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GRE METHODS FOR NONLINEAR MODEL OF EVOLUTION EQUATION AND LIMITED RESSOURCE ENVIRONMENT

Philippe Michel*

Ecole Centrale de Lyon, University Claude Bernard Lyon 1
CNRS UMR 5208, Ecully 69130, France

Bhargav Kumar Kakumani

T.I.F.R. Centre for Applicable Mathematics, Bangalore 560065, India

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Abstract. In this paper, we consider nonlocal nonlinear renewal equation (Markov chain, Ordinary differential equation and Partial Differential Equation). We show that the General Relative Entropy [29] can be extend to nonlinear problems and under some assumptions on the nonlinearity we prove the convergence of the solution to its steady state as time tends to infinity.

1. **Introduction.**

In [29], authors introduce the General Relative Entropy (GRE) which gives a natural Lyapunov structure in *linear* evolution equation from the linear ordinary differential equations to the linear partial differential equations (and for stochastic processes with Relative Entropy see Chapter 5 in [42]).

From Malthus to McKendrick - VonFoerster like equations. Under assumptions of homogeneity, size and unlimited resource, a population at time t that has a size $n(t)$ will evolve as follow

$$\frac{d}{dt}n(t) = (b-d)n(t), \quad i.e., \quad \mathcal{L} : g \mapsto (b-d)g, \tag{1}$$

where b is a birth rate and d a death rate. It is well known that the solution to (1) is given by $n(0)e^{(b-d)t}$. Considering that the population has different birth rates and death rates with respect to their age, therefore, a population at time t and age k of size $n(t,k)$ (with $k \in [0, \mathcal{N}]$) will evolve as follow

$$\frac{d}{dt}\begin{pmatrix} n(t,0) \\ n(t,1) \\ \vdots \\ n(t,\mathcal{N}) \end{pmatrix} = M \begin{pmatrix} n(t,0) \\ n(t,1) \\ \vdots \\ n(t,\mathcal{N}) \end{pmatrix}, \quad i.e., \quad \mathcal{L} : g \mapsto Mg,$$

where M is a Leslie-Usher matrix. We known (see [17]) that,

$$n(t) \sim Cst.e^{\lambda t}N,$$

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* Corresponding author: Philippe Michel.

where $\lambda = \sup_{\mu \in Sp(M)} Re(\mu)$ and N is a positive eigenvector associated to the eigenvalue λ (Perron Frobenius). When the class age has an infinitesimal length, we obtain McKendrick VonFoerster type of equations [32, 34]

$$\frac{\partial}{\partial t}n(t,x) = -\frac{\partial}{\partial x}n(t,x) - d(t,x)n(t,x) + \delta_0 \int b(t,x')n(t,x')dx',$$
$$i.e., \quad \mathcal{L} : g \mapsto -\frac{\partial}{\partial x}g - d(t,.)g + \delta_0 \int b(t,x')g(t,x')dx',$$

where b is a birth rate, d is a death rate and δ_x denotes the Dirac delta at age x.[1] The transport term $-\frac{\partial}{\partial x}$ correspond to the aging of the population. Here again, it is well known that n behaves as

$$n(t,.) \sim Cst.e^{\lambda t}N(.),$$

where $\lambda = \sup_{\mu \in Sp(\mathcal{L})} Re(\mu)$ and N is a positive eigenfunction associated to the eigenvalue λ. More generally, for a size structured population [29, 34], where n satisfies (with $n(t,0) = 0$, i.e., there is no individual of size 0)

$$\frac{\partial}{\partial t}n(t,x) = -\frac{\partial}{\partial x}n(t,x) - d(t,x)n(t,x) - \int_{y>x} b(x,y)n(t,y)dy + b(t,x)n(t,x),$$
$$i.e., \quad \mathcal{L} : g \mapsto -\frac{\partial}{\partial x}g - d(t,.)g - \int_{y>x} b(x,y)g(y)dy + b(t,.)g,$$

with b the division rate and d the death rate, we have proved the same behavior in long time asymptotic. And finally, when appears some randomness in the measure of the age (size or more generally trait), we have a diffusion terms

$$\frac{\partial}{\partial t}n(t,x) = -\frac{\partial}{\partial x}n(t,x) + C\frac{\partial^2}{\partial x^2}n(t,x) - d(t,x)n(t,x) + \delta_0 \int b(t,x')n(t,x')dx',$$
$$i.e., \quad \mathcal{L} : g \mapsto -\frac{\partial}{\partial x}g + C\frac{\partial^2}{\partial x^2}g - d(t,.)g + \delta_0 \int b(t,x')g(t,x')dx',$$

and we prove a similar result on the asymptotic behavior [1].[2] More generally, this results seems to hold for positive semigroups.

Positive Semigroups and "Perron Frobenius" results. The existence, of the eigenelements : (λ, N), is well known for irreducible positive matrix (Perron Frobenius), strongly positive and compact operators (Krein Rutmann). It is a general result on positive semigroups [14, 33] and we just recall that

Definition 1.1. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X is called positive if

$$0 \leq f \in X \;\; implies \;\; 0 \leq T(t)f \;\; \forall t \geq 0.$$

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[1] equivalent to the system with boundary condition $\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) = -d(t,x)n(t,x) \\ n(t,0) = \int b(t,x')n(t,x')dx' \end{cases}$

[2] equivalent to the system with boundary condition

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) = C\frac{\partial^2}{\partial x^2}n(t,x) - d(t,x)n(t,x) \\ n(t,0) - C\frac{\partial}{\partial x}n(t,0) = \int b(t,x')n(t,x')dx' \end{cases}$$

Theorem 1.2. *Let $(T(t))_{t\geq 0}$ be an irreducible, positive, strongly continuous semigroup with generator A on the Banach lattice X and assume that* $\sup\{Re\ \lambda : \lambda \in$ *Spectrum of* $A\} = 0$[3]*. If 0 is a pole of the resolvent $R(.,A)$, then the following properties hold.*

- $Ker(A) = Fixed\ Point\ (T(t))_t = lin\{N\}$, *for some positive function $N \in X$.*
- $Ker(A^*) = Fixed\ Point\ (T'(t))_t = lin\{\phi\}$ *(where A^* is the dual operator), for some positive function $\phi \in X^*$.*

We refer to [33] for more precise results.

General Relative Entropy results. The GRE gives a natural Lyapunov structure in an evolution equation such as

$$\frac{\partial}{\partial t} n = \mathcal{L}n, \quad n(t=0,.) = n_0(.). \tag{2}$$

More precisely, for $f = \frac{ne^{-\lambda t}}{N}$ with $\mathcal{L}N = \lambda N$ and $\mathcal{L}^*\phi = \lambda\phi$ [4] strictly positive eigenelement associated to the eigenvalue $\lambda = \sup\{\text{Re } \lambda : \lambda \in \text{Spectrum of } \mathcal{L}\}$ and for all H regular, positive and convex, we have

$$\frac{d}{dt}\mathcal{H}(f) = D_H^{\mathcal{L}}(f), \tag{3}$$

where $\mathcal{H}(f) = \langle H(f)N, \phi\rangle$ and, by direct computation,

$$D_H^{\mathcal{L}}(f) = \Big\langle H'(f)\mathcal{L}(fN) - H'(f)f\mathcal{L}(N) + H(f)\mathcal{L}(N) - \mathcal{L}\Big(H(f)N\Big), \phi\Big\rangle, \tag{4}$$

where $\langle u, v\rangle$ is the duality bracket between a Banach space and its dual. The dissipation can be rewritten, for convenience, as

$$D_H^{\mathcal{L}}(f) = \Big\langle \mathcal{L}\Big(\big(H'(f(x)(f(.) - f(x)) + H(f(x)) - H(f(.))\big)N(.)\Big)(x), \phi(x)\Big\rangle. \tag{5}$$

Therefore, we have the conservation law ($H = Id$)

$$\langle fN, \phi\rangle = \langle fN, \phi\rangle(t=0), \tag{6}$$

and for H positive and convex with $\mathcal{L}$ be a positive operator, we have

$$\frac{d}{dt}\mathcal{H}(f) = D_H^{\mathcal{L}}(f) \leq 0. \tag{7}$$

Using a LaSalle's principle, we see that the $\omega-$limit set of n belongs to the kernel of the entropy dissipation

$$Ker(D_H^{\mathcal{L}}) = \Big\{f \quad : \quad D_H^{\mathcal{L}}(f) = 0\Big\}, \tag{8}$$

and, under some assumptions on the kernel of $D_H^{\mathcal{L}}$ (irreducibility), we prove that $f(t) \xrightarrow{t\to\infty} Constant$, i.e.,

$$n(t,.) \sim Cst.e^{\lambda t}N(.). \tag{9}$$

We notice that the dissipation term is linear with respect to $\mathcal{L}$, this means that

$$\mathcal{L} = \sum_k a_k \mathcal{L}^k \Rightarrow D_H^{\mathcal{L}} = \sum_k a_k D_H^{\mathcal{L}}.$$

The formalism of the General Relative Entropy (GRE) (see [9, 16, 24, 29, 34, 35, 36, 26, 31, 40]) is an interesting tool to study semigroup of evolution equations (see [7, 14, 15, 17, 18, 33, 43, 45]). In particular, in linear renewal equations as in the

[3]True up to a translation of the spectrum by changing A to $A - cstId$

[4]$\mathcal{L}^*$ is the dual operator

McKendrick-VonFoerster (see [6, 16, 29]) the GRE has shown its easy computability and powerful results to study asymptotic behavior of solution to evolution equation.

Example of operators and their entropy dissipation. For instance, for the following operators (with H convex)

- Linear system of Ordinary Differential Equations : $\mathcal{L} = (a_{ij})_{i,j}$ is a matrix of transitions states (positive except on its diagonal) and n represents the states vector (see Leslie-Usher population matrix [5]), then
$$D_H^{\mathcal{L}}(f) = -\sum_{i,j} a_{ij} N_j \phi_i \big[H'(f_i)(f_j - f_i) + H(f_i) - H(f_j)\big] \leq 0.$$
- Differentiation : $\mathcal{L}^{\partial} f = f'$ corresponds to a transport (term) equation and $D_H^{\mathcal{L}^{\partial}}(f) = 0$ means that the "transport" operator gives no information on the dynamic of an evolution equation.
- Multiplication by r : $\mathcal{L}^{Mult} f = rf$ corresponds, for instance, to a death term in a population evolution equation and $D_H^{\mathcal{L}^{Mult}}(f) = 0$ means that the multiplication operator gives no information on the dynamic of an evolution equation.
- Diffusion : $\mathcal{L}^{Diff} f = Df''$ and $D_H^{\mathcal{L}^{Diff}}(f) = -\Big\langle DH''(f)(f')^2 N, \phi\Big\rangle \leq 0$. The Kernel $Ker(\mathcal{L}_H^{Diff})$ of the dissipation is the set $\{f \in X \quad : \quad f\,|_{Supp\ DN} = Cst\} \neq \emptyset$ if the support of D is not empty.
- Integral : (for instance in the Chapmann Kolmogorrov equation) : $\mathcal{L}^{Int} f = \int K(.,y) f(y) d\nu(y)$ corresponds to a mix states term and
$$\begin{aligned} D_H^{\mathcal{L}^{Int}}(f) &= -\Big\langle \int K(x,y)\Big[\Big(H'(f(x))(f(y) - f(x)) \\ &\qquad + H(f(x)) - H(f(y))\Big) N(y)\Big] d\nu(y), \phi(x)\Big\rangle \\ &\leq 0 \end{aligned}$$
In this case, we need that K mixes enough the variables x and y (irreducibility) to have a "useful" Kernel.
- Birth Term : $\mathcal{L}^{Birth} f = \delta_0 \int f d\nu$ and
$$\begin{aligned} D_H^{\mathcal{L}^{Birth}}(f) &= -\Big\langle \int B(y)\Big[\Big(H'(f(0))(f(y) - f(0)) \\ &\qquad + H(f(0)) - H(f(y))\Big) N(y)\Big] d\nu(y), \phi(0)\Big\rangle \\ &\leq 0. \end{aligned}$$
- and so on, by computation $\cdots$

The aim of this work is to extend this result to nonlinear evolution equation
$$\frac{d}{dt} n = \mathcal{L}(\langle n, \psi\rangle) n. \tag{10}$$
where ψ can be seen as a distribution function of ressources and $\langle n, \psi\rangle$ corresponds to the ressources consumption by the population (see [34, 7, 10, 5, 12, 11, 13]). We

[5] Leslie matrix with non null terms on the diagonal, such as death terms

show in section 2 (proofs are given in section 5) that we can decompose the entropy dissipation in two terms

$$\frac{d}{dt}Entropy(n) = -Entropy\_Dissipation^{\mathcal{L}}(n) + Entropy\_Increase^{\mathcal{L}}(n),$$

where the $Entropy\_Dissipation^{\mathcal{L}}(n)$ contains the linear part and the $+Entropy$ $\_Increase^{\mathcal{L}}(n)$ contains the nonlinear part of the dynamic. In section 3, we study theoretically three examples of application : Markov chains, an Ordinary Differential equation and a Partial Differential Equation. Finally we conclude in section 4.

2. **Entropy calculus and decomposition of its variation.** Let $\mathcal{B}$ a Banach space. For $\mathcal{L}$ nonlinear operator : $\mathcal{L} : n \in \mathcal{B} \mapsto \mathcal{L}(\langle n, \psi\rangle)n \in \mathcal{B}$, with $\psi \in \mathcal{B}^*$, such that, for any fixed n

$$\mathcal{L}^{\langle n,\psi\rangle} : m \mapsto \mathcal{L}(\langle n, \psi\rangle)m, \text{ is linear and compact operator} \tag{11}$$

which satisfies

$$\begin{cases} \forall z \in \mathbb{R}_+ \ \ \exists C(z) \in \mathbb{R} \ \ s.t. \ \ \mathcal{L}(z) + C(z)I_d \quad \text{is strongly positive,} \\ \sup Sp(\mathcal{L}(0)) > 0 \quad and \quad \sup Sp(\mathcal{L}(\infty)) < 0, \\ z \in \mathbb{R}_+ \mapsto \mathcal{L}(z) \text{ continuous.} \end{cases} \tag{12}$$

Assumptions (11)-(12) imply, by Krein Rutmman theorem [23], that for all $M \in \mathcal{B}$, there exists (N_M, ϕ_M) solution to $\mathcal{L}(\langle M, \psi\rangle)N_M = \lambda_M N_M$ and $\mathcal{L}(\langle M, \psi\rangle)^*\phi_M = \lambda_M\phi_M$. Moreover, compactness condition and condition on the spectrum in 0 and ∞ imply that there exists a fixed point to $M \mapsto N_M$ and $\lambda_M = 0$, i.e. $\mathcal{L}(\langle N, \psi\rangle)N = 0$ (and $\mathcal{L}(\langle N, \psi\rangle)^*\phi = 0$). We define the linear operator at the equilibrium

$$\mathcal{L}_{eq} := \mathcal{L}(\langle N, \psi\rangle), \tag{13}$$

where N satisfies $\mathcal{L}(\langle N, \psi\rangle)N = 0$. Denote $f = \frac{n}{N}$. For any convex function H and for all functional g, denote

$$\mathcal{H}(g) := \langle H(g)N, \phi\rangle. \tag{14}$$

Definition 2.1. We define the variation of $\mathcal{L}$ around its equilibrium N,

$$\forall g, \quad |\Delta\mathcal{L}_g| := -\frac{\mathcal{L}(\langle N, \psi\rangle + \langle gN, \psi\rangle) - \mathcal{L}(\langle N, \psi\rangle)}{\langle gN, \psi\rangle}. \tag{15}$$

Moreover, we define the following entropy dissipation

$$D_H^{\mathcal{L}inear}(g) := \Big\langle \mathcal{L}_{eq}\Big(u(x,y)\Big), \phi(x)\Big\rangle,$$

$$(E_H)_{\pm}^{\mathcal{L}}(g) := \pm\Big\langle |\Delta\mathcal{L}_g|(N(g+1))(x)\Big\langle\Big(g(s)H'(g(x))\Big)_{\mp}N(s), \psi(s)\Big\rangle, \phi\Big\rangle,$$ [6]

and

$$ND_H^{Non\ \ linear}(g) := -\Big\langle |\Delta\mathcal{L}_g|(N(g+1))(x)\langle u(x,y), \psi(y)\rangle, \phi(x)\Big\rangle,$$

where $u(x,y) := \Big[H'(g(x))\big(g(y) - g(x)\big) + H(g(x)) - H(g(y))\Big]N(y)$.

[6] We recall that $x_+ = \begin{cases} x, & if \quad x > 0, \\ 0, & if \quad x \leq 0 \end{cases}$ and $x_- = \begin{cases} -x, & if \quad x < 0, \\ 0, & if \quad x \geq 0 \end{cases}$.

Theorem 2.2. *(Entropy Calculus). Let $H \in C^1(\mathbb{R}, \mathbb{R}_+)$, convex and $H(0) = 0$. Then we have*

$$\frac{d}{dt}\mathcal{H}(f-1) = D_H^{\mathcal{L}inear}(f-1) + (E_H)_-^{\mathcal{L}}(f-1) + (E_H)_+^{\mathcal{L}}(f-1). \tag{16}$$

Now, assuming that, for all g, $|\Delta\mathcal{L}_g|$ is a positive operator, then we have

$$\frac{d}{dt}\mathcal{H}(f-1) = D_H^{\mathcal{L}}(f-1) \leq D_H^{\mathcal{L}inear}(f-1) + ND_H^{\mathcal{N}onlinear}(f-1). \tag{17}$$

Corollary 1. *(boundedness and Convergence). Assuming there exists $C > 0$ so that*

$$\exists Cst \in \mathbb{R},\ \forall -1 \leq g \leq C+1,\ |\Delta\mathcal{L}_g|(N\frac{g+1}{C}) \leq \inf_{u>0} \frac{1}{2}(\mathcal{L}_{eq} + CstI_d)\Big(\frac{u}{\langle u, \psi\rangle}\Big) \tag{18}$$

then $n_0 \leq CN$ implies that for all $t \geq 0$, $n(t,.) \leq CN(.)$. Moreover, if $n_0 \leq CN$ and

$$\exists Cst \in \mathbb{R}, \quad \forall -1 \leq g \leq C, \quad |\Delta\mathcal{L}_g|(N(g+1)) \leq \inf_{u>0}(\mathcal{L}_{eq} + CstI_d)\Big(\frac{u}{\langle u, \psi\rangle}\Big) \tag{19}$$

then $g(t,.) \to 0$ as $t \to \infty$, i.e., $n(t,.) \xrightarrow{t\to\infty} N$.[7]

Proofs of Theorem 2.2 and Corollary 1 are given in section 5. To show the usefulness of the methods we apply it to different types of evolution system.

3. Examples of application.

This section is subdivided in three paragraph where we give examples of application of the GRE method : a discrete time evolution equation (Markov Chain), in section 3.1, a continuous in time and discrete in "space" (Ordinary Differential Equation), in section 3.2 and finally a Partial Differential Equation in section 3.3.

3.1. Discrete time evolution equation : Application to non homogeneous Markov chains.

Let π a probability vector on $\mathbb{R}^{\mathscr{N}}$, $\psi \in \mathbb{R}_+^{\mathscr{N}}$ so that $\langle 1, \psi\rangle = 1$ and assume that $\mathcal{L}(\langle\pi, \psi\rangle)$ is a square $\mathscr{N} \times \mathscr{N}$ positive and irreducible matrix. We have, using Perron Frobenius theorem, that $\mathcal{L}(\langle\pi, \psi\rangle)$ (and $\mathcal{L}(\langle\pi, \psi\rangle)'$) admit a strictly positive eigenvector associated to the spectral radius of $\mathcal{L}(\langle\pi, \psi\rangle)$. Moreover, if $\mathcal{L}(\langle\pi, \psi\rangle)$ is stochastic then the spectral radius is 1 and we know that $\phi = (1, 1, ..., 1)$ is an eigenvector of $\mathcal{L}(\langle\pi, \psi\rangle)$ associated to 1, i.e. $\mathcal{L}(\langle\pi, \psi\rangle)\phi = \phi$ (see [3, 20]). Then, we can construct a, non homogeneous, Markov chain,

$$\pi^{k+1} = \pi^k \mathcal{L}(\langle\pi^k, \psi\rangle), \quad k \in \mathbb{N}, \tag{20}$$

where π^0 is a given probability vector. Then, by induction, for all k, π^k is a probability vector, i.e.,

$$\langle\pi^k, 1\rangle = \langle\bar{\pi}, 1\rangle = 1. \tag{21}$$

Moreover, assuming that $\pi \mapsto \mathcal{L}(\langle\pi, \psi\rangle)$ is continuous, we have by compactness, existence of $\bar{\pi}$ solution to the stationary equation

$$\bar{\pi} = \bar{\pi}\mathcal{L}(\langle\bar{\pi}, \psi\rangle). \tag{22}$$

We define

$$F_1(h) := \frac{1}{2}\Big\langle\Big\langle\Big[\Big(\frac{\frac{h}{\bar{\pi}}\bar{\pi}\mathcal{L}(\langle\bar{\pi}, \psi\rangle)}{\bar{\pi}}\Big)' - \frac{h}{\bar{\pi}}\Big]^2\Big(\frac{\bar{\pi}\mathcal{L}(\langle\bar{\pi}, \psi\rangle)}{\bar{\pi}}\Big)\Big(\frac{\bar{\pi}\mathcal{L}(\langle\bar{\pi}, \psi\rangle)}{\bar{\pi}}\Big)'\bar{\pi}, 1\Big\rangle, 1'\Big\rangle,$$

[7] We notice that the conditions (18) and (19) directly satisfy for a linear problem, i.e. $|\Delta\mathcal{L}_g| = 0$.

and

$$F_2(h) := \langle (\frac{h}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle^2$$

which are both, quadratic functions vanishing on $\bar{\pi}$. Then, we have the following result

Proposition 1. *If the variation of the transition matrix*

$$\Delta\mathcal{L} = \frac{\bar{\pi}(\mathcal{L}(\langle \pi^k, \psi\rangle) - \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}\langle (\frac{\pi^k}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle}$$

satisfies

$$F_2(h)[\sup_j 1(/\bar{\pi}_j)\|\Delta\mathcal{L}\|^2 + 2\sup_j(1/\psi_j)\|\Delta\mathcal{L}\|] \leq F_1(h), \quad \forall h \geq 0, \quad \sum_i h_i = 1, \tag{23}$$

and

$$F_1(h) = 0, \quad h \geq 0, \quad \sum_{i=1}^{\mathcal{N}} h_i = 1 \Longleftrightarrow h = \bar{\pi}. \tag{24}$$

Then, we have

$$\pi^k \xrightarrow{k\to\infty} \bar{\pi}.$$

Proof. We define

$$Entropy := \big\langle \big(\frac{\pi^k}{\bar{\pi}} - 1\big)^2\bar{\pi}, 1\big\rangle = \sum_j \big(\frac{\pi^k_j}{\bar{\pi}_j} - 1\big)^2\bar{\pi}_j,$$

and its variations by

$$D_2^{\mathcal{L}} := \big\langle \big(\frac{\pi^{k+1}}{\bar{\pi}} - 1\big)^2\bar{\pi}, 1\big\rangle - \big\langle \big(\frac{\pi^k}{\bar{\pi}} - 1\big)^2\bar{\pi}, 1\big\rangle.$$

Using (20), (21) and (22), we have

$$\Big\langle (\frac{\pi^k}{\bar{\pi}})^2\bar{\pi}, 1\Big\rangle = \sum_j (\frac{\pi^k_j}{\bar{\pi}_j})^2\bar{\pi}_j,$$

$$\Big\langle (\frac{\frac{\pi^k}{\bar{\pi}}\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}})^2\bar{\pi}, 1\Big\rangle = \sum_j \big(\sum_i \frac{\pi^k_i}{\bar{\pi}_i}\frac{\bar{\pi}_i\mathcal{L}(\langle \bar{\pi}, \psi\rangle)_{ij}}{\bar{\pi}_j}\big)^2\bar{\pi}_j = \sum_j \big(\sum_i \frac{\pi^k_i}{\bar{\pi}_i}\beta_{ij}\big)^2\bar{\pi}_j,$$

with $\beta_{ij} = \frac{\bar{\pi}_i\mathcal{L}(\langle \bar{\pi}, \psi\rangle)_{ij}}{\bar{\pi}_j}$ ($\sum_i \beta_{ij} = 1$) and so

$$\sum_{ii'}\sum_j \big(\frac{\pi^k_i}{\bar{\pi}_i} - \frac{\pi^k_{i'}}{\bar{\pi}_{i'}}\big)^2\beta_{ij}\beta_{i'j}\bar{\pi}_j = 2\sum_i\sum_j \big(\frac{\pi^k_i}{\bar{\pi}_i}\big)^2\bar{\pi}_j - 2\sum_j \big(\sum_i \frac{\pi^k_i}{\bar{\pi}_i}\beta_{ij}\big)^2\bar{\pi}_j,$$

$$\frac{1}{2}\Big\langle\Big\langle\Big[(\frac{\frac{\pi^k}{\bar{\pi}}\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}})' - \frac{\pi^k}{\bar{\pi}}\Big]^2\Big(\frac{\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}}\Big)\Big(\frac{\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}}\Big)'\bar{\pi}, 1\Big\rangle, 1'\Big\rangle$$

$$= \Big\langle (\frac{\pi^k}{\bar{\pi}})^2(\frac{\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}})\bar{\pi}, 1\Big\rangle - \Big\langle (\frac{\frac{\pi^k}{\bar{\pi}}\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}})^2(\frac{\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}})\bar{\pi}, 1\Big\rangle$$

$$= \Big\langle (\frac{\pi^k}{\bar{\pi}})^2\bar{\pi}, 1\Big\rangle - \Big\langle (\frac{\frac{\pi^k}{\bar{\pi}}\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}})^2\bar{\pi}, 1\Big\rangle.$$

Moreover, we can separate the nonlinear part and the linear part of the variation of the entropy, noticing that

$$D_2^{\mathcal{L}} := \Big\langle \big(\frac{\pi^k \mathcal{L}(\langle \pi^k, \psi \rangle)}{\bar{\pi}} - 1\big)^2 \bar{\pi}, 1 \Big\rangle - \Big\langle \big(\frac{\pi^k}{\bar{\pi}} - 1\big)^2 \bar{\pi}, 1 \Big\rangle,$$

and so we have

$$\begin{aligned} D_2^{\mathcal{L}} \quad &:= \langle (\frac{\pi^k(\mathcal{L}(\langle \pi^k, \psi\rangle) - \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}})(\frac{\pi^k(\mathcal{L}(\langle \pi^k, \psi\rangle) + \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}} - 2)\bar{\pi}, 1\rangle \\ &\quad + \langle (\frac{\frac{\pi^k}{\bar{\pi}}\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}} - 1)^2 \bar{\pi}, 1\rangle - \langle (\frac{\pi^k}{\bar{\pi}} - 1)^2 \bar{\pi}, 1\rangle \\ &= \langle (\frac{\pi^k(\mathcal{L}(\langle \pi^k, \psi\rangle) - \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}})(\frac{\pi^k(\mathcal{L}(\langle \pi^k, \psi\rangle) + \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}} - 2)\bar{\pi}, 1\rangle \\ &\quad - \frac{1}{2}\Big\langle \Big\langle \Big[\big(\frac{\frac{\pi^k}{\bar{\pi}}\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}}\big)' - \frac{\pi^k}{\bar{\pi}}\Big]^2 \Big(\frac{\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}}\Big)\Big(\frac{\bar{\pi}\mathcal{L}(\langle \bar{\pi}, \psi\rangle)}{\bar{\pi}}\Big)' \bar{\pi}, 1\Big\rangle, 1'\Big\rangle. \end{aligned}$$

Now, we have

$$\Big(\frac{\frac{\pi^k}{\bar{\pi}}\bar{\pi}(\mathcal{L}(\langle \pi^k, \psi\rangle) - \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}}\Big) = \frac{\pi^k}{\bar{\pi}}\Delta\mathcal{L}\langle (\frac{\pi^k}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle,$$

$$\begin{aligned} \frac{\pi^k(\mathcal{L}(\langle \pi^k, \psi\rangle) + \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}} - 2 &= \frac{\pi^k(\mathcal{L}(\langle \pi^k, \psi\rangle) - \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}} \\ &+ 2\frac{\pi^k - \bar{\pi}}{\bar{\pi}}\mathcal{L}(\langle \bar{\pi}, \psi\rangle) = \frac{\pi^k}{\bar{\pi}}\Delta\mathcal{L}\langle (\frac{\pi^k}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle + 2\frac{\pi^k - \bar{\pi}}{\bar{\pi}}\mathcal{L}(\langle \bar{\pi}, \psi\rangle), \end{aligned}$$

and the nonlinear part that satisfies the following inequality

$$\begin{aligned} &|\langle (\frac{\pi^k(\mathcal{L}(\langle \pi^k, \psi\rangle) - \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}})(\frac{\pi^k(\mathcal{L}(\langle \pi^k, \psi\rangle) + \mathcal{L}(\langle \bar{\pi}, \psi\rangle))}{\bar{\pi}} - 2)\bar{\pi}, 1\rangle| \\ &\quad \le |\langle (\frac{\pi^k}{\bar{\pi}}\Delta\mathcal{L})(\frac{\pi^k}{\bar{\pi}}\Delta\mathcal{L})\bar{\pi}, 1\rangle| \langle (\frac{\pi^k}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle^2 \\ &\qquad + 2|\langle (\tfrac{\pi^k}{\bar{\pi}}\Delta\mathcal{L})(\tfrac{\pi^k - \bar{\pi}}{\bar{\pi}}\mathcal{L}(\langle \bar{\pi}, \psi\rangle))\bar{\pi}, 1\rangle| \langle (\tfrac{\pi^k}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle \\ &\quad \le \sup_j (1/\bar{\pi}_j)\|\Delta\mathcal{L}\|^2 \langle (\frac{\pi^k}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle^2 + 2\sup_j (1/\psi_j)\|\Delta\mathcal{L}\| \langle (\frac{\pi^k}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle^2 \\ &\quad \le [\sup_j (1/\bar{\pi}_j)\|\Delta\mathcal{L}\|^2 + 2\sup_j (1/\psi_j)\|\Delta\mathcal{L}\|] \langle (\frac{\pi^k}{\bar{\pi}} - 1)\bar{\pi}, \psi\rangle^2 \end{aligned}$$

Using assumptions (23) and (24), we have the result. □

Remark 1. Markov chains : Assuming that $\mathcal{L}$ is a square $n \times n$ positive and irreducible matrix, we have, using Perron Frobenius theorem, that $\mathcal{L}$ (resp. $\mathcal{L}' =^t \mathcal{L}$) admit a strictly positive eigenvector associated to the spectral radius of $\mathcal{L}$. Moreover, if $\mathcal{L}$ is stochastic then the spectral radius is 1 and we know that $\phi = (1, 1, ..., 1)$ is an eigenvector of $\mathcal{L}'$ associated to 1 and $\bar{\pi}$ can be normalized to be a probability vector. Let $\mathcal{C}_i := \Big\{ j \quad : \quad a_{ij} > 0 \Big\}$, and we define the equivalence relation $\sim$ by

$$\mathcal{C}_i \sim \mathcal{C}_j \Leftrightarrow \exists i_0 = i,\ i_1,\ i_2, ..., i_r = j \quad : \quad \mathcal{C}_{i_k} \bigcap \mathcal{C}_{k+1} \neq \emptyset, \quad \forall k \in [0, r-1].$$

We note $\Omega_\sim := \{1, 2, 3, ..., n\}/\sim$ the quotient space states. Therefore, the aperiodic condition of convergence of Markov chains (see [3, 20]), can be seen as follows :

$$\sharp\Omega_\sim = 1 \Rightarrow \lim_{k\to\infty} \pi_k = \pi_\infty.$$

3.2. **A time continuous and discrete state : Application for an age structured model.** In this section, we are interested in the time evolution of a species which is state structured. More precisely, let $n(t) = (n_i(t))_{i=1}^{\mathcal{N}}$, at time t, a real vector in $\mathbb{R}_+^{\mathcal{N}}$, where $n_i(t)$ corresponds to the number of individuals at state i at time t, which follows the main evolution equation

$$\frac{d}{dt}n(t) = \mathcal{L}n(t), \quad \forall t \geq 0. \tag{25}$$

For example, in a discrete age structured model, we use a Leslie-Usher like matrix (see for discrete time application of the Leslie (or Leslie-Usher, with non null terms on the diagonal) matrix [2, 44, 38])

$$\mathcal{L} = \begin{pmatrix} b_1(t) - d_1 - p_1 & b_2(t) & b_3(t) & \cdots & b_{\mathcal{N}}(t) \\ p_1 & -d_2 - p_2 & 0 & \cdots & 0 \\ 0 & p_2 & -d_3 - p_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & p_{\mathcal{N}-1} & -d_{\mathcal{N}} \end{pmatrix}, \tag{26}$$

to modelize the aging with $(p_i)_i$, the death with $(d_i)_i$ and the birth process with $(b_i)_i > 0$. The linear evolution is classical and we only focus on the nonlinear problem : since resources are limited, the birth rate is depending on the number of individuals which use these resources, i.e., we have

$$b_i(t) = b_i h(w(t)), \tag{27}$$

where

$$w(t) = \sum_i \alpha_i n_i(t),$$

represents the total consumption of resource, assuming that individual of age i consume $\alpha_i > 0$ resources and h the decay of birth rate due to the lack of resources [38].

Proposition 2. *Assuming that h is a decreasing C^1 function which satisfies,*

$$-h'(\zeta)\frac{h^{-1}\Big(\frac{p_1 + d_1}{b_1 + \sum_{j\geq 2} b_j \prod_{k=2}^{j} \frac{p_{k-1}}{p_k + d_k}}\Big)}{\frac{p_1 + d_1}{b_1 + \sum_{j\geq 2} b_j \prod_{k=2}^{j} \frac{p_{k-1}}{p_k + d_k}}} \leq \frac{1}{2(C+1)}\frac{\inf(b_i)_i \inf(\alpha_j)_j}{\sup(b_j)_j \sup(\alpha_j)_j}, \tag{28}$$

$\forall \zeta \in [0, C\sum_i \alpha_i N_i]$ *and* $n(0) \leq CN$ $(C > 1)$ *where N is the stationary solution, i.e., solution to the following equation*

$$\begin{pmatrix} b_1(\sum_i \alpha_i N_i) - d_1 - p_1 & b_2(\sum_i \alpha_i N_i) & \cdots & b_n(\sum_i \alpha_i N_i) \\ p_1 & -d_2 - p_2 & \cdots & 0 \\ 0 & p_2 & \cdots & \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & p_{\mathcal{N}-1} & -d_{\mathcal{N}} \end{pmatrix} N = 0. \tag{29}$$

Then we have $n(t) \xrightarrow{t\to\infty} N$.

Proof. Using (29), we have, for all $j \in]1, \mathcal{N}]$, $N_j = N_{j-1}p_{j-1}/(d_j + p_j)$ (with the convention $p_{\mathcal{N}} = 0$) and $N_1(d_1 + p_1) = \sum_i b_i N_i h(\sum_i \alpha_i N_i)$. Thus, we find $N_j = N_1 \prod_{k=1}^{j-1} p_k/(d_k + p_k)$ for all $j \neq 1$ and finally $h(\sum_i \alpha_i N_i) = (d_1 + p_1)/(b_1 +$

$\sum_i b_i \prod_{k=1}^{j-1} p_k/(d_k+p_k))$ which means that $\sum_i \alpha_i N_i = h^{-1}((d_1+p_1)/(b_1+\sum_i b_i \prod_{k=1}^{j-1} p_k/(d_k+p_k)))$. We have

$$|\Delta\mathcal{L}_h| = -\begin{pmatrix} \frac{b_1(w(t))-\bar b_1}{w(t)-\bar w} & \cdots & \frac{b_n(w(t))-\bar b_n}{w(t)-\bar w} \\ 0 & \cdots & 0 \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} = -h'(\zeta)\begin{pmatrix} b_1 & \cdots & b_n \\ 0 & \cdots & 0 \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad (30)$$

for $\zeta \in [\min(\bar w, w(t)), \max(\bar w, w(t))]$. Moreover, a direct computation gives that

$$|\Delta\mathcal{L}_h|(N(h+1)) = -h'(\zeta)\begin{pmatrix} \sum_i b_i N_i(h_i+1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\leq -h'(\zeta)(C+1)\frac{\sup(b_j)_j}{\inf(\alpha_j)_j} h^{-1}\Big(\frac{p_1+d_1}{b_1+\sum_{j\geq 2} b_j \prod_{k=2}^{j} \frac{p_{k-1}}{p_k+d_k}}\Big), \quad \forall h \in [-1, C],$$

and for all $u \in (\mathbb{R}_+^*)^n$

$$(\mathcal{L}_{eq}+CstI_d)\Big(\frac{u}{\langle u,\psi\rangle}\Big) = \frac{\begin{pmatrix} \sum_i \bar b_i u_i + (Cst-d_1-p_1)u_1 \\ p_1u_1+(Cst-d_2-p_2)u_2 \\ p_2u_2+(Cst-d_3-p_3)u_3 \\ \vdots \\ p_{n-1}u_{n-1}+(Cst-d_n)u_n \end{pmatrix}}{\sum_j \alpha_j u_j} \geq \begin{pmatrix} \frac{\sum_i \bar b_i u_i}{\sum_j \alpha_j u_j} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\geq \frac{\inf(\bar b_i)_i}{\sup(\alpha_j)_j} = \frac{\inf(b_i)_i}{\sup(\alpha_j)_j}\frac{p_1+d_1}{b_1+\sum_{j\geq 2} b_j \prod_{k=2}^{j} \frac{p_{k-1}}{p_k+d_k}}.$$

Therefore, assumption (18)-(19) are satisfied as (28) is verified. □

3.3. **Partial differential equation : Application to renewal equation with diffusive effect on the age.** Renewal equation appears in mathematical biology to study the evolution of population structured in age (see [7, 17, 18, 25, 34, 43]). The density $n(t,x)$ at time t and age x follows the main equation (transport equation with loss due to a death term d and diffusion in age). According to the biologists, the matter of which sites are active on various chromosomes determines the true age of a biological entity [4]. This true age is a multidimensional variable and can be determined by time since birth. We are mainly concerned about the population and not on the individuals, hence we assume that average aging in the population is measured from time since birth (renewal). Because of lots of sources of variation in the vector valued age of individuals, the population as a whole diffuse in population age variable. We are interested to study the dynamics of the following renewal equation with diffusion.

$$\begin{cases} n_t(t,x) + n_x(t,x) + d(x, S(t,x))n(t,x) = Cn_{xx}(t,x), \qquad t > 0,\ x > 0, \\ n(t,0) - Cn_x(t,0) = \displaystyle\int_0^\infty b(x, S(t,x))n(t,x)dx, \qquad t > 0, \\ n(0,.) = n_0(.), \qquad n_0 \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+), \end{cases} \tag{31}$$

where $1/S(t,x)$ represents resource allocated to individuals of trait x at time t,

$$S(t,x) = \int_0^\infty \beta(x,y)n(t,y)dy, \quad \forall t, x. \tag{32}$$

Equation (31) with $C = 0$ is popularly known as McKendrick–Von Foerster (MV) equation (see [7, 39]). There are several mathematicians who worked on the stability estimates and longtime behavior of the MV equation ([7, 18, 43] and the references therein) or MV - like (see [19, 22] for instance). In [37, 41] the authors have discussed the existence and uniqueness of a weak solution and have also proved the linear stability around the nontrivial steady state of the nonlinear renewal equation. The linear version of equation (31) with $C = 1$ has been studied in [1]. Touaoula *et. al.*, proved the existence and uniqueness of a weak solution. They have used Poincaré Writinger's type inequality to prove the exponential decay of the solution for large times to a steady state. In [30], Michel *et. al.*, considered the nonlinearity in the boundary term in equation (31) and proved the convergence of the solution towards the steady state problem. In [21], Kakumani *et. al.*, proved the existence and uniqueness of a weak solution with $S(t) = \int \psi(y)n(t,y)dy$ and they have also proved the longtime behavior is some particular cases. We will prove that n converge to N solution to the corresponding steady state equation (of (31))

$$\begin{cases} N'(x) + d(x, \bar{S}(x))N(x) = CN''(x), \qquad x > 0, \\ N(0) - CN'(0) = \displaystyle\int_0^\infty b(x, \bar{S}(x))N(x)dx, \\ \displaystyle\int_0^\infty N(x)dx < \infty,\ \bar{S}(x) = \int_0^\infty \beta(x,y)N(y)dy. \end{cases} \tag{33}$$

Moreover, we will need the solution to the adjoint equation, i.e. ϕ solution to

$$\begin{cases} -\phi'(x) + d(x, \bar{S}(x))\phi(x) = C\phi''(x) + \phi(0)b(x, \bar{S}(x)), \qquad x > 0, \\ \phi'(0) = 0, \\ \displaystyle\int_0^\infty \phi(x)N(x)dx = 1. \end{cases} \tag{34}$$

Main results. Throughout this section, we assume that the functions d, b, β are nonnegative and continuous. Further we assume that there exists $L > 0$ such that for all $x,\ S_1,\ S_2$ we have

$$|b(x, S_1) - b(x, S_2)| \le L|S_1 - S_2|, \quad |d(x, S_1) - d(x, S_2)| \le L|S_1 - S_2|, \tag{35}$$

$$\frac{\partial}{\partial S} d(.,.) > 0,\ \frac{\partial}{\partial S} b(.,.) < 0, \tag{36}$$

$$0 < b_m \le b(.,.) \le b_M, \quad 0 < d_m \le d(.,.) \le d_M, \quad 0 \le \beta \le \beta_M \tag{37}$$

where $b_m, b_M, d_m, d_M,\ \beta_M$ are positive constants.

Proposition 3. *Assume (35)–(37), then there is a unique weak solution* $n \in C\big(\mathbb{R}^+; L^1(\mathbb{R}^+)\big) \cap L^2_{loc}(\mathbb{R}^+; W^{1,2}(\mathbb{R}^+))$ *solving (31) - (32). Moreover, assuming (36) and*

$$\underline{k} \le \beta \le \overline{k}, \quad 0 < \underline{k} \le \overline{k} < \infty. \tag{38}$$

$S_2 \mapsto b(x, S_2)$ *is strictly decreasing and*

$$S_1 \mapsto d(x, S_1) \text{ is strictly increasing on } [\alpha, \beta], \quad \alpha < \beta \tag{39}$$

and

$$b(x,0) - d(x,0) > 0 \quad b(x,\infty) - d(x,\infty) < 0 \tag{40}$$

are satisfied then there exists a solution to (33)-(34).

Since, in this work, we focus on the convergence of n to N, we give the proof of existence and uniqueness in annex 5. Now, we give assumptions on b, d and $n_0(.)$ which lead to the convergence of n, solution to (31) - (32), to N solution to (33).

Proposition 4. *Assuming that* $n(0,.) < KN(.)$ *for* $K > 0$ *and*

$$(Cbound)\begin{cases} \sup\limits_{S,\bar S} \Big|\dfrac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big| N(x)\dfrac{(K+2)}{K} < \dfrac{1}{2}\inf\limits_{u>0}\dfrac{\int b(y,\bar S)u(y)dy}{\int \beta(x,y)u(y)dy}, \\ \sup\limits_{S,\bar S} \Big|\dfrac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big| < \infty, \end{cases} \tag{41}$$

then for all $t > 0$, $n(t,.) \le KN(.)$. *Moreover, if we assume that*

$$(C1)\begin{cases} 2\displaystyle\int \Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\beta(x,y)N dx < b(y,\bar S)/K, \\ \displaystyle\int_s^\infty\!\!\int_0^s |x-y|\beta(x,y)N(y)dy\Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big| d\nu(x) < CN(s)\phi(s)/K, \end{cases} \tag{42}$$

or

$$(C2)\begin{cases} \Big|\dfrac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\Big[4\displaystyle\int \beta(x,y)dy\Big] < b(x,\bar S)/K, \\ \displaystyle\int_s^\infty \int_0^s |x-y|\beta(x,y)N(y)dy\Big[2\phi(0)\Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|N\Big]dx \\ \quad + \displaystyle\int_s^\infty \int_0^s |x-y|\beta(x,y)N(y)dy\Big[\Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|N\phi(x)\Big]dx \\ \qquad < CN(s)\phi(s)/K, \end{cases} \tag{43}$$

is satisfied, then $n(t,.) \xrightarrow{t\to\infty} N$, *i.e.* $\int_0^\infty (f(t)-1)^2 d\nu \to 0$.

Remark. The assumption (41) is the translation of assumption (18) to the problem (31) therefore the result holds (we notice that $\sup_{S,\bar S}\big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\big| < \infty$ implies the existence of Cst in assumption (18). First we decompose the variation of the entropy into negative part and positive part in Lemma 3.1. Then we show that under assumptions (41) and ((42) or(43)), the negative part (which forces the convergence) wins against the positive part (which creates oscillations).

Lemma 3.1. *Let n, N, ϕ be solution to equation (31), (33) and (34) respectively, $f = n/N$ and $d\nu(x) = N(x)\phi(x)dx$. Let the entropy defined as follows $\mathcal{H}(f(t)) := \int_0^\infty (f(t)-1)^2 d\nu$. Then we have*

$$\frac{d}{dt}\mathcal{H}(f(t)) = \underbrace{[D_2^{diff}(f) + D_2^{ren}(f) + E_2^-(f)]}_{\leq 0} + \underbrace{E_2^+(f)}_{\geq 0},$$

where the entropy dissipation due to diffusion and the renewal terms are

$$D_2^{diff}(f) = -2C\int_0^\infty \Big(\frac{\partial}{\partial x}f(t,x)\Big)^2 d\nu(x),$$

$$D_2^{ren}(f) = -\phi(0)\int_0^\infty \Big\{\Big(f(t,x)-1\Big)^2 - \Big(f(t,0)-1\Big)^2 - 2\big(f(t,0)\big)\Big[f(t,x)-f(t,0)\Big]\Big\} b(x,\bar S)N(x)dx,$$

$$E_2^-(f) = -\phi(0)\int_0^\infty \Big[2\Big(f(t,0)-1\Big)\big(b(x,S)-b(x,\bar S)\big)\Big]_- fN dx - 2\int_0^\infty \Big[(f(t,x)-1)\big[d(x,S)-d(x,\bar S)\big]f(t,x)\Big]_+ d\nu(x),$$

and the positive terms due to non linearities is given by

$$E_2^+(f) = 2\phi(0)\int_0^\infty \Big[\Big(f(t,0)-1\Big)\big(b(x,S)-b(x,\bar S)\big)\Big]_+ fN dx + \int_0^\infty \Big[2(f(t,x)-1)\big[d(x,S)-d(x,\bar S)\big]f(t,x)\Big]_- d\nu(x).$$

The proof of this lemma is a direct computation (application of Theorem 2.2). Now, we are ready to prove the Proposition 4.

Proof of Proposition 4. We recall that we denote $g = f-1$. Using that b (resp. d) decreases (resp. increases) with respect to S, we notice that

$$\begin{aligned}\Big[\Big(f(t,0)-1\Big)\big(b(x,S)-b(x,\bar S)\big)\Big]_+ &= \Big[-\Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|(S-\bar S)\Big(g(t,0)\Big)\Big]_+ \\ &= \Big[\Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|(S-\bar S)\Big(g(t,0)\Big)\Big]_-,\end{aligned}$$

and

$$\Big[2(f(t,x)-1)\big[d(x,S)-d(x,\bar S)\big]\Big]_- = \Big[2g(t,x)(S-\bar S)\Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|\Big]_-.$$

Thus, we have,

$$\begin{aligned}E_2^+(f) &= 2\phi(0)\int_0^\infty \Big[\Big(f(t,0)-1\Big)\big(b(x,S)-b(x,\bar S)\big)\Big]_+ fN dx \\ &\quad + \int_0^\infty \Big[2(f(t,x)-1)\big[d(x,S)-d(x,\bar S)\big]f(t,x)\Big]_- d\nu(x) \\ &= 2\phi(0)\int_0^\infty \Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\Big[(S-\bar S)\Big(g(t,0)\Big)\Big]_- fN dx \\ &\quad + \int_0^\infty \Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|\Big[2g(t,x)(S-\bar S)f(t,x)\Big]_- d\nu(x).\end{aligned}$$

Since, for all $a, b \in \mathbb{R}$, we have[8] : $(ab)_- \le (a-b)^2$, then, using the Jensen inequality, we obtain that

$$\begin{aligned}
&\Big(\int \beta(x,y)g(t,y)g(t,x)N(y)dy\Big)_-\frac{\int \beta(x,y')N(y')dy'}{\int \beta(x,y')N(y')dy'}\\
&\qquad\le \Big(g(t,x)-\int g(t,y)\frac{\beta(x,y)N(y)}{\int \beta(x,y')N(y')dy'}dy\Big)^2\int \beta(x,y')N(y')dy'\\
&\qquad\le \int (g(t,x)-g(t,y))^2\beta(x,y)N(y)dy,
\end{aligned}\tag{44}$$

with $g = f - 1$. Therefore, we have

$$\begin{aligned}
&\int_0^\infty \Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\Big(\int \beta(x,y)N(y)g(t,y)g(t,0)dy\Big)_- f(t,x)N(x)dx\\
&\quad\le 2\iint \Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\beta(x,y)N(y)\Big(g(t,x)-g(t,0)\Big)^2 f(t,x)N(x)dxdy\\
&+2\iint \Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\beta(x,y)N(y)\Big(g(t,x)-g(t,y)\Big)^2 f(t,x)N(x)dxdy,
\end{aligned}\tag{45}$$

and

$$\begin{aligned}
&\int_0^\infty \Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\Big(\int \beta(x,y)N(y)g(t,y)g(t,0)dy\Big)_- f(t,x)N(x)dx\\
&\quad\le \iint \Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\beta(x,y)N(y)\Big(g(t,y)-g(t,0)\Big)^2 f(t,x)N(x)dxdy.
\end{aligned}\tag{46}$$

Moreover, using (44), we find

$$\begin{aligned}
&\int_0^\infty \Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|\Big(\int \beta(x,y)g(t,y)g(t,x)N(y)dy\Big)_- f(t,x)d\nu(x)\\
&\qquad\le \iint \Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big| f(t,x)(g(t,x)-g(t,y))^2\beta(x,y)N(y)dyd\nu(x).
\end{aligned}$$

Since, $g(t,x)-g(t,y)=\int_y^x \frac{\partial}{\partial s}g(t,s)ds$, we have (Poincare inequality)

$$\begin{aligned}
&\int_0^\infty \Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|\Big(\int \beta(x,y)g(t,y)g(t,x)N(y)dy\Big)_- f(t,x)d\nu(x)\\
&\qquad\le \int \Big(\frac{\partial}{\partial s}g(t,s)\Big)^2\\
&\qquad\Big[\int_0^s\int_s^\infty |x-y|\beta(x,y)N(y)dy\Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big| f(t,x)d\nu(x)\Big]ds,
\end{aligned}$$

and, using Fubini Tonelli theorem, we have

$$\begin{aligned}
&\int_0^\infty \Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|\Big(\int \beta(x,y)g(t,y)g(t,x)N(y)dy\Big)_- f(t,x)d\nu(x)\\
&\le \int\Big(\frac{\partial}{\partial s}g(t,s)\Big)^2\Big[\int_s^\infty\int_0^s |x-y|\beta(x,y)N(y)dy\Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big| f(t,x)d\nu(x)\Big]ds.
\end{aligned}\tag{47}$$

[8] $(ab)_- \le (a-b)^2,\ if\ sgn(ab) > 0$ and $(ab)_- \le (a-b)^2-(|a|^2+|b|^2) \le (a-b)^2,\ if\ sgn(ab) \le 0$.

Moreover, we find

$$\iint \Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big| f(t,x)(g(t,x)-g(t,y))^2\beta(x,y)N(y)dyd\nu(x)$$
$$\le 2\int (g(t,x)-g(t,0))^2\Big[\int \Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big| f(t,x)\beta(x,y)N(y)dy\Big]d\nu(x)$$
$$+2\int (g(t,x)-g(t,0))^2\Big[\int \Big|\frac{d(y,S)-d(y,\bar S)}{S-\bar S}\Big| f(t,y)\beta(y,x)N(x)d\nu(y)\Big]dx. \quad (48)$$

Then, using (46)-(47), we find

$$E_2^+(g)\le\phi(0)\int\Big(g(t,y)-g(t,0)\Big)^2$$
$$\Big[2\int\Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\beta(x,y)N(y)f(t,x)N(x)dx\Big]dy$$
$$+2\iint\Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|f(t,x)(g(t,x)-g(t,y))^2\beta(x,y)N(y)dyd\nu(x),$$

or,

$$E_2^+(g)\le\phi(0)\int\Big(g(t,y)-g(t,0)\Big)^2$$
$$\Big[2\int\Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\beta(x,y)f(t,x)N(x)dx\Big]N(y)dy$$
$$+\int\Big(\frac{\partial}{\partial s}g(t,s)\Big)^2$$
$$\Big[2\int_s^\infty\int_0^s|x-y|\beta(x,y)N(y)dy\Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|f(t,x)d\nu(x)\Big]ds,$$

or, using (45)-(47), we find

$$E_2^+(g)\le\phi(0)\int\Big(g(t,x)-g(t,0)\Big)^2$$
$$\Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|\Big[4\int\beta(x,y)N(y)dy\Big]f(t,x)N(x)dx$$
$$+\int\Big(\frac{\partial}{\partial s}g(t,s)\Big)^2\Big[4\phi(0)\int_s^\infty\int_0^s|x-y|\beta(x,y)N(y)dy\Big|\frac{b(x,S)-b(x,\bar S)}{S-\bar S}\Big|fNdx\Big]ds$$
$$+\int\Big(\frac{\partial}{\partial s}g(t,s)\Big)^2$$
$$\Big[2\int_s^\infty\int_0^s|x-y|\beta(x,y)N(y)dy\Big|\frac{d(x,S)-d(x,\bar S)}{S-\bar S}\Big|f(t,x)d\nu(x)\Big]ds.$$

Since, we have,

$$D_2^{ren}(g)=-\phi(0)\int_0^\infty\Big(g(t,y)-g(t,0)\Big)^2b(y,\bar S)N(y)dy,$$

$$D_2^{diff}(g)=-2C\int_0^\infty\Big(\frac{\partial}{\partial s}g(t,s)\Big)^2d\nu(s),$$

both conditions (42) and (43) leads to the decay of the entropy and, so, to the convergence of f to 1 as $t\to\infty$. $\square$

4. **Conclusion.** We show in this paper that the GRE is a powerful method to study the dynamic of solutions of evolution equations (from positive semigroups) which appears in biology (where population stays positive). We prove that the study of the kernel of the entropy dissipation is the key to study the dynamic. We see that the variation of the entropy can be decomposed in a negative part which participates to the convergence to the equilibrium (containing the linear part around the equilibrium) and the positive part which participates to the oscillations (coming form the nonlinear part of the evolution equation). The difficulties (and so the assumptions that would be find), to prove the convergence, come from the comparison between these two effects : oscillation versus back to the equilibrium. We do not claim that assumptions we give here are optimal but are sufficient to obtain the convergence in each models. It could be interesting to study optimal assumption in order to have the convergence and so to compare more accurately L^2 norms which appear in the GRE computation.

5. **Annex.**

5.1. **Proof of Theorem 2.2.**

Proof. Using the main equation (10), we have

$$\frac{d}{dt}n = \mathcal{L}(\langle N,\psi\rangle)n + \frac{\mathcal{L}(\langle n,\psi\rangle) - \mathcal{L}(\langle N,\psi\rangle)}{\langle n,\psi\rangle - \langle N,\psi\rangle} n\langle (n-N),\psi\rangle.$$

Now, noticing that $(\mathcal{L}(\langle N,\psi\rangle)N) = 0$, we get

$$\frac{d}{dt}nN^{-1} = \Big[\mathcal{L}(\langle N,\psi\rangle)n + \frac{\mathcal{L}(\langle n,\psi\rangle) - \mathcal{L}(\langle N,\psi\rangle)}{\langle n,\psi\rangle - \langle N,\psi\rangle} n\langle (n-N),\psi\rangle\Big]N^{-1} - (\mathcal{L}(\langle N,\psi\rangle)N)nN^{-1}N^{-1}.$$

Let $\tilde{H} : z \mapsto H(z-1)$ a C^1, function, we find

$$\frac{d}{dt}\tilde{H}(nN^{-1}) = \tilde{H}'(nN^{-1})\Big[\big[\mathcal{L}(\langle N,\psi\rangle)n + \frac{\mathcal{L}(\langle n,\psi\rangle) - \mathcal{L}(\langle N,\psi\rangle)}{\langle n,\psi\rangle - \langle N,\psi\rangle} n\langle (n-N),\psi\rangle\big]N^{-1} - (\mathcal{L}(\langle N,\psi\rangle)N)nN^{-2}\Big].$$

Then, we have directly

$$\begin{aligned}\frac{d}{dt}\langle \tilde{H}(nN^{-1})N,\phi\rangle = {}& \langle \tilde{H}'(nN^{-1})\Big[\big[\mathcal{L}(\langle N,\psi\rangle)n + \frac{\mathcal{L}(\langle n,\psi\rangle) - \mathcal{L}(\langle N,\psi\rangle)}{\langle n,\psi\rangle - \langle N,\psi\rangle} n\langle (n-N),\psi\rangle\big]N^{-1} - (\mathcal{L}(\langle N,\psi\rangle)N)nN^{-2}\Big]N,\phi\rangle \\ &- \langle \tilde{H}(nN^{-1})N, \mathcal{L}(\langle N,\psi\rangle)^*\phi\rangle + \langle \tilde{H}(nN^{-1})\mathcal{L}(\langle N,\psi\rangle)N,\phi\rangle,\end{aligned}$$

and replacing nN^{-1} by f we find

$$\begin{aligned}\frac{d}{dt}\langle \tilde{H}(f)N,\phi\rangle = {}& \langle \tilde{H}'(f)\Big[\mathcal{L}(\langle N,\psi\rangle)(fN) - (\mathcal{L}(\langle N,\psi\rangle)N)f\Big],\phi\rangle \\ &- \langle \tilde{H}(f)N, \mathcal{L}(\langle N,\psi\rangle)^*\phi\rangle + \langle \tilde{H}(f)\mathcal{L}(\langle N,\psi\rangle)N,\phi\rangle \\ &+ \langle \tilde{H}'(f)\frac{\mathcal{L}(\langle fN,\psi\rangle) - \mathcal{L}(\langle N,\psi\rangle)}{\langle (f-1)N,\psi\rangle}(Nf)\langle (f-1)N,\psi\rangle,\phi\rangle,\end{aligned}$$

and finally we obtain

$$\frac{d}{dt}\langle \tilde{H}(f)N, \phi\rangle = \langle \mathcal{L}(\langle N,\psi\rangle)\Big(\big(\tilde{H}'(f(x))(f(y)-f(x)) + \tilde{H}(f(x)) - \tilde{H}(f(y))\big)N(y)\Big), \phi(x)\rangle + \langle \tilde{H}'(f)\frac{\mathcal{L}(\langle fN,\psi\rangle) - \mathcal{L}(\langle N,\psi\rangle)}{\langle (f-1)N,\psi\rangle}(Nf)\langle (f-1)N,\psi\rangle, \phi\rangle.$$

with $\langle \tilde{H}'(f)\frac{\mathcal{L}(\langle fN,\psi\rangle)-\mathcal{L}(\langle N,\psi\rangle)}{\langle (f-1)N,\psi\rangle}(Nf)\langle (f-1)N,\psi\rangle,\phi\rangle = (E_H)^{\mathcal{L}}_+(f-1) + (E_H)^{\mathcal{L}}_-(f-1)$. This proves that $\frac{d}{dt}\mathcal{H}(f-1) = D^{\mathcal{L}}_H(f-1) = D^{\mathcal{L}inear}_H(f-1) + (E_H)^{\mathcal{L}}_-(f-1) + (E_H)^{\mathcal{L}}_+(f-1)$. Since H is convex, positive and $H(0)=0$ we have directly that

$$\begin{aligned}(H'(f(x))f(s))_+ &\le H'(f(x))f(s) + \underbrace{-H(0) + H(f(x)) - H'(f(x))f(x)}_{\le 0}\underbrace{-H(f(s))}_{\le 0} \\ &= \Big(H'(f(x))(f(s)-f(x)) + H(f(x)) - H(f(s))\Big),\end{aligned}$$

and so $(E_H)^{\mathcal{L}}_+(g) \le ND^{\mathcal{N}on\ linear}_H(g)$. ∎

5.2. Proof of Corollary 1.

Proof. Let $C>0$ and $H : x \mapsto ((x-C)_+)^2$, then, we have directly that

$$\begin{aligned}&\big(2(g(x)-C)_+(g(y)-g(x)) + ((g(x)-C)_+)^2 - ((g(y)-C)_+)^2\big) \\ &\qquad = -\big((g(x)-C)_+ - (g(y)-C)_+\big)^2 - 2(g(x)-C)_+(g(y)-C)_-,\end{aligned}$$

and $(g(s)H'(g(x)))_- = 2(g(x)-C)_+ g(s)_-$. Therefore, we find

$$\begin{aligned} D^{\mathcal{L}inear}_H(g) &:= \langle \mathcal{L}_{eq}\Big(\big(H'(g(x))(g(y)-g(x)) + H(g(x)) - H(g(y))\big)N(y)\Big), \phi(x)\rangle \\ &= -\big\langle \mathcal{L}_{eq}\Big(\big((g(x)-C)_+ - (g(y)-C)_+\big)^2 N(y)\Big), \phi(x)\big\rangle \\ &\qquad - 2\langle \mathcal{L}_{eq}\Big((g(y)-C)_- N(y)\Big)(g(x)-C)_+, \phi(x)\rangle,\end{aligned}$$

and

$$\begin{aligned}(E_H)^{\mathcal{L}}_+(g) &:= \langle |\Delta\mathcal{L}_g|(N(g+1))(x)\langle (g(s)H'(g(x)))_- N(s), \psi(s)\rangle, \phi\rangle \\ &= 2\langle |\Delta\mathcal{L}_g|(N(g+1))(x)(g(x)-C)_+, \phi\rangle\langle (g(s)_- N(s), \psi(s)\rangle.\end{aligned}$$

Since $g \ge -1$, we have $(g(y)-C)_- \ge Cg_-(y)$, therefore, we obtain the following inequality,

$$C\langle g(s)_- N(s), \psi(s)\rangle \le \langle (g(y)-C)_- N(s), \psi(s)\rangle$$

and so we find
$(E_H)^{\mathcal{L}}_+(g) \le 2\langle |\Delta\mathcal{L}_g|(N\frac{(g+1)}{C})(x)(g(x)-C)_+, \phi\rangle\langle (g(s)-C)_- N(s), \psi(s)\rangle$.

Under assumption (18), we have

$\langle |\Delta\mathcal{L}_g|(N\frac{2(g+1)}{C})(x), (g(x)-C)_+\phi\rangle \le \langle \frac{\mathcal{L}_{eq}\Big((g(y)-C)_- N(y)\Big)}{\langle (g(s)-C)_- N(s), \psi(s)\rangle}, (g(x)-C)_+\phi(x)\rangle$. Now, we assume that $g(t=0,.) < C$ and we let

$$T^* = \sup_{t>0}\{g(s,.) \le C, \quad \forall s \in [0,t[\}.$$

Assuming that $T^* < \infty$, then in a neighbourhood of T^* : $]T^* - \mu, T^* + \mu[$, $0 \leq (g - C)_+ \leq \epsilon \leq 1$ and so

$$\frac{d}{dt}\langle (g(x) - C)_+^2 N, \phi\rangle \leq -\langle |\Delta \mathcal{L}_g|(N\frac{(g+1)}{C})(x), (g(x) - C)_+\phi\rangle \leq 0, \quad \forall t \in [0, T^* + \mu[,$$

which means that $g(t, .) \leq C$, for all $t \in [0, T^* + \mu[$ (absurd) and so we have $T^* = \infty$.

For the convergence result, with $H(z) = z^2$, it suffices to notice that under assumption (19), we have

$$D_H^{\mathcal{L}inear}(f - 1) + N D_H^{\mathcal{N}onlinear}(f - 1) \leq 0,$$

and so, using the inequality (17) we have the decay of the entropy and the convergence to the equilibrium $f = 1$ (LaSalle's principle). $\square$

5.3. **Proof of Proposition 3.** In this section, we prove existence and uniqueness result of solution to (31)–(32), (33) and (34). We use the same definition of weak solution and follow the similar arguments which are used in [21] to prove the existence and uniqueness result to (31)–(32). We start with the following *a priori* estimate of n.

Lemma 5.1. *Assume that* $S(.) \in L^\infty_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^+)$, *then there exists a unique weak solution* $n \in C\big(\mathbb{R}^+; L^1(\mathbb{R}^+)\big) \cap L^2_{loc}(\mathbb{R}^+; W^{1,2}(\mathbb{R}^+))$ *which solves (31). Moreover, we have* $n \geq 0$, *and*

$$\int_0^\infty |n(t, x)| dx \leq e^{||(B-d)_+||_\infty t} \int_0^\infty |n_0(x)| dx. \tag{49}$$

Theorem 5.2. *Assume* (35) $-$ (37), *then there is a unique weak solution*

$$n \in C\big(\mathbb{R}^+; L^1(\mathbb{R}^+)\big) \cap L^2_{loc}(\mathbb{R}^+; W^{1,2}(\mathbb{R}^+))$$

solving (31).

Proof's of Lemma 5.1 and Theorem 5.2 goes in similar lines that are given in [21]. So we omit the proofs.

Now we prove the existence and uniqueness of (33) and (34). First we observe that for a given S, we consider the associated eigenvalue problem of (33) and (34).

Before we prove Proposition 3, we prove some lemmas which are helpful. We notice that for a given $\bar{S}$ there exists $(\lambda_{\bar{S}}, N_{\bar{S}}, \phi_{\bar{S}})$ solution to the eigenproblem (see [1, 8] for details),

$$\begin{cases} \partial_x N_{\bar{S}} = C\Delta N_{\bar{S}} - d(x, \bar{S}) N_{\bar{S}} - \lambda_{\bar{S}} N_{\bar{S}}, \\ N_{\bar{S}}(0) - C N'_{\bar{S}}(0) = \displaystyle\int B(x, \bar{S}) N_{\bar{S}}(x) dx, \quad N_{\bar{S}} \in W^{1,2}(\mathbb{R}_+), \\ -\partial_x \phi_{\bar{S}} = C\Delta\phi_{\bar{S}} - d(x, \bar{S})\phi_S + \phi_{\bar{S}}(0) B(x, \bar{S}) - \lambda_{\bar{S}}\phi_{\bar{S}}, \ \phi_{\bar{S}} \in W^{1,2}(\mathbb{R}_+), \\ \phi'_{\bar{S}}(0) = 0 \quad and \quad \displaystyle\int \phi_{\bar{S}} N_{\bar{S}}(x) dx = 1. \end{cases} \tag{50}$$

Lemma 5.3. *Assume (36) then we have*

$$\frac{\partial}{\partial \bar{S}}\lambda_{\bar{S}} = -\int \Big(\frac{\partial}{\partial \bar{S}} d\Big) N_{\bar{S}}\phi_{\bar{S}} dx + \phi_{\bar{S}}(0) \int \Big(\frac{\partial}{\partial \bar{S}} B\Big) N_{\bar{S}} dx < 0. \tag{51}$$

Proof. The proof goes in similar lines that are given in [27, 28]. Therefore we skip the proof. $\square$

Lemma 5.4. *Assume (36),* $\lambda_0 > 0$, $\lambda_\infty < 0$ *and (38) then there exists a solution to (33).*

Proof. Using that $\lambda_0 > 0$, $\lambda_\infty < 0$ and the decay (51), we have the existence of Γ decreasing regular function defined on $[0, \bar{S}^*[$ (with $\bar{S}^* \in [0, \infty]$) so that $\{\bar{S} \,:\, \lambda_{\bar{S}} = 0\} = \{(\bar{S}, \Gamma(\bar{S})) :\ \bar{S} \in \mathbb{R}_+\} \subset \mathbb{R}^2_+$ (1-dimension manifold). ☐

Remark 2. It is easy to check that Proposition 3 is an immediate consequence of Lemma 5.4. Notice that using (40), we have $\lambda_0 > 0$, $\lambda_\infty < 0$ are satisfied.

Uniqueness of U: Let U solution of the eigenproblem given by the Proposition 3 and V an another positive solution to

$$V' + d(x, \int V\psi)V = CV'', \quad V(0) - V'(0) = \int B(x, \int V\psi)V(x)dx$$

with $\int V\psi \neq \int U\psi$. Then there exists $\bar{V}, \bar{\phi}, \bar{\lambda}$ solution to the eigenproblem

$$\bar{V}' + d(x, \int V\psi)\bar{V} = C\bar{V}'' - \bar{\lambda}\bar{V}, \quad \bar{V}(0) - \bar{V}'(0) = \int B(x, \int V\psi)\bar{V}(x)dx$$

$$-\bar{\phi}' + d(x, \int V\psi)\bar{\phi} = C\bar{\phi}'' - \bar{\lambda}\bar{\phi} + B(x, \int V\psi)\phi(0), \quad \bar{\phi}'(0) = 0$$

with $\bar{\lambda} \neq 0$ (since $\frac{\partial}{\partial S}\lambda_S < 0$). Therefore by integration, we have $\bar{\lambda}\int V\bar{\phi} = 0$ and hence $V = 0$.

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E-mail address: philippe.michel@ec-lyon.fr
E-mail address: bharu4hcu@gmail.com