is a supersolution if

$$\chi'(y) - \chi'(y)\lambda\chi(y) = \chi'(y)(1 - \chi(y)\lambda) \geq 0,$$

$$\chi(0) = \infty,$$

which is satisfied for $\chi(y) = (1/y + 1/\lambda)$ and C large enough, thus $\phi(y) \leq Cy$ for $y \geq 1/\lambda$.

Step3. Bound on $(0, \infty)$ Finally, we have,

$$\phi(y) \leq C_0(\lambda)(y + 1). \qquad \square$$

5. Extension to cell division problem with non constant growth speed

In this section we consider the eigenproblem

$$\begin{cases} \frac{\partial}{\partial y}(V(y)N(y)) + [\lambda + B(y)]N(y) = \int_y^\infty b(y, y')N(y')dy', \quad y \geq 0, \\ N(y = 0) = 0, \quad N \geq 0, \quad \int N(y)dy = 1, \\ -V(y)\frac{\partial}{\partial y}\phi(y) + [\lambda + B(y)]\phi(y) = \int_0^y b(y', y)\phi(y')dy', \quad y \geq 0, \\ \phi \geq 0, \quad \int N(y)\phi(y)dy = 1. \end{cases} \tag{5.1}$$

where $B(y) = y^\gamma$, $V(y) = y^\mu$.

Proposition 5.1. *Assume (2.1), (2.2), (2.3) and*

$$\gamma > 0, \quad \gamma + 1 - \mu > 0, \tag{5.2}$$

$$\sup_y \int_0^y b(y', y)e^{-\int_{y'}^y \frac{B(s)}{V(s)}ds} dy' < \infty, \tag{5.3}$$

$$c_k(y) := \int_0^y b(y', y)(y'/y)^k / B(y)dy' < 1, \quad k \geq \gamma - \mu + 1, \tag{5.4}$$

then there exists a solution to (5.1).

Proof. Here, we are in the case $B > 0$ at ∞, then we prove some a priori bound on (λ, N, ϕ) as in the proposition 3.2. If we assume there exists a regularized solution, defined as in the proposition 3.2 then we prove this solution belongs a compact set. Assume (λ, N, ϕ) is a solution to (5.1) then

Step1. λ bound: Proceeding analogously to similar bound (4.7) in the lemma 4.1, we have the following bound on λ,

$$0 \le \lambda \le \sup_y \int_0^y b(y',y) e^{-\int_{y'}^y \frac{B(s)}{V(s)} ds} dy' < \infty.$$

Step2. N bound: We have the following bound on N

$$\int N(y) y^{k+1} dy \int y^\gamma N(y) dy \le \int y^{\mu+k} N(y) dy, \quad \forall k > 0,$$

which gives a bound $\int y^\alpha N(y) dy \le \gamma/(\kappa-1)$, $\alpha = \gamma - \mu + 1 > 0$.
Indeed multiplying by y^{k+1}, we have

$$\frac{\partial}{\partial y}(V(y)N(y)) + [\lambda + B(y)]N(y) = \int_y^\infty b(y,y')N(y')dy',$$

and integrating from 0 to ∞ (for $k+\gamma+1>0$) then

$$-(k+\mu+1)\int y^{k+\gamma} N(y) dy + \lambda \int y^{k+1} N(y) dy + \int y^{k+\gamma+1} N(y) dy \le c_k \int y^{\gamma+k+1} N(y) dy,$$

and $\lambda = (\kappa - 1)\int y^\gamma N(y)dy$. Thus, using $c_k = \int_0^{y'} b(y,y') y'^k dy'/B(y')y'^k < 1$,

$$(\kappa-1)\int y^\gamma N(y) dy \int y^{k+1} N(y) dy \le (k+\mu+1)\int y^{k+\mu} N(y) dy,$$

and for $k+\mu = \gamma$, i.e., $k+1 = \gamma - \mu + 1 > 0$ we have

$$\int y^{\gamma-\mu+1} N(y) dy \le \gamma/(\kappa-1).$$

Step3. ϕ bound: We have

$$V(y)\partial_y \phi(y) - (\lambda + B(y))\phi(y) = -\int_0^y b(y',y)\phi(y')dy'. \tag{5.5}$$

Similarly to the lemma 4.4, we bound ϕ in a neighborhood of ∞ and 0.

Step3a. Bound in a neighborhood of ∞ Let $\overline{\phi}(y) = \phi(1/y)$ then

$$y^2 V(1/y)\partial_y \overline{\phi}(y) + (\lambda + B(1/y))\overline{\phi}(y) = \int_0^{1/y} b(y', 1/y)\phi(y')dy', \tag{5.6}$$

$\overline{v}(y) = 1/y$ is a supersolution (5.6) in a neighbourhood of 0 if we have

$$\overline{V}(y)\partial_y \overline{v}(y) \ge -\frac{1}{y^2}\Big((\lambda + \overline{B}(y))\overline{v}(y) - \int_0^{1/y} b(y',y)\overline{v}(1/y')dy'\Big),$$

where $\overline{V}(y) = V(1/y)$ and $\overline{B}(y) = B(1/y)$. Thus $\overline{v}(y) = 1/y$ is a supersolution (5.6) in a neighbourhood of 0 if

$$-\overline{V}(y)/y^2 \ge -\lambda/y^3,$$

so if

$$y\overline{V}(y) \le \lambda.$$

It remains to study the different possibilities for V, i.e., the different μ:

If $\mu < 1$ then $\overline{v}$ is an supersolution in a neighbourhood of 0.

If $\mu = 1$ then $\lambda = 2 > 1$ and $\overline{v}$ is a global supersolution.

If $\mu > 1$ then we let $\overline{v}(y) = (1/y)^k$ and prove $\overline{v}$ is a supersolution. Indeed, if we assume that $c_k = \int_0^y b(y', y)(y'/y)^k / B(y)dy'$ is well defined, $c_0 = 2$, $c_1 = 1$ and $c_k < 1$ as k large enough then (using $\gamma > \mu + 1$) we have

$$ky\overline{V}(y) \le \lambda + (1 - c_k)\overline{B}(y),$$

thus

$$-k\overline{V}(y)/y^{k+1} \ge -\frac{\lambda}{y^{k+2}} - (1 - c_k)\overline{B}(y)/y^{2+k},$$

and so $\overline{v}$ is a supersolution in a neighbourhood of 0.

Step3b. Bound in a neighborhood of 0 Now, we prove that ϕ is bounded in a neighbourhood of 0 for the different μ:

If $\mu < 1$ then $\phi(y)e^{-\int \frac{(\lambda+B(s))}{V(s)}ds}$ is decreasing and $1/V(y)$ belongs to $L^1([0, A])$ for all $A > 0$, thus, as in the case $V(y) = 1,\ \ \mu = 0$ we can prove that ϕ is bounded in a neighbourhood of 0, the same proof holds.

If $\mu = 1$ then by the last point $\overline{v}(y) = 1/y$ is a supersolution and $\phi(y) \le y$ (more precisely $\phi(y) = y$).

If $\mu > 1$ then let ξ defined by $\xi'(y) = V(\xi(y))$, i.e., $\xi(x) = (\frac{1-x}{1-\mu})^{\frac{1}{1-\mu}}$, $\underline{\phi}(y) = \phi(\xi(y))$, thus

$$-\partial_y \underline{\phi} + (\lambda + B(\xi(y)))\underline{\phi}(y) = \int_{\xi^{-1}(0)}^{y} b(\xi(y'), \xi(y))\underline{\phi}(y')dw(y'),$$

$$-\partial_y \underline{\phi} + (\lambda + \overline{B}(y))\underline{\phi}(y) = \int_{\xi^{-1}(0)}^{y} b(\xi(y'), \xi(y))\underline{\phi}(y')dw(y'),$$

where $dw(y') = \partial_y \xi(y')dy'$, $\underline{B}(y) = B(\xi(y))$ and $\underline{b}(y, y') = b(\xi(y), \xi(y'))$, thus using (2.2), we obtain

$$\int_{\xi^{-1}(0)}^{y} \underline{b}(y', y)dw(y') = \kappa \underline{B}(y)$$

with $\xi^{-1}(0) = \infty$ because $\xi(x) = (\frac{1-x}{1-\mu})^{\frac{1}{1-\mu}}$ with $\mu > 1$,

$$\int_{\xi^{-1}(0)}^{y} \xi(y')\underline{b}(y', y)dw(y') = \xi(y)\underline{B}(y).$$

Therefore, for x_0 such that $0 < alpha = \int_{-\infty}^{x_0} B(\xi(y))dy < 1/2$ (which exists because $\gamma/(\mu - 1) > 1$), proceeding similarly to the lemma 4.4,

$$\sup_{(\infty,x_0)} \underline{\phi} \leq \underline{\phi}(x_0)/(1 - \alpha),$$

and finally ϕ is bounded in a neighbourhood of 0.

Step4. Compactness: Using the construction given in Ref. 19 (or like in lemmas 3.1, 3.17) we have the existence of a sequence of solution to a regularized problem of (5.1). Then we use the bounds we find (step1. to step3.) to prove the compactness (see below Λ, S_{dens}, S_{dual}) of this sequence and conclude to the existence of a solution to (5.1).
- λ belongs to the compact set $\Lambda = [0, \overline{\lambda}]$,
- N belongs to the compact set S_{dens} of $L^1([0, \infty[)$ given by

$$S_{dens} = \{f \in L^1 : \int_0^\infty f(y)dy = 1, \int_0^\infty f(y)y^{\gamma-\mu+1}dy \leq \gamma/(\kappa - 1), \\ \int_0^\infty \mid \partial_y V(y)f(y) \mid dy \leq \frac{2\kappa}{\kappa - 1}\overline{\lambda}\}, \quad (5.7)$$

- ϕ belongs to the compact set S_{dual} of $L^1([0, \infty[, N(y)dy)$ given by

$$S_{dual} = \{g \in L^1 : \int_0^\infty g(y)N(y)dy \geq 1, \;\; g(y) \leq C_1(y), \\ \mid V(y)\partial_y g(y) \mid \leq C_1(y)(\overline{\lambda} + \frac{2\kappa}{\kappa - 1}B(y))\}, \quad (5.8)$$

with $C_1(y)$ defined above and $L_l^\infty oc(]0, \infty[)$. Proceeding similarly to the theorems 2.1, 2.2, we conclude the proof of this theorem. □

6. More precise results

In this paper we prove the existence of (λ, N, ϕ) for a large class of B, b. Now we are interesting in more precise results as regularity, existence in the case where $b(y, y') = y'^\gamma \wp(y/y')/y'$ (mitosis or some more general cell divison). Moreover we construct explicitely the solution (λ, N, ϕ) in the case where for instance $Supp_{y'} b(y', y) \in [\eta y, (1 - \eta)y]$ and prove that $N(y) \sim_{y\to\infty} e^{-\int_0^y (\lambda+B)(s)ds}$.

6.1. *Homogeneous cell division* $b(y,y') = y'^{\gamma}\wp(y/y')/y'$

Identities (2.1), (2.2), (2.3) become

$$\int_0^{y'} \wp(\frac{y}{y'})\frac{dy}{y'} = \kappa, \tag{6.1}$$

$$\int_0^{y'} y\wp(\frac{y}{y'})\frac{dy}{y'} = y', \tag{6.2}$$

$$\int_{(1-\eta)y'}^{y'} \wp(\frac{y}{y'})\frac{dy}{y'} \le 1. \tag{6.3}$$

Theorem 6.1.
We assume that $\wp, \gamma$ satisfy (6.1), (6.2), (6.3), (2.10), (2.11), then assumptions (2.5), (2.6), (2.7), (2.9) (for b, B) are fulfilled and Theorem 2.1 holds.

Proof. Assumptions (2.10) on $\gamma, \wp$ imply directly the conditions (2.6), (2.9) on b, B. Assume (2.7) and let

$$\int_0^{y'} b(y,y')e^{-\int_y^{y'} B(s)ds}dy = y'^{\gamma}\int_0^1 \wp(\theta)e^{-\frac{y'^{\gamma+1}}{\gamma+1}(1-\theta^{\gamma+1})}d\theta,$$

then

$$\sup_{y'} y'^{\gamma}e^{-\frac{y'^{\gamma+1}}{\gamma+1}(1-\theta^{\gamma+1})} = \frac{\gamma^{\gamma/(\gamma+1)}e}{(1-\theta^{\gamma+1})^{\gamma/(\gamma+1)}} = \frac{C_\gamma}{(1-\theta^{\gamma+1})^{\gamma/(\gamma+1)}},$$

and finally

$$\overline{\lambda} \le \int_0^1 \wp(\theta)\frac{C_\gamma}{(1-\theta^{\gamma+1})^{\gamma/(\gamma+1)}}d\theta \le C_\gamma\overline{\lambda'} < \infty.$$ □

6.2. *Regularity*

The regularity of the solution is given by the behaviour of $b(y,y')$. For example for the age structured model, we have a discontinuity at $y=0$ given by the behaviour of $b(y,y') = B(y')(\delta_{(y'=y)} + \delta_{(y=0)})$. Indeed

$$\lim_{y_1\to 0} N(y_1) - N(0) = \int_0^\infty \lim_{y_1\to 0}\int_0^{\min(y_1,y')} b(y,y')dyN(y')dy' = \int_0^\infty B(y')N(y')dy'.$$

More precisely we have

Lemma 6.1. *Assume (2.6) and the existence of (λ, N, ϕ) solution to (1.2) then*

$$\forall y_0 \ge 0, \quad \lim_{y_1\to y_0, y_1>y_0} \mid N(y_1) - N(y_0) \mid = \int_0^\infty \lim_{y_1\to y_0}\int_{y_0}^{\min(y_1,y')} b(y,y')dyN(y')dy'.$$

The same result holds for ϕ.

Proof. Let $0 \leq y_0 < y_1$ and integrate the equation

$$\frac{\partial}{\partial y}N(y) + [\lambda + B(y)]N(y) = \int_y^\infty b(y,y')N(y')dy', \quad y \geq 0,$$

from y_0 to y_1 then

$$\mid N(y_1) - N(y_0) \mid = -\int_{y_0}^{y_1} (B(y)+\lambda)N(y)dy + \int_{y_0}^\infty \int_{y_0}^{\min(y_1,y')} b(y,y')dyN(y')dy'.$$

Therefore the regularity of N is given by the term $\int_{y_0}^{\min(y_1,y')} b(y,y')dy$. Indeed the existence of N solution to (1.2) is $L^1(\mathbb{R}_+)$ by definition, λ is bounded and $Ne^{\int_0^y (B(s)+\lambda)ds}$ is increasing, thus N is $L^\infty_{loc}(\mathbb{R}_+)$ and

$$\int_{y_0}^{y_1} (B(y)+\lambda)N(y)dy = C(y_1 - y_0), \quad \lim_{z\to 0} C(z) = 0,$$

$$\int_{y_0}^\infty N(y')dy' = D > 0,$$

and finally

$$\lim_{y_1 \to y_0} \mid N(y_1) - N(y_0) \mid = \int_{y_0}^\infty \lim_{y_1\to y_0} \int_{y_0}^{\min(y_1,y')} b(y,y')dyN(y')dy'.$$

Using the same proof, we obtain

$$\lim_{y_1 \to y_0} \mid \phi(y_1) - \phi(y_0) \mid = \int_0^{y_1} \lim_{y_1\to y_0} \int_{\max(y_0,y')}^{y_1} b(y,y')dy\phi(y')dy'. \qquad \square$$

Remark 6.1. The regularity of N in 0 in the case $Supp_{y'}\ b(y',y) \subset [\eta y, (1-\eta)y]$, $0 < \eta < 1$, by the last point we have

$$\forall k \geq 0,\ \exists C_k > 0, \quad N(x) \leq C_k x^k,$$

and more precisely,

$$\forall k \geq 0,\ \exists D_k > 0, \quad N(x) \leq D_k e^{|ln(x)/ln(\eta)|^k ln(\eta)/D_k},$$

which gives more information on the behavior of N near 0, which is much better than a polynomial term but not better than e^{-1/x^k}.

6.3. *Example of construction of solution*

We recall the eigenproblem

$$\frac{\partial N}{\partial y} + (B(y) + \lambda)N(y) = \int_y^\infty b(y, y')N(y')dy', \quad y \geq 0, \tag{6.4}$$

$$-\frac{\partial \phi}{\partial y} + (B(y) + \lambda)\phi(y) = \int_0^y b(y', y)\phi(y')dy', \quad y \geq 0, \tag{6.5}$$

$$N(0) = 0, \quad N(y) \geq 0, \quad y \geq 0, \quad \int_0^\infty N(y')dy' = 1, \tag{6.6}$$

$$\phi(y) \geq 0, \quad y \geq 0, \quad \int_0^\infty N(y')\phi(y')dy' = 1. \tag{6.7}$$

In some case we can construct explicitely the solution of this problem. The strategy we adopt is to construct a solution N, ϕ to (6.4), (6.5) for all $\lambda \geq 0$, then prove the existence of $\lambda > 0$ such that N, ϕ are positive and satisfy (6.6), (6.7).
In order to construct a solution to (6.4), (6.5), we notice that it is equivalent to construct W and ψ solution to

$$\begin{cases} \frac{\partial W}{\partial y} = \int_y^\infty b(y, y')e^{-\int_y^{y'}(B(s)+\lambda)ds}W(y')dy', \quad y \geq 0, \\ -\frac{\partial \psi}{\partial y} = \int_0^y b(y', y)e^{-\int_{y'}^{y}(B(s)+\lambda)ds}\psi(y')dy', \quad y \geq 0, \\ W\,, \psi \in L^\infty(]0, \infty[). \end{cases} \tag{6.8}$$

We search W under the form

$$W(x) = \sum_{i=0}^\infty (-1)^i w_i(x),$$

where w_i is defined recursively by

$$w_{i+1}(x) = \int_x^\infty \int_y^\infty b(y, y')e^{-\int_y^{y'}(B(s)+\lambda)ds}w_i(y')dy'dy \geq 0,$$

$$w_0(y) = 1.$$

Similarly, we set

$$\psi(x) = \sum_{i=0}^\infty (-1)^i \psi_i(x),$$

where ψ_i is defined recursively by

$$\psi_{i+1}(x) = \int_0^x \int_0^y b(y', y)e^{-\int_{y'}^{y}(B(s)+\lambda)ds}\psi_i(y')dy'dy \geq 0,$$

$$\psi_0(y) = 1.$$

In order to study the convergence and the properties of these sums, we let Z, Z^* positive operators used in the definition of the sequences w_i, ψ_i

$$\begin{cases} Z(f)(x) = \int_x^\infty \int_y^\infty b(y,y')e^{-\int_y^{y'}(B(s)+\lambda)ds} f(y')dy'dy \geq 0, \\ Z^*(g)(x) = \int_0^x \int_0^y b(y',y)e^{-\int_{y'}^{y}(B(s)+\lambda)ds} g(y')dy'dy \geq 0. \end{cases} \tag{6.9}$$

Under some assumptions on B and b the operators are well defined onto the bounded valued functions and the sequences and their sums are well defined, derivable and by construction satisfy equation (6.8).

$$\int_y^\infty b(y,y')e^{-\int_y^{y'} B(s)ds} dy \in L^1([0,\infty[), \tag{6.10}$$

$$\lim_{r\to 0} \int_0^\infty \int_y^{y(1+r)} b(y,y')e^{-\int_y^{y'} B(s)ds} dy'dy = 0, \tag{6.11}$$

$$y \int_0^y b(y',y)e^{-\int_{y'}^{y} B(s)ds} dy' \in L^1([0,\infty[). \tag{6.12}$$

These assumptions cover several cases where the probability of dividing a cell of size y' into two cells of size y and $y'-y$ only depends on $\frac{y}{y'}$, $b(y,y') = \wp(\frac{y}{y'})B(y')/y'$ with *Supp* $\wp \subset [\theta_0,\theta_1] \subset]0,1[$ and either (2.8) or (2.9). When $B(y') = {y'}^\gamma$ the assumptions can be written in a simpler way,

$$\gamma \geq 0,\ \wp \in M^1([\theta_0,\theta_1]), \quad 0 < \theta_0 < \theta_1 < 1, \tag{6.13}$$

or

$$\gamma \geq 0,\ \wp \in M^1([0,1]), \tag{6.14}$$

$$\int_0^1 \frac{\wp(\theta)}{(1-\theta)^2\theta^{\gamma+2}(1-\theta^{\gamma+1})^{\gamma/(\gamma+1)}} d\theta < \infty. \tag{6.15}$$

Finally we prove

Theorem 6.2. *Assume (2.2), (2.3), (2.6), (6.10), (6.11), (6.12), either (2.8) or (2.9) then there exists a solution of (6.4), (6.5) satisfying (6.6), (6.7). Moreover in the compact support case (2.8) with* $A = \sup$ *Supp* B, *we have,*

$$\phi(z) = 0, \ \ \forall z \geq A. \tag{6.16}$$

Remark 6.2. We notice that assumption (2.5) is not used in this theorem, we only use it to prove the L^∞ bound of ϕ and thus compactness of ϕ^ϵ or ϕ^ϵ_A.

Step1. Construction The construction of W is given by the lemma

Lemma 6.2. *Let A, B two linear operators such that $Ker(A) - Ker(B)$ is a non-empty set. If there exists $(x_i)_{i\geq 0}$ such that*
1) $x_0 \in Ker(A) - Ker(B)$ and $A(x_{i+1}) = B(x_i)$,
2) $X = \sum_{i\geq 0} x_i$ converges in $D(A), D(B)$ with $A(X) = \sum_{i\geq 0} A(x_i)$ and $B(X) = \sum_{i\geq 0} B(x_i)$,
then it is clear that $A(X) = B(X)$.

Here $A(f)(y) = \frac{\partial f}{\partial y}(y)$ and $B(f)(y) = \int_y^\infty b(y, y') e^{\int_y^{y'} (B(s)+\lambda)ds} f(y')dy'$,

Lemma 6.3. *Assume (2.6), (6.10),(6.11), then for all $\lambda > 0$, there exists $(x_i)_{i\geq 0}$ such that*
1) $x_0 \in Ker(A) - Ker(B)$ and $A(x_{i+1}) = B(x_i)$,
2) $X = \sum_{i\geq 0} x_i$ converges in $D(A), D(B)$ with $A(X) = \sum_{i\geq 0} A(x_i)$ and $B(X) = \sum_{i\geq 0} B(x_i)$,
thus there exists a solution to (6.8).

Proof. Assumptions (2.6), (6.10) prove the existence of the sequence w_i in $L^\infty(]0, \infty[)$. Now we notice that assumption (6.10) gives the existence of the sum of w_i in a neigbourhood of infinity, indeed in this case the operator is strictly bounded by 1 into the bounded valued functions space, moreover $\frac{d}{dy} W(y) = \sum_{i=0}^\infty (-1)^i \frac{d}{dy} w_i(y)$. The assumptions lead to the proof of convergence on $]0, \infty[$. Indeed let $x > 0$ and

$$Z(f)(x) = Z_1(f)(x) + Z_2(f)(x), \quad f \in L^\infty(]0, \infty[), \tag{6.17}$$

$$Z_1(f)(x) = \int_x^\infty \int_{y'}^{y'(1+r)} b(y, y') e^{-\int_y^{y'} (B(s)+\lambda)ds} f(y)dydy', \tag{6.18}$$

$$Z_2(f)(x) = \int_x^\infty \int_{y'(1+r)}^\infty b(y, y') e^{-\int_y^{y'} (B(s)+\lambda)ds} f(y)dydy'. \tag{6.19}$$

We define recursively the operator $Z^{(k+1)}(f) = Z(Z^{(k)}(f))$ and $Z^{(0)}(f) = f$ (similary for Z_1, Z_2) then

$$| Z^{(n)}(f)(x) | \leq \sum_0^n \binom{n}{k} | Z_1^{(n-k)}(f)(x) || Z_2^{(k)}(f)(x) | .$$

Now, let $\epsilon > 0$, the function $Z_1(1)$ is uniformly bounded by $\frac{\epsilon}{2}$ if r is small enough (assumption (6.11)) and there exists $x_0 > 0$ such that the operator Z_2 satisfies $| Z_2(f)(x_1) | \leq \frac{\epsilon}{2} | f |_\infty$ for all $x_1 \geq x_0$ (see (6.10)). Moreover,

$$\begin{aligned} | Z_2^{(m_0+m_1)}(1)(x) | &\leq | Z_2^{(m_1)}(1)(x) || Z_2^{(m_0)}(1)((1+r)^{m_1}x) | \\ &\leq C_{m_1} | Z_2^{(m_0)}(1)((1+r)^{m_1}x) |, \end{aligned}$$

thus if $m_1 \geq \frac{ln(x_0/x)}{ln(1+r)}$ we have $\mid Z_2^{(m_0+m_1)}(1)(x) \mid \leq C_{m_1}(\frac{\epsilon}{2})^{m_0}$. Finally, let $m = \max(0, \frac{ln(x_0/x)}{ln(1+r)})$

$$\mid Z^{(n)}(1)(x) \mid \leq C_m \sum_0^n C_k^n \mid (\frac{\epsilon}{2})^{n-k} \mid\mid (\frac{\epsilon}{2})^{k-m} \mid = C_m (\frac{\epsilon}{2})^{-m} \epsilon^n,$$

and the sum of w_i is absolutly convergent, moreover $\frac{d}{dy}W(y) = \sum_{i=0}^{\infty}(-1)^i \frac{d}{dy}w_i(y)$. A similar proof gives the convergence of the sum of general term ψ_i. Thus we have proved the existence of w, ψ for all $\lambda > 0$. □

Step2. Positivity We prove the existence of λ such that the constructed solution satisfies (6.6).
Under the assumptions (2.6), (6.10),(6.11) we have (by Step1. lemma 6.3) the existence of W, ψ solution of (6.8) for all $\lambda \geq 0$,

Lemma 6.4. *Moreover, if we assume that (2.8) or (2.9) is satisfied then there exists λ such that $N(0) = W(0) = 0$ and W, ψ are non negative and different from* 0.

Proof. If λ is large enough, the solution satisfies

$$W \textit{ is strictly positive on } [0, \infty[\textit{ and strictly increasing.} \tag{6.20}$$

$$\psi \textit{ is strictly positive on } [0, \infty[\textit{ and decreasing.} \tag{6.21}$$

Moreover, using *Lemma* 6.6, we have $W(0) = \psi(\infty)$. Thus, if there exists λ_0 such that $W(0) < 0$, then there exists λ_c such that $W(.)$, $\psi(.)$ are positive and $W(0) = \psi(\infty) = 0$.
We prove the existence of λ_0 under assumption (2.9), assume there does not exist such a λ then for all $\lambda \geq 0$ there exists N strictly positive and bounded by $e^{-\int_0^y B(s)ds}$ (which is $L^1([0, \infty[)$ by (2.9)) satisfying (6.4). But integrating (6.4), we find

$$0 > -N(0) = \int_0^\infty N(y')((\kappa - 1)B(y') - \lambda)dy',$$

which is absurd if $N > 0$, (2.6) and $\lambda = 0$, thus there exists $\lambda_0 > 0$ such that $N(0) < 0$. Now if we assume (2.9), we have, using the proof of Lemma 3.1, $\lambda \geq \underline{\lambda} > 0$. Thus using $N(y) = W(y)e^{-\int_0^y(\lambda + B(s))ds}$, we have the sign of N and W which are the same and $W(0) = N(0)$, therefore the lemma is proved. □

Step3. Integrability Now we prove integrability of $W\psi$, i.e. $N\phi$, we have

Lemma 6.5. *Assume (6.12) then for λ such that $W(0) = \psi(\infty) = 0$ and W, ψ non negative, we have*

$$\int_0^\infty N(y)\phi(y)dy = \int_0^\infty W(y)\psi(y)dy < \infty.$$

Proof. We have,

$$N(y)e^{\int_0^y (\lambda+B(s))ds} \quad \textit{is bounded by } 1, \textit{ positive and increasing}, \tag{6.22}$$

$$\phi(y)e^{-\int_0^y (\lambda+B(s))ds} \quad \textit{is bounded by } 1, \textit{ positive and decreasing to } 0. \tag{6.23}$$

Hence, integrate $N(x)\phi(x)$ from 0 to z and use (6.22), we find

$$\int_0^z (N(x)e^{\int_0^x (\lambda+B(s))ds})(\phi(x)e^{-\int_0^x (\lambda+B(s))ds})dx \le \int_0^z (\phi(x)e^{-\int_0^x (\lambda+B(s))ds})dx.$$

But $\big(\phi(y)e^{-\int_0^y (\lambda+B(s))ds}\big)$ vanishes at infinity and satisfies the equation

$$\partial_{y'}\big(\phi(y')e^{-\int_0^{y'} (\lambda+B(s))ds}\big) = -\int_0^{y'} b(y,y')\big(\phi(y)e^{-\int_0^y (\lambda+B(s))ds}\big)e^{-\int_y^{y'} (\lambda+B(s))ds}dy,$$

so integrating this equation we find

$$\phi(x)e^{-\int_0^x (\lambda+B(s))ds} = \int_x^\infty \int_0^{y'} b(y,y')\big(\phi(y)e^{-\int_0^y (\lambda+B(s))ds}\big)e^{-\int_y^{y'} (\lambda+B(s))ds}dydy',$$

and

$$\big(\phi(x)e^{-\int_0^x (\lambda+B(s))ds}\big) \le \int_x^\infty \int_0^{y'} b(y,y')e^{-\int_y^{y'} (\lambda+B(s))ds)}dydy'.$$

Finally, by integration

$$\int_0^\infty N(x)\phi(x)dx \le \int_0^\infty y' \int_0^{y'} b(y,y')e^{-\int_y^{y'} (\lambda+B(s))ds}dydy'.$$

Thus by (6.12), ψW is integrable and we can normalize

$$N(y) = \frac{W(y)e^{-\int_0^y (B(s)+\lambda)ds}}{\int_0^\infty W(z)e^{-\int_0^z (B(y')+\lambda)dy'}dz},$$

$$\phi(y) = \frac{\psi(y)(y)e^{\int_0^y (B(y')+\lambda)dy'}}{\int_0^\infty N(z)\psi(z)e^{\int_0^z (B(y')+\lambda)dy'}dz},$$

and these functions satisfy (6.4), (6.6). □

Remark 6.3. If B has a compact support with $A = \sup\ Supp\ B$ then $W(y)$ is constant equal to 1 for $y \ge A$ and $\psi(y)$ must be constant for $y \ge A$, but $W\psi$ belongs to L^1, thus $\psi(y) = 0$ for $y \ge A$.

Lemma 6.6. *Assume that the constant function* 1 *satisfy* $Z(1) \in L^1([0,\infty[)$, *(Z defined in (6.9)) and for all* $k > 0$

$$Z^{(k)}(1)(.),\;\; Z^{*(k)}(1)(.) \in L^1([0,\infty[),$$

$$Z^{*(k+1)}(1)(.) = Z^*(Z^{*(k)}(1))(.),$$

$$Z^{(k+1)}(1)(.) = Z(Z^{(k)}(1))(.),$$

$$Z^{(0)}(1)(.) = Z^{*0}(1)(.) = 1,$$

then we have

$$Z^{*(k)}(1)(\infty) = \int_0^\infty Z^{*(k-j)}(1)(y)Z^{(j-1)}(1)(y)dy, \quad \forall k \geq 1, 1 \leq j \leq k,$$

$$and \quad Z^{*(k)}(1)(\infty) = Z^{(k)}(1)(0).$$

Proof. Let $k \geq 1$ then we prove

$$Z^{*(k)}(1)(\infty) = \int_0^\infty Z^{*(k-j)}(1)(y)Z^{(j-1)}(1)(y)dy, \quad \forall 1 \leq j \leq k. \tag{6.24}$$

Step1. We first prove the equation for $j = 1$

$$\begin{aligned} Z^{*(k)}(1)(\infty) &= \int_0^\infty \int_0^y Z^{*(k-1)}(1)(y')b(y',y)e^{-\int_{y'}^y (B(s)+\lambda)ds}dy'dy \\ &= \int_0^\infty Z^{*(k-1)}(1)(y)Z(1)(y)dy. \end{aligned}$$

Step2. Assume (6.24) is true for $j < k$ then

$$\begin{aligned} Z^{*(k)}(1)(\infty) &= \int_0^\infty Z^{*(k-j)}(1)(y)Z^{(j-1)}(1)(y)dy, \\ &= \int_0^\infty \int_0^y h(y',y)Z^{*(k-(j+1))}(1)(y')dy'Z^{(j-1)}(1)(y)dy, \\ &= \int_{y'=0}^\infty \int_{y=y'}^\infty h(y',y)Z^{*(k-(j+1))}(1))(y')Z^{(j-1)}(1)(y)dydy', \\ &= \int_{y'=0}^\infty Z^*(F^{*(k-(j+1))}(1))(y')\Big(\int_{y=y'}^\infty h(y',y)Z^{(j-1)}(1)(y)dy\Big)dy', \\ &= \int_{y'=0}^\infty Z^{*(k-(j+1))}(1)(y')Z^{(j)}(1)(y')dy', \end{aligned}$$

where $h(y',y) = b(y',y)e^{-\int_{y'}^{y}(B(s)+\lambda)ds}$, then (6.24) is true for $j+1 \leq k$.

Step3. Thus we prove (6.24) for all $1 \leq j \leq k$ and if $j = k$ we find $Z^{*(k)}(1)(\infty) = Z^{(k)}(1)(0)$.

The result is independent of $k \geq 1$, thus (6.24) is true for all $k \geq 1$ and $1 \leq j \leq k$ □

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