

# NUMERICAL SCHEMES FOR KINETIC EQUATIONS IN THE ANOMALOUS DIFFUSION LIMIT. PART I: THE CASE OF HEAVY-TAILED EQUILIBRIUM

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**Abstract.** In this work, we propose some numerical schemes for linear kinetic equations in the anomalous diffusion limit. When the equilibrium distribution function is a Maxwellian distribution, it is well known that for an appropriate time scale, the small mean free path limit gives rise to a diffusion type equation. However, when a heavy-tailed distribution is considered, another time scale is required and the small mean free path limit leads to a fractional anomalous diffusion equation. Our aim is to develop numerical schemes for the original kinetic model which works for the different regimes, without being restricted by stability conditions of standard explicit time integrators. Starting from some numerical schemes for the diffusion asymptotics, their extension to the anomalous diffusion limit is then studied. In this case, it is crucial to capture the effect of the large velocities of the heavy-tailed equilibrium, so that some important transformations of the schemes derived for the diffusion asymptotics are needed. As a result, we obtain numerical schemes which enjoy the Asymptotic Preserving property in the anomalous diffusion limit, that is: they do not suffer from the restriction on the time step and they degenerate towards the fractional diffusion limit when the mean free path goes to zero. We also numerically investigate the uniform accuracy and construct a class of numerical schemes satisfying this property. Finally, the efficiency of the different numerical schemes is shown through numerical experiments.

**Key words.** BGK equation, Diffusion limit, Anomalous diffusion equation, Asymptotic preserving scheme.

**AMS subject classifications.** 35B25, 41A60, 65L04, 65M22.

**1. Introduction.** The modeling and numerical simulation of particles systems is a very active field of research. Indeed, they provide the basis for applications in neutron transport, thermal radiation, medical imaging or rarefied gas dynamics. According to the physical context, particles systems can be described at different scales. When the mean free path of the particles (*i.e.* the crossed distance between two collisions) is large compared to typical macroscopic length, the system is described at a microscopic level by kinetic equations; kinetic equations consider the time evolution of a distribution function which gives the probability of a particle to be at a given state in the six dimensional phase space at a given time. Conversely, when the mean free path is small, a macroscopic description (such as diffusion or fluid equations) can be used. It makes evolve macroscopic quantities which depends only on time and on the three dimensional spatial variable. In some situations, this description can be sufficient and leads to faster numerical simulations.

Mathematically, the passage from kinetic to macroscopic models is performed by asymptotic analysis. From a numerical point of view, considering a small mean free path, the kinetic equation then contains stiff terms which make the numerical simulations very expensive for stability reasons. In fact, a typical example is the presence of multiple spatial and temporal scales which intervene in different positions and at different times. These behaviors make the construction of efficient numerical methods a real challenge.

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In this work, we are interested in the time evolution of the distribution function  $f$  which depends on the time  $t \geq 0$ , the space variable  $x \in \Omega \subset \mathbb{R}^d$  and the velocity  $v \in \mathbb{R}^d$ , with  $d = 1, 2, 3$ . Particles undergo the effect of collisions which are modeled here by a linear operator  $L$  acting on  $f$  through

$$L(f)(t, x, v) = \rho(t, x)M_\beta(v) - f(t, x, v),$$

where

$$\rho(t, x) = \langle f(t, x, v) \rangle =: \int_{\mathbb{R}^d} f(t, x, v) dv,$$

and where the equilibrium  $M_\beta$  is defined by

$$M_\beta(v) = \begin{cases} \frac{1}{(2\pi)^{d/2}} \exp(-|v|^2/2) & \text{if } \beta = 2 + d, \\ \underset{|v| \rightarrow \infty}{\sim} \frac{m}{|v|^\beta} & \text{if } \beta \in (d, d + 2), \end{cases} \quad (1.1)$$

where  $m$  is a normalization factor. In the sequel, we will always denote by brackets the integration over  $v$ . In order to capture a nontrivial asymptotic model, a suitable scaling has to be considered. The good scaling of the kinetic equation (see [24, 23, 2]) satisfied by the distribution function  $f$  is given by

$$\varepsilon^\alpha \partial_t f + \varepsilon v \cdot \nabla_x f = L(f), \quad \text{with } \alpha = \beta - d \in (0, 2], \quad (1.2)$$

where  $\varepsilon > 0$  is the Knudsen number (ratio between the mean free path and a typical macroscopic length). With the equilibrium  $M_\beta$  defined in (1.1), the case  $\alpha = 2$  refers to the classical diffusion limit whereas the case  $\alpha \in (0, 2)$  refers to the anomalous diffusion limit. Although our analysis could be applied to general operators  $L$ , we will consider for a sake of simplicity a BGK type operator. We suppose that  $M_\beta$  given by (1.1) is an even positive function such that  $\langle M_\beta(v) \rangle = 1$  and  $\langle v M_\beta(v) \rangle = 0$ . Equation (1.2) has to be supplemented with an initial condition  $f(0, x, v) = f_0(x, v)$  and spatial periodic boundary conditions are considered.

The main goal of this work is to construct numerical schemes for (1.2) which are uniformly stable along the transition from kinetic to macroscopic regime (*i.e.* for all  $\varepsilon > 0$ ). More precisely, we consider here the asymptotic of anomalous diffusion ( $\alpha \in (0, 2)$ ). We shall also consider the asymptotic of diffusion ( $\alpha = 2$ ) to highlight the differences between the two cases. For the diffusion limit, several numerical schemes have already been proposed but in the anomalous diffusion limit, to the best of author's knowledge, the only other approach was proposed in [29] for the case  $\alpha \in (1, 2)$  at the same time this paper was submitted. Here, we deal with  $\alpha \in (0, 2)$  and the method we propose is robust enough to deal with other cases giving fractional diffusion, such as the case of a degenerate collision frequency which will be detailed in [8]. The results we present here were announced in [9].

Mathematically, the derivation of diffusion type equation from kinetic equations such as (1.2) when  $\alpha = 2$  was first investigated in [30], [3], [21] or [12]. When  $M_\beta$  decreases quickly enough for large  $|v|$ , the solution  $f$  of (1.2) converges, when  $\varepsilon$  goes to zero, to an equilibrium function  $f(t, x, v) = \rho(t, x)M_\beta(v)$  where  $\rho(t, x)$  is solution of the diffusion equation

$$\partial_t \rho(t, x) - \nabla_x \cdot (D \nabla_x \rho(t, x)) = 0, \quad (1.3)$$

and where the diffusion matrix  $D$  is given, in the simple case of the BGK operator, by the formula

$$D = \int_{\mathbb{R}^d} v \otimes v M_\beta(v) dv. \quad (1.4)$$

In particular, a crucial assumption on  $M_\beta$  for  $D$  to be finite is that

$$\int_{\mathbb{R}^d} |v|^2 M_\beta(v) dv < +\infty;$$

this is the case when  $M_\beta$  is Maxwellian (first case of (1.1)).

Hence, if the equilibrium has no finite second order moment, the asymptotic of rescaled kinetic equation with  $\alpha = 2$  is not able to capture a nontrivial dynamics and the value of  $\alpha$  should be tuned with  $M_\beta$  in order to recover this macroscopic equation when  $\varepsilon \rightarrow 0$ . Typically, let  $M_\beta$  be a heavy-tailed distribution function, corresponding to the second case of (1.1). In the sequel, we will consider as an example of such a distribution function  $M_\beta(v) = \frac{m}{1+|v|^\beta}$  with  $m$  chosen such that  $\langle M_\beta(v) \rangle = 1$  and  $\beta = d + \alpha$ ,  $\alpha \in (0, 2)$ .

In the case of astrophysical plasmas, it often occurs that the equilibrium  $M_\beta$  is a heavy-tailed distribution of particles which typically generates an anomalous diffusion behaviour (see [25, 28]). In particular, the diffusion matrix  $D$  is no longer well-defined and this requires to adapt the value of  $\alpha$  in terms of the tail decreasing rate  $\beta$ , in order to capture a nontrivial dynamics.

When  $\varepsilon$  goes to 0, the solution of this equation with BGK operator converges to  $\rho(t, x)M_\beta(v)$  where  $\rho$  is the solution of the anomalous diffusion equation which can be written in Fourier variable

$$\partial_t \hat{\rho}(t, k) = -\kappa |k|^\alpha \hat{\rho}(t, k), \quad (1.5)$$

where  $\hat{\rho}$  stands for the space Fourier variable of  $\rho$ ,  $k$  is the Fourier variable and  $\kappa$  is a constant which depends only on the size of the tail of the equilibrium  $M_\beta(v)$  and is defined by

$$\kappa = \int_{\mathbb{R}^d} \frac{(w \cdot e)^2}{1 + (w \cdot e)^2} \frac{m}{|w|^\beta} dw, \quad (1.6)$$

for any  $e \in \mathbb{R}^d$  such that  $|e| = 1$  (note that  $\kappa$  does not depend on  $e$ ). The anomalous diffusion equation can also be written in the space variable

$$\partial_t \rho(t, x) = -\frac{m\Gamma(\alpha + 1)}{c_{d,\alpha}} (-\Delta_x)^{\frac{\alpha}{2}} \rho(t, x), \quad (1.7)$$

where  $m$  is the normalization factor of  $M_\beta$  in (second case in 1.1),  $\Gamma$  is the usual function defined by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt,$$

and  $c_{d,\alpha}$  is a normalization constant given by

$$c_{d,\alpha} = \frac{\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{2\pi^{\frac{d}{2}+\alpha} \Gamma\left(1 - \frac{\alpha}{2}\right)}. \quad (1.8)$$

The fractional operator of the anomalous diffusion is easily defined by its Fourier transform

$$\widehat{((-\Delta_x)^{\frac{\alpha}{2}} \rho)}(k) = |k|^\alpha \hat{\rho}(k),$$

but has also an integral definition

$$(-\Delta_x)^{\frac{\alpha}{2}} \rho(x) = c_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{\rho(x+y) - \rho(x)}{|y|^{d+\alpha}} dy, \quad (1.9)$$

where *P.V.* denotes the principal value of the integral.

The anomalous diffusion occurs not only in astrophysical plasmas but also in the study of granular media (see [14, 5, 4]), tokamaks (see [13]) or even in economy or social science: as detailed in [20], the Pareto distribution is a power tail distribution that satisfies the second case of (1.1). This kind of distribution is a common object used to modelise the repartition of sizes in a given set (asteroids, cities, income, opinions. . .). At a microscopic level, these velocity distributions arise when the motion of the particles is no longer governed by a brownian process but by a Levy process (see [10], [11]).

In this work, we want to construct a numerical scheme over the kinetic equation using a heavy-tailed distribution equilibrium, which is robust when the stiffness parameter  $\varepsilon$  goes to 0. Asymptotic Preserving (AP) schemes already exist in the case of the diffusion limit where the equilibrium is a Maxwellian (see [16, 6, 7, 17, 19, 22, 18, 27]) but the strategy used to obtain them cannot be rendered word for word to the anomalous diffusion case. As in the diffusion case, the difficulties when numerically solving the kinetic equation in the case of anomalous diffusion come from the stiff terms which require a severe condition on the numerical parameters. But in the anomalous diffusion case, there is an additional difficulty since one needs to take into account the large velocities arising in the equilibrium  $M_\beta$ . Indeed, the role of these high velocities is crucial and they have to be captured numerically in order to produce the anomalous diffusion operator when  $\varepsilon \rightarrow 0$ .

To derive a numerical scheme which enjoys the AP property, different strategies are investigated in this work. The first one is based on a fully implicit scheme in time (both the transport and the collision term are considered implicit). Using this *brute force* strategy, the so-obtained numerical scheme is obviously AP in the diffusion limit but not in the anomalous diffusion one. Therefore, a suitable transformation of this implicit scheme is proposed in this work and leads to a scheme enjoying the AP property in the anomalous diffusion limit.

From a computational point of view, a fully implicit scheme may be very expensive especially if one deals with high-dimensional problems. To avoid this impication, we propose another strategy which is based on a micro-macro decomposition. The distribution function  $f$  is written as a sum of an equilibrium part  $\rho M_\beta$  plus a remainder  $g$ . A model equivalent to (1.2) is then derived, composed of a kinetic equation satisfied by  $g$  and a macroscopic equation satisfied by  $\rho$ . The macroscopic flux is then reformulated to obtain a numerical scheme which enjoys the AP property and which does not require the impication of the transport term.

The last strategy proposed in this work is based on a Duhamel formulation of the equation from which a AP numerical scheme solving the kinetic equation is obtained, this approach bears similarities with [15]. This numerical scheme is efficient for the diffusion case and has to be adapted also to get an AP scheme in anomalous diffusion

limit. Moreover, there are two main advantages in using such strategy. First we derive a closed equation on  $\rho$  and the computation of the full distribution function is not needed at any time. Second, this strategy ensures that the numerical scheme has uniform accuracy in  $\varepsilon$ . Eventually, it would be easier to construct high order (in time) schemes on this formulation.

The paper is organized as follows. In the next section we shall start with formal computations recalling how the anomalous diffusion limit can be obtained from the kinetic equation. Section 3 is devoted to the derivation of the different numerical schemes: a fully implicit scheme, a micro-macro scheme and a scheme based on a Duhamel formulation of (1.2). After the theoretical study of their properties, several numerical tests are presented in Section 4 in a one-dimensional case with periodic boundary conditions in space to illustrate and compare the efficiency of the various numerical schemes.

**2. A closed equation for the density and its diffusion asymptotics.** This section is devoted to the derivation from (1.2) of a closed equation satisfied by  $\rho$ . This equation will be the basic equation from which we formally derive the anomalous diffusion equation. Of course, the formal derivation of anomalous diffusion equation is classical but, for a sake of clarity, we present one way (among many others) to perform such derivation. In particular, the formal computation below will drive the construction of a class of our numerical schemes. We recall here the hypothesis on the equilibrium  $M_\beta(v)$  introduced in (1.1): it is an even function such that  $\langle M_\beta(v) \rangle = 1$ ,  $\langle v M_\beta(v) \rangle = 0$ . Then we will consider the two following cases mentioned in (1.1):  $\alpha = 2$  and  $0 < \alpha < 2$  with the following equilibrium functions

CASE 1.  $\langle v \otimes v M_{d+2}(v) \rangle < +\infty$  and  $\alpha = 2$ . We will consider

$$M_{d+2}(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}.$$

CASE 2.  $\langle v \otimes v M_\beta(v) \rangle = \infty$  and  $\alpha \in (0, 2)$ . We will consider the heavy-tailed function  $M_\beta$  defined by

$$M_\beta(v) = \frac{m}{1 + |v|^\beta}, \quad \beta = d + \alpha, \quad (2.1)$$

with a positive normalization factor  $m$  ensuring  $\langle M_\beta \rangle = 1$ .

**2.1. Space Fourier transform based computations.** Starting from (1.2), we propose here a method based on spatial Fourier transform to formally derive the asymptotic equation in both diffusion and anomalous diffusion limits. It will also be the basis of the numerical methods we set in Section 3.3.

Eventually, we remind that we denote by  $\hat{f}(t, k, v)$  (resp.  $\hat{\rho}(t, k)$ )

$$\hat{f}(t, k, v) = \int_{\mathbb{R}^d} e^{-ik \cdot v} f(t, x, v) dx,$$

the space Fourier transform of  $f(t, x, v)$  (resp.  $\rho(t, x)$ ). We have

PROPOSITION 2.1.

(i) Equation (1.2) formally implies the following exact equation on  $\hat{\rho}$

$$\hat{\rho}(t, k) = \left\langle e^{-\frac{t}{\varepsilon^\alpha}(1+i\varepsilon k \cdot v)} \hat{f}(0, k, v) \right\rangle + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta(v) \rangle \hat{\rho}(t - \varepsilon^\alpha s, k) ds. \quad (2.2)$$

(ii) *In case 2 (anomalous diffusion scaling), when  $\varepsilon \rightarrow 0$ ,  $\hat{\rho}$  solves the anomalous diffusion equation*

$$\partial_t \hat{\rho}(t, k) = -\kappa |k|^\alpha \hat{\rho}(t, k),$$

with  $\kappa$  defined by (1.6) and  $\beta - d = \alpha$ .

*Proof.*

(i) **Formal derivation of the expression (2.2) from (1.2)**

We start by taking the Fourier transform of (1.2)

$$\partial_t \hat{f} + i\varepsilon^{1-\alpha} v \cdot k \hat{f} = \frac{1}{\varepsilon^\alpha} (\hat{\rho} M_\beta - \hat{f}).$$

We solve in  $\hat{f}$  (assuming  $\hat{\rho}$  is given)

$$\hat{f}(t, k, v) = e^{-\frac{t}{\varepsilon^\alpha}(1+i\varepsilon k \cdot v)} \hat{f}(0, k, v) + \frac{1}{\varepsilon^\alpha} \int_0^t e^{-\frac{t-s}{\varepsilon^\alpha}(1+i\varepsilon k \cdot v)} \hat{\rho}(s, k) M_\beta(v) ds,$$

which can be written after a change of variables  $s \rightarrow (t-s)/\varepsilon^\alpha$

$$\hat{f}(t, k, v) = e^{-\frac{t}{\varepsilon^\alpha}(1+i\varepsilon k \cdot v)} \hat{f}_0 + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s(1+i\varepsilon k \cdot v)} \hat{\rho}(t - \varepsilon^\alpha s, k) M_\beta(v) ds.$$

Still denoting by brackets the integration over the velocity space, we can integrate the previous expression with respect to  $v$  to get (2.2). Then, we can write it to make the limit equation appear

$$\begin{aligned} \hat{\rho} &= \hat{A}_0(t, k) + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle ds \hat{\rho} \\ &\quad + \varepsilon^\alpha \int_0^{\frac{t}{\varepsilon^\alpha}} s e^{-s} \frac{\hat{\rho}(t - \varepsilon^\alpha s, k) - \hat{\rho}(t, k)}{\varepsilon^\alpha s} \langle e^{-i\varepsilon s k \cdot v} M_\beta(v) \rangle ds, \end{aligned} \quad (2.3)$$

where

$$\hat{A}_0(t, k) = \left\langle e^{-\frac{t}{\varepsilon^\alpha}(1+i\varepsilon k \cdot v)} \hat{f}_0 \right\rangle, \quad (2.4)$$

denotes the initial layer term.

Here, in case 1, Taylor expansions of the exponentials and of  $\hat{\rho}$  with respects to  $\varepsilon$  formally leads to the asymptotic diffusion equation (1.3) in Fourier variable. In case 2 (anomalous diffusion case) a finer analysis must be done to find the asymptotic equation.

(ii) **The anomalous diffusion limit**

Now we will show that when  $M_\beta$  fulfills the conditions of case 2 (anomalous diffusion scaling), the solution of (1.2) converges to the solution of the anomalous diffusion equation when  $\varepsilon$  goes to zero.

At first, let us remark that the first term has exponential decay when  $\varepsilon$  goes to zero,  $\hat{A}_0 = o(\varepsilon^\infty)$ , for all  $t > 0$ . With assumptions of smoothness in time for  $\hat{\rho}$ , we get the following behaviour for the third term of (2.3)

$$\int_0^{\frac{t}{\varepsilon^\alpha}} s e^{-s} \frac{\hat{\rho}(t - \varepsilon^\alpha s, k) - \hat{\rho}(t, k)}{\varepsilon^\alpha s} \langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle ds \underset{\varepsilon \rightarrow 0}{\sim} -\partial_t \hat{\rho}(t, k) + o(1).$$

We now have to make appear the fractional diffusion in the second term of (2.3). Since the moment of order 2 of  $M_\beta$  is not finite, we cannot expand the exponential term into a power series in  $\varepsilon$  anymore. We decompose this term as

$$\int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle ds = \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} ds + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle (e^{-i\varepsilon s k \cdot v} - 1) M_\beta \rangle ds,$$

and we focus on the second term on the right hand side of this last equality.

Here we still suppose that  $M_\beta(v) = \frac{m}{1+|v|^\beta}$ , where  $m$  is chosen to ensure that  $\langle M_\beta \rangle = 1$ . Still, the general case dealing with  $M_\beta \underset{|v| \rightarrow \infty}{\sim} \frac{m}{|v|^\beta}$  is a bit more tedious but can be done similarly, the details of the computations are proposed in the supplementary materials of this paper.

We consider the integral  $\langle (e^{-i\varepsilon s k \cdot v} - 1) M_\beta \rangle$  and, for nonzero  $k$ , we perform the change of variables  $w = \varepsilon|k|v$

$$\left\langle (e^{-i\varepsilon s k \cdot v} - 1) \frac{m}{1+|v|^\beta} \right\rangle = (\varepsilon|k|)^{\beta-d} \int_{\mathbb{R}^d} \left( e^{-is \frac{k}{|k|} \cdot w} - 1 \right) \frac{m}{(\varepsilon|k|)^\beta + |w|^\beta} dw.$$

Since the last integral has rotational symmetry, for any unitary vector  $e$  of  $\mathbb{R}^d$  we have

$$\begin{aligned} \left\langle (e^{-i\varepsilon s k \cdot v} - 1) \frac{m}{1+|v|^\beta} \right\rangle &= (\varepsilon|k|)^{\beta-d} \int_{\mathbb{R}^d} (e^{-ise \cdot w} - 1) \frac{m}{(\varepsilon|k|)^\beta + |w|^\beta} dw \quad (2.5) \\ &= (\varepsilon|k|)^{\beta-d} \mathcal{C}(s) + (\varepsilon|k|s)^{\beta-d} R(\varepsilon, s, k), \end{aligned}$$

where

$$\mathcal{C}(s) = \int_{\mathbb{R}^d} (e^{-ise \cdot w} - 1) \frac{m}{|w|^\beta} dw, \quad (2.6)$$

and

$$R(\varepsilon, s, k) = -m (\varepsilon s |k|)^\beta \int_{\mathbb{R}^d} (e^{-ie \cdot w} - 1) \frac{1}{|w|^\beta \left( (\varepsilon s |k|)^\beta + |w|^\beta \right)} dw.$$

In particular,  $R(\varepsilon, s, k)$  tends to 0 when  $\varepsilon$  go to zero and is bounded. If we set  $R(\varepsilon, s, 0) = 0$ , equality (2.5) is also true for  $k = 0$ .

From (2.3) and (2.5), we have to set  $\alpha = \beta - d$  to get the fractional diffusion equation when  $\varepsilon \rightarrow 0$  and then we get

$$\hat{\rho} \underset{\varepsilon \rightarrow 0}{=} \hat{\rho} + \varepsilon^\alpha |k|^\alpha \int_0^{+\infty} \left( \int_{w \in \mathbb{R}^d} (e^{-isw \cdot e} - 1) \frac{m}{|w|^\beta} dw \right) e^{-s} ds \hat{\rho} - \varepsilon^\alpha \partial_t \hat{\rho} + o(\varepsilon^\alpha), \quad \forall t > 0.$$

From the equality

$$\int_0^{+\infty} \left( \int_{w \in \mathbb{R}^d} (e^{-isw \cdot e} - 1) \frac{m}{|w|^\beta} dw \right) e^{-s} ds = - \int_{w \in \mathbb{R}^d} \frac{(w \cdot e)^2}{1 + (w \cdot e)^2} \frac{m}{|w|^{d+\alpha}} dw := -\kappa,$$

we derive, when  $\varepsilon$  goes to 0, the fractional diffusion equation (1.5) given in [24, 2].

□

**2.2. Computations in the space variable.** The aim of this section is to generalize the previous computations to the original space variable. We have the following proposition

PROPOSITION 2.2.

(i) Equation (1.2) formally implies the following exact equation on  $\rho$

$$\rho(t, x) = \left\langle e^{-\frac{t}{\varepsilon^\alpha}} f_0(x - \varepsilon^{1-\alpha} t v, v) \right\rangle + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle \rho(t - \varepsilon^\alpha s, x - \varepsilon s v) M_\beta(v) \rangle ds. \quad (2.7)$$

(ii) In case 2 (anomalous diffusion scaling), when  $\varepsilon \rightarrow 0$ ,  $\rho$  solves the anomalous diffusion equation

$$\partial_t \rho(t, x) = -\frac{m\Gamma(\alpha+1)}{c_{d,\alpha}} (-\Delta_x)^{\frac{\alpha}{2}} \rho(t, x),$$

where  $\Gamma$  is the usual Euler Gamma function,  $m$  the normalization constant appearing in the expression of the equilibrium  $M_\beta$ ,  $c_{d,\alpha}$  is defined by (1.8) and  $\beta - d = \alpha$ .

*Proof.*

(i) **Formal derivation of the expression (2.7) from (1.2)**

As in the previous section, we can integrate (1.2) to get

$$\rho = A_0 + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle \rho(t - \varepsilon^\alpha s, x - \varepsilon s v) M_\beta \rangle ds,$$

with

$$A_0(t, x) = \left\langle e^{-\frac{t}{\varepsilon^\alpha}} f_0(x - \varepsilon^{1-\alpha} t v, v) \right\rangle. \quad (2.8)$$

Once again, in case 1, the diffusion equation can be derived from the (2.7) with Taylor expansions with respect to  $\varepsilon$  in the expression of  $\rho$ , whereas in case 2 the asymptotic equation is more complicated to obtain because of the infinite moment of order 2 of  $M_\beta$ .

(ii) **The anomalous diffusion limit**

To find the limit equation in the case of the anomalous diffusion limit, we rewrite (2.7) as follows

$$\begin{aligned} \rho(t, x) &= A_0(t, x) + \varepsilon^\alpha \int_0^{\frac{t}{\varepsilon^\alpha}} s e^{-s} \left\langle \frac{\rho(t - \varepsilon^\alpha s, x - \varepsilon v s) - \rho(t, x - \varepsilon v s)}{\varepsilon^\alpha s} M_\beta(v) \right\rangle ds \\ &\quad + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle (\rho(t, x - \varepsilon v s) - \rho(t, x)) M_\beta(v) \rangle ds + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} ds \langle M_\beta(v) \rangle \rho(t, x). \end{aligned} \quad (2.9)$$

Under smoothness assumptions on  $\rho$ , the first integral of (2.9) becomes, for small  $\varepsilon$

$$\int_0^{\frac{t}{\varepsilon^\alpha}} s e^{-s} \left\langle \frac{\rho(t - \varepsilon^\alpha s, x - \varepsilon v s) - \rho(t, x - \varepsilon v s)}{\varepsilon^\alpha s} M_\beta(v) \right\rangle ds = -\partial_t \rho(t, x) + o(1).$$

Since the last term of (2.9) can be explicitly computed, let us focus on the second integral of (2.9). As in the Fourier variable, we cannot expand  $\rho$  with respect to  $\varepsilon$  since  $M_\beta$  has an infinite moment of order 2. Hence, we must refine the analysis to



find the limit of this term. We perform the change of variables  $y = x - \varepsilon v s$  in the integral over  $v$  and exchanging the integrals in  $t$  and  $y$  to get

$$\int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle (\rho(t, x - \varepsilon v s) - \rho(t, x)) M_\beta(v) \rangle ds = \int_{\mathbb{R}^d} \frac{\rho(t, y) - \rho(t, x)}{|x - y|^\beta} a(\varepsilon, x - y) dy, \quad (2.10)$$

where

$$a(\varepsilon, z) = \int_0^{\frac{t}{\varepsilon^\alpha}} |z|^\beta \frac{e^{-s}}{(\varepsilon s)^d} M_\beta\left(\frac{z}{\varepsilon s}\right) ds. \quad (2.11)$$

Now, we want to find an equivalent of  $a(\varepsilon, z)$  for small values of  $\varepsilon$ . Here we consider the equilibrium  $M_\beta(v) = \frac{m}{1+|v|^\beta}$ . Once again, the general case with  $M_\beta$  satisfying (2.1) can be done similarly, we refer for the supplementary materials of this article for more details. The integral  $a$  becomes

$$a(\varepsilon, z) = m \int_0^{\frac{t}{\varepsilon^\alpha}} |z|^\beta \frac{(\varepsilon s)^\beta}{(\varepsilon s)^\beta + |z|^\beta} \frac{e^{-s}}{(\varepsilon s)^d} ds,$$

and we consider, for a nonzero  $z$ , the following quantity

$$\begin{aligned} a(\varepsilon, z) - m\varepsilon^{\beta-d}\Gamma(\beta-d+1) &= -m \int_0^{+\infty} (\varepsilon s)^{\beta-d} e^{-s} \frac{(\varepsilon s)^\beta}{(\varepsilon s)^\beta + |z|^\beta} ds \\ &\quad - m \int_{\frac{t}{\varepsilon^\alpha}}^{+\infty} |z|^\beta \frac{(\varepsilon s)^\beta}{(\varepsilon s)^\beta + |z|^\beta} \frac{e^{-s}}{(\varepsilon s)^d} ds. \end{aligned}$$

Then, for almost every  $z \in \mathbb{R}^d$ ,  $\frac{1}{\varepsilon^{\beta-d}} (a(\varepsilon, z) - m\varepsilon^{\beta-d}\Gamma(\beta-d+1))$  converges to 0 when  $\varepsilon$  goes to zero. We also have the following estimation

$$|a(\varepsilon, z) - m\varepsilon^{\beta-d}\Gamma(\beta-d+1)| \leq K \left( \varepsilon^{\beta-d} + \varepsilon^{\beta-d} e^{-\frac{t}{2\varepsilon^\alpha}} \right). \quad (2.12)$$

Let us remark that for a reasonable regularity of  $\rho$ , the integral appearing in (2.10) is well defined as a principal value because  $\beta \in (d, d+2)$

$$P.V. \int_{\mathbb{R}^d} \frac{\rho(t, y) - \rho(t, x)}{|y - x|^\beta} dy = \int_{\mathbb{R}^d} \frac{\rho(t, y) - \rho(t, x) - (y - x) \cdot \nabla_x \rho(t, x)}{|y - x|^\beta} dy.$$

Hence, using (2.12) we obtain that for almost all  $y \in \mathbb{R}^d$

$$\left| \frac{1}{\varepsilon^{\beta-d}} \frac{\rho(t, y) - \rho(t, x) - (y - x) \cdot \nabla_x \rho(t, x)}{|y - x|^\beta} (a(\varepsilon, z) - m\varepsilon^{\beta-d}\Gamma(\beta-d+1)) \right|,$$

is dominated by an integrable function. We also obtained that for almost all  $y \in \mathbb{R}^d$  it tends to zero when  $\varepsilon$  goes to zero, by using the dominated convergence theorem for this function.

We have, from (2.10)

$$\begin{aligned} &\int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle (\rho(t, x - \varepsilon v s) - \rho(t, x)) M_\beta(v) \rangle ds \\ &= \int_{\mathbb{R}^d} \frac{\rho(t, y) - \rho(t, x) - (y - x) \cdot \nabla_x \rho(t, x)}{|x - y|^\beta} a(\varepsilon, x - y) dy, \end{aligned}$$

where we used that  $a$  is even. So we have

$$\begin{aligned} & \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle (\rho(t, x - \varepsilon v s) - \rho(t, x)) M_\beta(v) \rangle ds \\ & \underset{\varepsilon \rightarrow 0}{\sim} m \varepsilon^{\beta-d} \Gamma(\beta - d + 1) \int_{\mathbb{R}^d} \frac{\rho(t, y) - \rho(t, x) - (y - x) \nabla_x \rho(t, x)}{|y - x|^\beta} dy \\ & \underset{\varepsilon \rightarrow 0}{\sim} m \varepsilon^{\beta-d} \Gamma(\beta - d + 1) P.V. \int_{\mathbb{R}^d} \frac{\rho(t, y) - \rho(t, x)}{|y - x|^\beta} dy. \end{aligned}$$

We remark that we have to set  $\beta - d = \alpha$  to recover the limit equation, so we get

$$\rho(t, x) \underset{\varepsilon \rightarrow 0}{=} -\varepsilon^\alpha \partial_t \rho(t, x) + m \varepsilon^\alpha \Gamma(\alpha + 1) P.V. \int_{\mathbb{R}^d} \frac{\rho(t, y) - \rho(t, x)}{|y - x|^{\alpha+d}} dy + \rho(t, x) + o(\varepsilon^\alpha),$$

that is when  $\varepsilon$  goes to 0

$$\partial_t \rho(t, x) = -m \Gamma(\alpha + 1) P.V. \int_{\mathbb{R}^d} \frac{\rho(t, x) - \rho(t, y)}{|y - x|^{\alpha+d}} dy,$$

which is the expected anomalous diffusion equation.

□

**3. Numerical schemes.** This section is devoted to the presentation of appropriate numerical schemes to capture the solution of (1.2). We want these schemes to be Asymptotic Preserving (AP), that is, for arbitrary initial condition  $f_0$

1. to be consistent with the kinetic equation (1.2) when the time step  $\Delta t$  tends to 0, with a fixed  $\varepsilon$ ,
2. to degenerate into a scheme solving the asymptotic equation (here, anomalous diffusion) when  $\varepsilon$  goes to 0 with a fixed time step  $\Delta t$ .

In the sequel, three different numerical schemes are presented: a fully implicit scheme, a micro-macro decomposition based scheme and a Duhamel based scheme. We will consider a time discretization  $t_n = n \Delta t, n = 0, \dots, N$  such that  $N \Delta t = T$  where  $T$  is the final time and we will set  $f^n \simeq f(t_n)$ .

As we used Fourier transform in the previous formal computations, we will consider a bounded domain  $\Omega$  for the spatial domain with periodic conditions. Hence, we will be able to use the discrete Fourier transform in the algorithm. The schemes we present in this section are semi-discrete-in-time schemes. However, in the section devoted to the numerical tests, we will consider the following discretization for the space and velocity

- The space domain will be  $\Omega = [-1, 1]$ , discretized with  $N_x$  points such that, denoting  $\Delta x = \frac{2}{N_x}$ ,

$$x_i = -1 + (i - 1) \Delta x, \quad 1 \leq i \leq N_x. \quad (3.1)$$

The discrete Fourier transform will be computed with the modes

$$-\frac{N_x}{2} \leq k \leq \frac{N_x}{2}, \quad k \in \mathbb{Z}. \quad (3.2)$$

- For the velocities, we want to compute the integrals with very simple discretization. Here, we consider the bounded velocity space  $V = [-v_{max}, v_{max}]$

and  $2N_v + 1$  points of discretization such that denoting  $\Delta v = \frac{v_{max}}{N_v}$ , we define the points

$$v_j = -v_{max} + j\Delta v, \quad 0 \leq j \leq 2N_v, \quad (3.3)$$

to ensure the symmetry property of the discretization. The numerical integration in  $v$  will be denoted by  $\langle \cdot \rangle_{N_v}$ . In the case  $d = 1$  with (3.3), we will consider

$$\langle g(v) \rangle_{N_v} = \Delta v \sum_{j=0}^{2N_v} g(v_j). \quad (3.4)$$

**3.1. Implicit scheme.** The first idea to design a scheme for the kinetic equation (1.2) is to set an implicit scheme over the Fourier formulation of the kinetic equation. It is known that in the case of the diffusion limit, it preserves easily the asymptotic equation. But in the case of the anomalous diffusion limit, the effect of large velocities must be taken into account to obtain the good asymptotic and this needs a suitable modification of this fully implicit scheme.

We start with (1.2) written in the spatial Fourier variable and consider a fully implicit time discretization

$$\frac{\hat{f}^{n+1} - \hat{f}^n}{\Delta t} + \frac{1}{\varepsilon^\alpha} (1 + i\varepsilon k \cdot v) \hat{f}^{n+1} = \frac{1}{\varepsilon^\alpha} \hat{\rho}^{n+1} M_\beta.$$

We have

$$\hat{f}^{n+1} = \frac{1 - \lambda}{1 + i\lambda\varepsilon k \cdot v} \hat{f}^n + \frac{\lambda}{1 + i\lambda\varepsilon k \cdot v} \hat{\rho}^{n+1} M_\beta, \quad (3.5)$$

with

$$\lambda = \frac{\Delta t}{\varepsilon^\alpha + \Delta t}. \quad (3.6)$$

Note that  $0 < \lambda < 1$ ,  $\lambda \xrightarrow{\Delta t \rightarrow 0} 0$  and that  $\lambda \xrightarrow{\varepsilon \rightarrow 0} 1$ .

To compute  $\hat{f}^{n+1}$  from (3.5),  $\hat{\rho}^{n+1}$  has to be determined first. To do that we integrate (3.5) with respect to  $v$  at the discrete level to get

$$\hat{\rho}^{n+1} = \lambda \left\langle \frac{M_\beta}{1 + i\lambda\varepsilon k \cdot v} \right\rangle_{N_v} \hat{\rho}^{n+1} + (1 - \lambda) \left\langle \frac{\hat{f}^n}{1 + i\lambda\varepsilon k \cdot v} \right\rangle_{N_v}, \quad (3.7)$$

that is

$$\hat{\rho}^{n+1} = \left\langle \frac{\hat{f}^n}{1 + i\lambda\varepsilon k \cdot v} \right\rangle_{N_v} \frac{1}{\left\langle \frac{M_\beta}{1 + i\lambda\varepsilon k \cdot v} \right\rangle_{N_v} + \frac{1}{1 - \lambda} \left\langle \frac{i\lambda\varepsilon k \cdot v M_\beta}{1 + i\lambda\varepsilon k \cdot v} \right\rangle_{N_v}}. \quad (3.8)$$

In case 1 of the diffusion scaling, this scheme is consistent with the kinetic equation and degenerates into a scheme solving the diffusion equation when  $\varepsilon$  becomes small.

In case 2 of anomalous diffusion scaling, the equilibrium  $M_\beta$  is heavy-tailed and  $D = \langle v \otimes v M_\beta(v) \rangle = +\infty$ . Hence, at the continuous level, the limit of

$$\left\langle \frac{i\lambda\varepsilon k \cdot v M_\beta}{1 + i\lambda\varepsilon k \cdot v} \right\rangle = -\varepsilon^2 \lambda^2 \left\langle \frac{(k \cdot v)^2 M_\beta}{1 + \varepsilon^2 \lambda^2 (k \cdot v)^2} \right\rangle,$$

which appears in (3.8) is not obvious but uses similar calculations as in proof of Prop. 2.1. It reads

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^\alpha} \left\langle \frac{\varepsilon^2 (k \cdot v)^2 \lambda^2}{1 + \lambda^2 \varepsilon^2 (k \cdot v)^2} M_\beta(v) \right\rangle \hat{\rho}^{n+1} = |k|^\alpha \kappa \hat{\rho}^{n+1},$$

with  $\kappa$  defined in (1.6). Unfortunately, as we want to use a discretization of the velocities using a bounded domain for  $v$ , a quadrature of the integral involved in this limit does not capture the heavy tail of the equilibrium. It means that the effect of large velocities is completely missed. Namely, once computed with (3.4), the above integral in velocity reads

$$\varepsilon^{2-\alpha} \Delta v \sum_{j=0}^{2N_v} \frac{k^2 v_j^2 \lambda^2}{1 + \lambda^2 \varepsilon^2 k^2 v_j^2} M_\beta(v_j) \underset{\varepsilon \rightarrow 0}{\sim} \varepsilon^{2-\alpha} \Delta v k^2 \sum_{j=0}^{2N_v} v_j^2 M_\beta(v_j).$$

In fact, with a finite  $N_v$ , the discretized coefficient  $\sum_{j=0}^{2N_v} v_j^2 M_\beta(v)$  might be large but is finite. Thus,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{2-\alpha} \Delta v k^2 \sum_{j=0}^{2N_v} v_j^2 M_\beta(v_j) = 0,$$

and then, when  $\varepsilon$  goes to zero the implicit kinetic scheme we wrote in (3.5)-(3.8) degenerates into the following scheme

$$\hat{\rho}^{n+1} = \hat{\rho}^n, \quad \hat{f}^{n+1} = \hat{\rho}^{n+1} M_\beta,$$

which is not the correct asymptotics. Hence we have to transform its expression to make the right limit clearly appear in the scheme. The scheme is presented in the following proposition.

**PROPOSITION 3.1.** *In the case 2 (anomalous diffusion scaling), we consider the following scheme defined for all  $k$  and for all time index  $0 \leq n \leq N$ ,  $N\Delta t = T$  by*

$$\left\{ \begin{array}{l} \hat{\rho}^{n+1}(k) = \frac{\left\langle \frac{\hat{f}^n}{1 + i\lambda\varepsilon k \cdot v} \right\rangle_{N_v}}{\left\langle \frac{M_\beta}{1 + i\lambda\varepsilon k \cdot v} \right\rangle_{N_v} + \frac{(\varepsilon\lambda|k|)^\alpha}{1-\lambda} \left\langle \frac{m}{(\varepsilon\lambda|k|)^{\alpha+d} + |v|^{\alpha+d}} \frac{(v \cdot e)^2}{1 + (v \cdot e)^2} \right\rangle_{N_v}} \\ \hat{f}^{n+1} = \frac{1}{1 + i\lambda\varepsilon k \cdot v} \left[ (1-\lambda)\hat{f}^n + \lambda\hat{\rho}^{n+1}M_\beta \right], \end{array} \right.$$

with  $\lambda = \Delta t / (\varepsilon^\alpha + \Delta t)$ , where  $e$  is any unitary vector and with the initial condition  $\hat{f}^0(k, v) = \hat{f}_0(k, v)$ . This scheme has the following properties:

(i) The scheme is of order 1 for any fixed  $\varepsilon$  and preserves the total mass

$$\forall n \in \llbracket 1, N \rrbracket, \hat{\rho}^n(0) = \hat{\rho}^0(0).$$

(ii) The scheme is AP: for a fixed  $\Delta t$ , the scheme solves the anomalous diffusion equation when  $\varepsilon$  goes to zero

$$\frac{\hat{\rho}^{n+1}(k) - \hat{\rho}^n(k)}{\Delta t} = -\kappa |k|^\alpha \hat{\rho}^{n+1}(k), \quad (3.9)$$

where  $\kappa$ , defined by (1.6), is computed with (3.4).

*Proof.* As (i) is straightforward, let us prove (ii). We start by considering the expression of  $\hat{\rho}^{n+1}$  given in (3.8) with continuous integrations in  $v$ , and more precisely the integral

$$\left\langle \frac{i\varepsilon\lambda k \cdot v M_\beta}{1 + i\varepsilon\lambda k \cdot v} \right\rangle = \int_{\mathbb{R}^d} M_\beta(v) \frac{\varepsilon^2 \lambda^2 (k \cdot v)^2}{1 + \varepsilon^2 \lambda^2 (k \cdot v)^2} dv.$$

As in the previous section, we perform the change of variables  $w = \varepsilon\lambda|k|v$  to obtain

$$\begin{aligned} \int_{\mathbb{R}^d} M_\beta(v) \frac{\varepsilon^2 \lambda^2 (k \cdot v)^2}{1 + \varepsilon^2 \lambda^2 (k \cdot v)^2} dv &= (\varepsilon\lambda|k|)^{-d} \int_{\mathbb{R}^d} M_\beta \left( \frac{w}{\varepsilon\lambda|k|} \right) \frac{\left( w \cdot \frac{k}{|k|} \right)^2}{1 + \left( w \cdot \frac{k}{|k|} \right)^2} dw, \quad (3.10) \\ &= (\varepsilon\lambda|k|)^{-d} \int_{\mathbb{R}^d} M_\beta \left( \frac{w}{\varepsilon\lambda|k|} \right) \frac{(w \cdot e)^2}{1 + (w \cdot e)^2} dw, \end{aligned}$$

with  $e = k/|k|$ ; note that the last integral does not depend on this unitary vector  $e$  thanks to its rotational invariance.

We consider explicitly the case  $M_\beta(v) = \frac{m}{1+|v|^{\alpha+d}}$  with  $m$  such that  $\langle M_\beta \rangle = 1$  and with  $\alpha \in ]0, 2[$ . We have

$$\left\langle M_\beta \frac{\varepsilon^2 \lambda^2 (k \cdot v)^2}{1 + \varepsilon^2 \lambda^2 (k \cdot v)^2} \right\rangle = (\varepsilon\lambda|k|)^\alpha \int_{\mathbb{R}^d} \frac{m}{(\varepsilon\lambda|k|)^{\alpha+d} + |w|^{\alpha+d}} \frac{(w \cdot e)^2}{1 + (w \cdot e)^2} dw,$$

where  $e$  is any unitary vector. Inserting this formula in (3.8) with discrete integration, we get the expression for  $\hat{\rho}^{n+1}$ . The equation satisfied by  $\hat{f}^{n+1}$  in (3.5) is unchanged, then the numerical scheme of the proposition is derived. Now, it remains to show that this scheme enjoys the AP property. From the following form of the equation on  $\hat{\rho}^{n+1}$

$$\begin{aligned} \frac{\hat{\rho}^{n+1} - \hat{\rho}^n}{\Delta t} &= -\lambda^\alpha |k|^\alpha \left\langle \frac{m}{(\varepsilon\lambda|k|)^{\alpha+d} + |v|^{\alpha+d}} \frac{(v \cdot e)^2}{1 + (v \cdot e)^2} \right\rangle_{N_v} \hat{\rho}^{n+1} \quad (3.11) \\ &\quad - \left\langle \frac{i\lambda\varepsilon k \cdot v}{1 + i\lambda\varepsilon k \cdot v} \frac{\hat{f}^n}{\Delta t} \right\rangle_{N_v}, \end{aligned}$$

we easily observe that it degenerates when  $\varepsilon$  goes to zero to an implicit scheme for the anomalous diffusion equation. This concludes the proof.

□

**3.2. Scheme based on a micro-macro decomposition.** In the previous part, we constructed a fully implicit scheme enjoying the AP property. The implicit character of the transport may induce a high computational cost. Therefore, we propose another scheme, which is based on a micro-macro decomposition of the kinetic equation and in which the transport part is explicit in time. In the diffusion case, such a scheme has been set in [22]. We first recall here the way to derive it. Then, we consider more precisely the case of the anomalous diffusion and show how to take into account the effect of the large velocities induced by the heavy-tailed structure of the equilibrium  $M_\beta(v)$ .

The distribution  $f$  is decomposed as

$$f = \langle f \rangle M_\beta + g = \rho M_\beta + g,$$

where  $\langle g \rangle = 0$ . Injecting this decomposition into (1.2), we have

$$\partial_t \rho M_\beta + \partial_t g + \frac{\varepsilon}{\varepsilon^\alpha} v \cdot \nabla_x \rho M_\beta + \frac{\varepsilon}{\varepsilon^\alpha} v \cdot \nabla_x g = -\frac{1}{\varepsilon^\alpha} g. \quad (3.12)$$

To derive an equation on  $\rho$ , we integrate with respect to  $v$  to obtain

$$\partial_t \rho + \frac{\varepsilon}{\varepsilon^\alpha} \langle v \cdot \nabla_x g \rangle = 0. \quad (3.13)$$

Now we replace  $\partial_t \rho$  in (3.12) to get an equation satisfied by  $g$

$$\partial_t g + \frac{\varepsilon}{\varepsilon^\alpha} v \cdot \nabla_x \rho M_\beta + \frac{\varepsilon}{\varepsilon^\alpha} (v \cdot \nabla_x g - \langle v \cdot \nabla_x g \rangle M_\beta) = -\frac{1}{\varepsilon^\alpha} g. \quad (3.14)$$

From this formulation the semi-implicit scheme proposed in [22] writes

$$\begin{cases} \frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{\varepsilon}{\varepsilon^\alpha} \langle v \cdot \nabla_x g^{n+1} \rangle_{N_v} = 0 \\ \frac{g^{n+1} - g^n}{\Delta t} + \frac{\varepsilon}{\varepsilon^\alpha} v \cdot \nabla_x \rho^n M_\beta + \frac{\varepsilon}{\varepsilon^\alpha} (v \cdot \nabla_x g^n - \langle v \cdot \nabla_x g^n \rangle_{N_v} M_\beta) = -\frac{1}{\varepsilon^\alpha} g^{n+1}, \end{cases} \quad (3.15)$$

which is of order 1 for any fixed  $\varepsilon$  and preserves the asymptotic of diffusion equation when considered in case 1. However, as in the last section, when case 2 is considered, formulation (3.15) does not work because the fractional laplacian does not appear when  $\varepsilon$  goes to 0. We then have to modify the micro-macro scheme. As  $M_\beta$  is even and the discrete velocities are symmetrically spread around 0 we have

$$\langle v \cdot \nabla_x g^{n+1} \rangle_{N_v} = \langle v \cdot \nabla_x \rho^{n+1} M_\beta \rangle_{N_v} + \langle v \cdot \nabla_x g^{n+1} \rangle_{N_v} = \langle v \cdot \nabla_x f^{n+1} \rangle_{N_v},$$

and so the first equation of (3.15) can be rewritten as

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{\varepsilon}{\varepsilon^\alpha} \langle v \cdot \nabla_x f^{n+1} \rangle_{N_v} = 0.$$

Now we use an implicit formulation of the kinetic equation to express  $f^{n+1}$

$$\varepsilon^\alpha \frac{f^{n+1} - f^n}{\Delta t} + \varepsilon v \cdot \nabla_x f^{n+1} = \rho^{n+1} M_\beta - f^{n+1}.$$

As we did in the previous part we introduce the variable  $\lambda$  defined by (3.6) and write this last equation as

$$(I + \varepsilon \lambda v \cdot \nabla_x) f^{n+1} = \lambda \rho^{n+1} M_\beta + (1 - \lambda) f^n,$$

that is

$$f^{n+1} = \lambda (I + \varepsilon \lambda v \cdot \nabla_x)^{-1} \rho^{n+1} M_\beta + (1 - \lambda) (I + \varepsilon \lambda v \cdot \nabla_x)^{-1} f^n.$$

As  $\lambda \underset{\Delta t \rightarrow 0}{=} O(\Delta t)$ , we will use the following approximated expression for  $f^{n+1}$

$$f^{n+1} = \lambda (I + \varepsilon \lambda v \cdot \nabla_x)^{-1} \rho^{n+1} M_\beta + (1 - \lambda) (I - \varepsilon \lambda v \cdot \nabla_x) f^n + O(\Delta t),$$

to avoid a costly inversion of the transport operator. As we will use a bounded domain for the velocities in the numerical computations, this approximation induces an error

of order  $O(\Delta t)$ . However, since we are writing an order 1 scheme, its accuracy is not changed. In this expression, the terms we removed also vanish when  $\varepsilon$  goes to zero, independently of  $\Delta t$ . Scheme (3.15) can be rewritten in the anomalous diffusion limit as

$$\left\{ \begin{array}{l} \frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{\varepsilon}{\varepsilon^\alpha} \lambda \langle v \cdot \nabla_x (I + \varepsilon \lambda v \cdot \nabla_x)^{-1} \rho^{n+1} M_\beta \rangle_{N_v} \\ \quad + \frac{\varepsilon}{\varepsilon^\alpha} (1 - \lambda) \langle v \cdot \nabla_x (I - \varepsilon \lambda v \cdot \nabla_x) f^n \rangle_{N_v} = 0, \\ \frac{g^{n+1} - g^n}{\Delta t} + \frac{\varepsilon}{\varepsilon^\alpha} v \cdot \nabla_x \rho^n M_\beta + \frac{\varepsilon}{\varepsilon^\alpha} (v \cdot \nabla_x g^n - \langle v \cdot \nabla_x g^n \rangle_{N_v} M_\beta) = -\frac{1}{\varepsilon^\alpha} g^{n+1}. \end{array} \right. \quad (3.16)$$

We can make additional simplifications. First, we remark that,

$$\langle v \cdot \nabla_x (I - \varepsilon \lambda v \cdot \nabla_x) f^n \rangle_{N_v} = \langle v \cdot \nabla_x f^n \rangle_{N_v} + O(\Delta t) = \langle v \cdot \nabla_x g^n \rangle_{N_v} + O(\Delta t),$$

where we used  $\langle v M_\beta \rangle_{N_v} = 0$  and  $\lambda = O(\Delta t)$  at fixed  $\varepsilon$ . When performing this approximation, we removed terms of order  $O(\Delta t)$ . It is consistent with the kinetic equation, since the magnitude of the numerical error is not changed. Moreover, this simplification is also valid when considering the AP character of the scheme. Indeed, for a fixed  $\Delta t$  the terms we removed can also be neglected in the small  $\varepsilon$  limit. As the fractional diffusion limit of the scheme is now ensured (at the semi-discrete level), all the other terms of order  $\Delta t$  can also be neglected with no incidence on the precision and on the AP character of the scheme. Then, the last term in the first line of the scheme (3.16) can be replaced by  $\varepsilon^{1-\alpha} (1-\lambda) \langle v \cdot \nabla_x g^n \rangle_{N_v}$ . Second, we will reformulate the second term of the first line of the scheme (3.16) (which produces the fractional diffusion when  $\varepsilon$  goes to zero) in order to get the right limit when  $\varepsilon \rightarrow 0$  when the integrals in  $v$  are computed with (3.4). To simplify the presentation, we present the reformulation using the Fourier variable (because of the non-local character of the fractional laplacian in the space coordinates, the case in these coordinates is more delicate). The second term of (3.16) writes in the Fourier variable and with continuous integration in  $v$

$$\frac{\varepsilon}{\varepsilon^\alpha} \lambda \left\langle \frac{i k \cdot v}{1 + i \varepsilon \lambda k \cdot v} M_\beta \right\rangle \hat{\rho}^{n+1}(k) = \frac{1}{\varepsilon^\alpha} \left\langle \frac{\varepsilon^2 \lambda^2 (k \cdot v)^2}{1 + \varepsilon^2 \lambda^2 (k \cdot v)^2} M_\beta \right\rangle \hat{\rho}^{n+1}(k).$$

Here, as in the previous section, we remark that this term degenerates into the fractional laplacian  $\kappa |k|^\alpha \hat{\rho}^{n+1}(k)$  with  $\kappa$  defined in (1.6) for small  $\varepsilon$ . However, once again, the passage to the numerical integration misses the high velocities effects. To make the fractional diffusion appear in the scheme, we then make the change of variables  $w = \varepsilon \lambda |k| v$  in the above expression before discretizing in velocity. Hence, denoting  $e$  any unitary vector and  $\mathcal{F}^{-1}$  the inverse of the Fourier transform, the following proposition holds:

**PROPOSITION 3.2.** *In case 2 (anomalous diffusion scaling), we introduce the following micro-macro the scheme defined for all  $x \in \Omega$ ,  $v \in \mathbb{R}^d$  and all time index  $0 \leq n \leq N$ ,  $N \Delta t = T$  by*

$$\left\{ \begin{array}{l} \frac{\rho^{n+1} - \rho^n}{\Delta t} + \lambda^\alpha \mathcal{F}^{-1} \left( |k|^\alpha \left\langle \frac{(v \cdot e)^2}{1 + (v \cdot e)^2} \frac{m}{(\varepsilon \lambda |k|)^\beta + |v|^\beta} \right\rangle_{N_v} \hat{\rho}^{n+1}(k) \right) \\ \quad + \frac{\varepsilon}{\varepsilon^\alpha + \Delta t} \langle v \cdot \nabla_x g^n \rangle_{N_v} = 0 \\ \frac{g^{n+1} - g^n}{\Delta t} + \frac{\varepsilon}{\varepsilon^\alpha} v \cdot \nabla_x \rho^n M_\beta + \frac{\varepsilon}{\varepsilon^\alpha} (v \cdot \nabla_x g^n - \langle v \cdot \nabla_x g^n \rangle_{N_v} M_\beta) = -\frac{g^{n+1}}{\varepsilon^\alpha}, \end{array} \right. \quad (3.17)$$

where  $e$  is any unitary vector, and where initial conditions are  $\rho^0(x) = \rho(0, x)$  and  $g^0(x, v) = f(0, x, v) - \rho(0, x)M_\beta$ . This scheme has the following properties:

(i) The scheme is of order 1 for any fixed  $\varepsilon$  and preserves the total mass

$$\forall n \in \llbracket 1, N \rrbracket, \hat{\rho}^n(0) = \hat{\rho}^0(0).$$

(ii) The scheme is AP: for a fixed  $\Delta t$ , the scheme solves the diffusion equation when  $\varepsilon$  goes to zero

$$\frac{\hat{\rho}^{n+1}(k) - \hat{\rho}^n(k)}{\Delta t} = -\kappa |k|^\alpha \hat{\rho}^{n+1}(k),$$

where  $\kappa$ , defined by (1.6), is computed with (3.4).

*Proof.*

REMARK 1. In the macro part of (3.17), for a fixed  $\varepsilon$ , we have

$$\left\langle \frac{(v \cdot e)^2}{1 + (v \cdot e)^2} \frac{m}{(\varepsilon \lambda |k|)^\beta + |v|^\beta} \right\rangle_{N_v} = \left\langle \frac{(v \cdot e)^2}{1 + (v \cdot e)^2} \frac{m}{|v|^\beta} \right\rangle_{N_v} + O(\Delta t^\beta).$$

Hence, as  $\beta > 1$ , we could have replaced directly the term treated in Fourier variable by its asymptotics for small  $\varepsilon$  with no impact on the accuracy of the scheme neither on its AP property. This rewriting permits to write a micro-macro scheme with no use of the Fourier variable. Since the discretization of the fractional laplacian in the space variable is a consequent piece of work, we decided, for a sake of simplicity, to present here the strategy in Fourier variable.

The proof of (i) is immediate, let us prove (ii). We have when  $\varepsilon$  goes to zero with a fixed  $\Delta t$ , the scheme solves the anomalous diffusion equation. The equation on  $g$  gives, when  $\varepsilon$  goes to zero.

$$g^{n+1} = O(\min(\varepsilon, \varepsilon^\alpha)).$$

The equation on  $\rho$  gives, in the Fourier variable

$$\frac{\hat{\rho}^{n+1} - \hat{\rho}^n}{\Delta t} + \lambda^\alpha |k|^\alpha \left\langle \frac{(v \cdot e)^2}{1 + (v \cdot e)^2} \frac{m}{(\varepsilon \lambda |k|)^\beta + |v|^\beta} \right\rangle_{N_v} \hat{\rho}^{n+1} = O(\min(\varepsilon^2, \varepsilon^{1+\alpha})).$$

Thanks to the following relations,

$$\lambda^\alpha = \left( \frac{\Delta t}{\varepsilon^\alpha + \Delta t} \right)^\alpha = 1 + O(\varepsilon^\alpha), \quad \text{and} \quad \frac{m}{(\varepsilon \lambda |k|)^\beta + |v|^\beta} = \frac{m}{|v|^\beta} + O(\varepsilon^{\alpha+d}),$$

we get the implicit discretization of the anomalous diffusion equation (3.9) when  $\varepsilon \rightarrow 0$ . Hence the scheme solves the anomalous diffusion equation when  $\varepsilon$  goes to zero with a fixed  $\Delta t$ .  $\square$

**3.3. Scheme based on an integral formulation of the equation.** In the previous parts, we wrote two AP schemes solving (1.2) in the case of anomalous diffusion limit, both of them were of order 1 in time. Here we present a scheme based on a Duhamel formulation of (1.2) which has uniform accuracy in  $\varepsilon$ . This approach bears similarities with the UGKS scheme (see [31, 26]). As in the previous parts, the case of the classical diffusion gives easily an AP scheme but the large velocities effects require a specific treatment for the anomalous diffusion case.



Considering in a first time that the integrals in velocity are done continuously, we start from (2.2)

$$\hat{\rho}(t, k) = \left\langle e^{-\frac{t}{\varepsilon^\alpha}(1+i\varepsilon k \cdot v)} \hat{\rho}(0, k, v) \right\rangle + \int_0^{\frac{t}{\varepsilon^\alpha}} e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta(v) \rangle \hat{\rho}(t - \varepsilon^\alpha s, k) ds,$$

hence evaluating at time  $t = t_{n+1}$  leads to

$$\hat{\rho}(t_{n+1}, k) = \hat{A}_0(t_{n+1}, k) + \sum_{j=0}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle \hat{\rho}(t_{n+1} - \varepsilon^\alpha s, k) ds, \quad (3.18)$$

where  $\hat{A}_0$  is defined by (2.4). We perform a quadrature of order 2 in the integrals. Assuming that the time derivatives of  $\hat{\rho}$  are uniformly bounded in  $\varepsilon$ , we have:

$$\forall j \in \llbracket 1, N-1 \rrbracket, \forall s \in \left[ \frac{t_j}{\varepsilon^\alpha}, \frac{t_{j+1}}{\varepsilon^\alpha} \right],$$

$$\hat{\rho}(t_{n+1} - \varepsilon^\alpha s, k) = a_j(\varepsilon, s) \hat{\rho}(t_{n+1} - t_j, k) + (1 - a_j(\varepsilon, s)) \hat{\rho}(t_{n+1} - t_{j+1}, k) + O(\Delta t^2), \quad (3.19)$$

uniformly in  $\varepsilon$ , with  $a_j(\varepsilon, s) = 1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}$ . Inserting (3.19) in the integral term of (3.18) leads to

$$\begin{aligned} \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle \hat{\rho}(t_{n+1} - \varepsilon^\alpha s, k) ds &= c_j(k) \hat{\rho}(t_{n+1} - t_j, k) \\ &\quad + b_j(k) \hat{\rho}(t_{n+1} - t_{j+1}, k) + O(\Delta t^2), \end{aligned} \quad (3.20)$$

uniformly in  $\varepsilon$ , where we used the following notations  $\forall j \in \llbracket 0, N \rrbracket$

$$b_j(k) = \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t} e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle ds, \quad (3.21)$$

$$c_j(k) = \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left( 1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t} \right) e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle ds. \quad (3.22)$$

We use the quadrature (3.20) in (3.18) to write

$$\hat{\rho}(t_{n+1}, k) = \hat{A}_0(t_{n+1}, k) + \sum_{j=0}^n (c_j \hat{\rho}(t_{n+1-j}, k) + b_j \hat{\rho}(t_{n-j}, k) + O(\Delta t^2)), \quad (3.23)$$

and as  $n\Delta t \leq T$  then  $\sum_{j=0}^n \Delta t^2 = O(\Delta t)$  and we get a first order scheme that writes

$$\hat{\rho}^{n+1}(k) = \frac{\hat{A}_0(t_{n+1}, k) + \sum_{j=1}^n (c_j \hat{\rho}^{n+1-j}(k) + b_j \hat{\rho}^{n-j}(k)) + b_0 \hat{\rho}^n(k)}{1 - c_0}. \quad (3.24)$$

In the case of diffusion or when the integrals in  $v$  of the coefficients  $b_j$  and  $c_j$  are computed exactly, this scheme enjoys the AP property. However, always because the high velocities cannot be taken into account in the numerical computations, it does

not preserve the asymptotic in the case of the fractional diffusion limit with discrete integrations in  $v$ . Therefore, we have to suitably transform the integrals  $c_j$  and  $b_j$  before discretizing in velocity. We have

$$\begin{aligned} b_j &= \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t} e^{-s} \langle (e^{-i\varepsilon s k \cdot v} - 1) M_\beta \rangle ds + \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t} e^{-s} ds, \\ c_j &= \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}\right) e^{-s} \langle (e^{-i\varepsilon s k \cdot v} - 1) M_\beta \rangle ds \\ &\quad + \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}\right) e^{-s} ds, \end{aligned}$$

and in the velocity integrations we set  $w = \varepsilon|k|v$  before discretizing the velocities as in (2.5), so when  $M_\beta(v) = m/(1 + |v|^\beta)$  we have with (3.4)

$$\begin{aligned} b_j &= \varepsilon^\alpha |k|^\alpha \left\langle \frac{m}{(\varepsilon|k|)^\beta + |w|^\beta} \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t} e^{-s} (e^{-isw \cdot e} - 1) ds \right\rangle_{N_v} \\ &\quad + \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t} e^{-s} ds, \\ c_j &= \varepsilon^\alpha |k|^\alpha \left\langle \frac{m}{(\varepsilon|k|)^\beta + |w|^\beta} \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}\right) e^{-s} (e^{-isw \cdot e} - 1) ds \right\rangle_{N_v} \\ &\quad + \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}\right) e^{-s} ds, \end{aligned}$$

with  $e$  any unitary vector. To ensure the AP property, the time integrations are computed exactly to get

$$\begin{aligned} b_j &= e^{-\frac{t_j}{\varepsilon^\alpha}} \left( \frac{1 - e^{-\frac{\Delta t}{\varepsilon^\alpha}}}{\frac{\Delta t}{\varepsilon^\alpha}} - e^{-\frac{\Delta t}{\varepsilon^\alpha}} \right) \left( 1 - \varepsilon^\alpha |k|^\alpha \left\langle \frac{m}{(\varepsilon|k|)^\beta + |v|^\beta} \right\rangle_{N_v} \right) \\ &\quad + \varepsilon^\alpha |k|^\alpha \left\langle \left( -\frac{e^{-\frac{t_{j+1}}{\varepsilon^\alpha}(1+iv \cdot e)}}{1+iv \cdot e} + \frac{\varepsilon^\alpha}{\Delta t} e^{-\frac{t_j}{\varepsilon^\alpha}(1+iv \cdot e)} \frac{1 - e^{-\frac{\Delta t}{\varepsilon^\alpha}(1+iv \cdot e)}}{(1+iv \cdot e)^2} \right) \frac{m}{(\varepsilon|k|)^\beta + |v|^\beta} \right\rangle_{N_v}, \end{aligned} \quad (3.25)$$

$$\begin{aligned} c_j &= e^{-\frac{t_j}{\varepsilon^\alpha}} \left( 1 - \frac{1 - e^{-\frac{\Delta t}{\varepsilon^\alpha}}}{\frac{\Delta t}{\varepsilon^\alpha}} \right) \left( 1 - \varepsilon^\alpha |k|^\alpha \left\langle \frac{m}{(\varepsilon|k|)^\beta + |v|^\beta} \right\rangle_{N_v} \right) \\ &\quad + \varepsilon^\alpha |k|^\alpha \left\langle \left( \frac{e^{-\frac{t_j}{\varepsilon^\alpha}(1+iv \cdot e)}}{1+iv \cdot e} - \frac{\varepsilon^\alpha}{\Delta t} e^{-\frac{t_j}{\varepsilon^\alpha}(1+iv \cdot e)} \frac{1 - e^{-\frac{\Delta t}{\varepsilon^\alpha}(1+iv \cdot e)}}{(1+iv \cdot e)^2} \right) \frac{m}{(\varepsilon|k|)^\beta + |v|^\beta} \right\rangle_{N_v}. \end{aligned} \quad (3.26)$$

We have the following proposition

**PROPOSITION 3.3.** *In case 2 (anomalous diffusion scaling), with the notations  $b_j$  and  $c_j$  defined in (3.25)-(3.26) and  $\hat{A}_0$  defined in (2.4) and computed with (3.4), we consider the following scheme defined for all  $k$  and for all time index  $0 \leq n \leq N$*

such that  $N\Delta t = T$  by

$$\hat{\rho}^{n+1}(k) = \frac{\hat{A}_0(t_{n+1}, k) + \sum_{j=1}^n (c_j \hat{\rho}^{n+1-j}(k) + b_j \hat{\rho}^{n-j}(k)) + b_0 \hat{\rho}^n(k)}{1 - c_0},$$

with the initial condition  $\hat{\rho}^0(k) = \hat{\rho}(0, k)$ . This scheme has the following properties:

(i) The scheme is of order 1 and preserves the total mass

$$\forall n \in \llbracket 1, N \rrbracket, \hat{\rho}^n(0) = \hat{\rho}^0(0).$$

(ii) The scheme is AP: for a fixed  $\Delta t$ , the scheme solves the anomalous diffusion equation when  $\varepsilon$  goes to zero

$$\frac{\hat{\rho}^{n+1}(k) - \hat{\rho}^n(k)}{\Delta t} = -\kappa |k|^\alpha \hat{\rho}^{n+1}(k),$$

where  $\kappa$ , defined by (1.6), is computed with (3.4).

(iii) Moreover, the semi-discrete-in-time scheme has uniform accuracy with respect to  $\Delta t$ , that is

$$\|\hat{\rho}^n(\cdot) - \hat{\rho}(n\Delta t, \cdot)\|_{L^\infty} \leq C\Delta t,$$

with  $C$  independent of  $\varepsilon$ .

REMARK 2. The numerical tests (see Fig. 4.6) suggest that, for all  $\varepsilon > 0$ , this scheme is of order 2 and degenerates into an order 1 scheme when  $\varepsilon$  becomes small. However, we are only able to prove the uniform order 1 of the scheme.

*Proof.* The conservation of the mass is obtained by induction, and the fact that it is of order 1 comes from the Taylor expansion we performed in the integrals. let us prove that for a fixed  $\Delta t$  the scheme solves the anomalous diffusion equation when  $\varepsilon$  goes to zero. From the exponential decay of  $A_0$  and  $b_j, c_j, j \geq 1$  we have when  $\varepsilon$  goes to zero

$$(1 - c_0) \hat{\rho}^{n+1}(k) = b_0(k) \hat{\rho}^n(k) + o(\varepsilon^\infty).$$

On the one side we have

$$b_0(k) = \frac{\varepsilon^\alpha}{\Delta t} + o(\varepsilon^\alpha),$$

and on the other side we have

$$\begin{aligned} c_0 &= \varepsilon^\alpha |k|^\alpha \left\langle \left( \frac{1}{1 + i v \cdot e} - 1 \right) \frac{m}{(\varepsilon |k|)^\beta + |v|^\beta} \right\rangle_{N_v} + 1 - \frac{\varepsilon^\alpha}{\Delta t} + o(\varepsilon^\alpha) \\ &= -\varepsilon^\alpha |k|^\alpha \left\langle \frac{m}{|v|^\beta} \frac{(v \cdot e)^2}{1 + (v \cdot e)^2} \right\rangle_{N_v} + 1 - \frac{\varepsilon^\alpha}{\Delta t} + o(\varepsilon^\alpha) \\ &= -\varepsilon^\alpha |k|^\alpha \kappa + 1 - \frac{\varepsilon^\alpha}{\Delta t} + o(\varepsilon^\alpha). \end{aligned}$$

We deduce that when  $\varepsilon$  goes to zero, the scheme degenerates into

$$\frac{\hat{\rho}^{n+1}(k) - \hat{\rho}^n(k)}{\Delta t} = -\kappa |k|^\alpha \hat{\rho}^{n+1}(k),$$

where  $\kappa$  is defined by (1.6) and computed with (3.4). This is an implicit scheme for the anomalous diffusion equation (1.5).

Now, let us prove the uniform accuracy of the semi-discrete scheme. Here, we consider that all the velocity integrations are done continuously. Firstly, we need to precise the remainder  $O(\Delta t^2)$  in (3.20). Taylor expansions of  $\hat{\rho}(t_{n+1} - \varepsilon^\alpha s, k)$  leads to

$$\begin{aligned} \hat{\rho}(t_{n+1} - \varepsilon^\alpha s, k) &= a_j(\varepsilon, s)\hat{\rho}(t_{n+1} - t_j, k) + (1 - a_j(\varepsilon, s))\hat{\rho}(t_{n+1} - t_{j+1}) \\ &\quad - a_j(\varepsilon, s)\frac{(\varepsilon^\alpha s - t_j)^2}{2}\partial_t^2\hat{\rho}(t_{n+1} - \varepsilon^\alpha \xi_1(s), k) \\ &\quad - (1 - a_j(\varepsilon, s))\frac{(\varepsilon^\alpha s - t_{j+1})^2}{2}\partial_t^2\hat{\rho}(t_{n+1} - \varepsilon^\alpha \xi_2(s), k), \end{aligned}$$

with  $s \in \left[\frac{t_j}{\varepsilon^\alpha}, \frac{t_{j+1}}{\varepsilon^\alpha}\right]$ ,  $\xi_1(s) \in \left[\frac{t_j}{\varepsilon^\alpha}, s\right]$ ,  $\xi_2(s) \in \left[s, \frac{t_{j+1}}{\varepsilon^\alpha}\right]$  and  $a_j = 1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}$ . Hence, (3.18) rewrites

$$\hat{\rho}(t_{n+1}, k) = \hat{A}_0(t_{n+1}, k) + \sum_{j=0}^n c_j \hat{\rho}(t_{n+1} - t_j) + b_j \hat{\rho}(t_{n+1} - t_{j+1}) + \Delta t^2 F, \quad (3.27)$$

where

$$\begin{aligned} F &= -\frac{1}{2} \sum_{j=0}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left( a_j(\varepsilon, s) \left( \frac{\varepsilon^\alpha s - t_j}{\Delta t} \right)^2 \partial_t^2 \hat{\rho}(t_{n+1} - \varepsilon^\alpha \xi_1(s), k) \right. \\ &\quad \left. + (1 - a_j(\varepsilon, s)) \left( \frac{t_{j+1} - \varepsilon^\alpha s}{\Delta t} \right)^2 \partial_t^2 \hat{\rho}(t_{n+1} - \varepsilon^\alpha \xi_2(s), k) \right) e^{-s} \langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle ds, \end{aligned} \quad (3.28)$$

and with  $b_j$  and  $c_j$  defined in (3.21)-(3.22). As  $a_j(\varepsilon, s) \geq 0$  and  $1 - a_j(\varepsilon, s) \geq 0$  for  $s \in \left[\frac{t_j}{\varepsilon^\alpha}, \frac{t_{j+1}}{\varepsilon^\alpha}\right]$ , we have

$$\begin{aligned} |F| &\leq \mathcal{C} \sum_{j=0}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left( a_j(\varepsilon, s) \left( \frac{\varepsilon^\alpha s - t_j}{\Delta t} \right)^2 + (1 - a_j(\varepsilon, s)) \left( \frac{t_{j+1} - \varepsilon^\alpha s}{\Delta t} \right)^2 \right) e^{-s} ds \\ &\leq \mathcal{C} \sum_{j=0}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left( \frac{\varepsilon^\alpha s - t_j}{\Delta t} \right) \left( 1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t} \right) e^{-s} ds \\ &\leq \mathcal{C} \sum_{j=0}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left( \frac{\varepsilon^\alpha s - t_j}{\Delta t} \right) e^{-s} ds, \end{aligned}$$

where we denoted  $\mathcal{C} = \frac{1}{2} \sup_{t,k} \|\partial_t^2 \hat{\rho}\|$  and remarked that  $\frac{\varepsilon^\alpha s - t_j}{\Delta t} \in [0, 1]$ .

Now, let us denote  $E_n = \hat{\rho}(t_n, k) - \hat{\rho}^n(k)$  and suppose that  $E_0 = 0$ . From (3.24) and (3.27), there is a relation linking the  $E_j$

$$(1 - c_0) E_{n+1} = b_0 E_n + \sum_{j=1}^n (c_j E_{n+1-j} + b_j E_{n-j}) + \Delta t^2 F. \quad (3.29)$$

To obtain a bound of  $E_{n+1}$ , we will need two different estimates of  $1/(1-c_0)$ . On the one hand, we remark that with the expression of  $c_0$  in (3.22) we have

$$|c_0| \leq \int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s}{\Delta t}\right) e^{-s} ds. \quad (3.30)$$

On the other hand the inequality  $0 \leq 1 - \frac{\varepsilon^\alpha s}{\Delta t} \leq 1$  for  $s \in [0, \frac{\Delta t}{\varepsilon^\alpha}]$  also gives  $|c_0| \leq 1 - e^{-\frac{\Delta t}{\varepsilon^\alpha}}$ , where we remark that in particular  $|c_0| \leq 1$ . Eventually, the inequality  $|1 - c_0| \geq 1 - |c_0|$  implies the two following bounds

$$\left| \frac{1}{1 - c_0} \right| \leq \frac{1}{1 - \int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s}{\Delta t}\right) e^{-s} ds} \quad \text{and} \quad \left| \frac{1}{1 - c_0} \right| \leq \frac{1}{e^{-\frac{\Delta t}{\varepsilon^\alpha}}}. \quad (3.31)$$

In (3.29) let us focus on the term containing  $F$ . It writes

$$\left| \frac{F}{1 - c_0} \right| \leq \frac{\|\partial_t^2 \hat{\rho}\|_\infty \int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \left(\frac{\varepsilon^\alpha s}{\Delta t}\right) e^{-s} ds}{|1 - c_0|} + \frac{\|\partial_t^2 \hat{\rho}\|_\infty \sum_{j=1}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(\frac{\varepsilon^\alpha s - t_j}{\Delta t}\right) e^{-s} ds}{|1 - c_0|}, \quad (3.32)$$

using (3.31) we find an upper bound for the first term of (3.32)

$$\frac{\int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \left(\frac{\varepsilon^\alpha s}{\Delta t}\right) e^{-s} ds}{|1 - c_0|} \leq \frac{\int_0^{\frac{\Delta t}{\varepsilon^\alpha}} e^{-s} ds - \int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s}{\Delta t}\right) e^{-s} ds}{1 - \int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s}{\Delta t}\right) e^{-s} ds} \leq 1, \quad (3.33)$$

and reminding that  $\frac{\varepsilon^\alpha s - t_j}{\Delta t} \leq 1$  for  $s \in \left[\frac{t_j}{\varepsilon^\alpha}, \frac{t_{j+1}}{\varepsilon^\alpha}\right]$ , the second term of (3.32) reads, using (3.31)

$$\frac{\sum_{j=1}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(\frac{\varepsilon^\alpha s - t_j}{\Delta t}\right) e^{-s} ds}{|1 - c_0|} \leq \frac{\sum_{j=1}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} e^{-s} ds}{|1 - c_0|} \leq \frac{\int_{\frac{\Delta t}{\varepsilon^\alpha}}^{+\infty} e^{-s} ds}{|1 - c_0|} \leq 1. \quad (3.34)$$

Eventually, (3.33) and (3.34) give the following upper bound for  $\frac{F}{1 - c_0}$

$$\left| \frac{F}{1 - c_0} \right| \leq 2\mathcal{C}. \quad (3.35)$$

Let us focus now on the remaining term in the right hand side of (3.29) and denote

$\mathcal{E}_n = \max_{\substack{j=0 \dots n \\ \varepsilon \in (0,1]}} |E_j|$ . We have from (3.21) and (3.22)

$$\begin{aligned}
\left| b_0 E_n + \sum_{j=1}^n (c_j E_{n+1-j} + b_j E_{n-j}) \right| &\leq \int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s}{\Delta t} e^{-s} |\langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle E_n| ds \\
&+ \sum_{j=1}^n \left( \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left( 1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t} \right) e^{-s} |\langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle E_{n+1-j}| ds \right. \\
&\left. + \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t} e^{-s} |\langle e^{-i\varepsilon s k \cdot v} M_\beta \rangle E_{n-j}| ds \right) \\
&\leq \left( \int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s}{\Delta t} e^{-s} ds + \sum_{j=1}^n \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} e^{-s} ds \right) \mathcal{E}_n \\
&\leq \left( 1 - \int_0^{\frac{\Delta t}{\varepsilon^\alpha}} \left( 1 - \frac{\varepsilon^\alpha s}{\Delta t} \right) e^{-s} ds \right) \mathcal{E}_n,
\end{aligned}$$

where we have used  $\langle M_\beta \rangle = 1$ . Hence, using (3.31), we get

$$\left| \frac{b_0 E_n + \sum_{j=1}^n (c_j E_{n+1-j} + b_j E_{n-j})}{1 - c_0} \right| \leq \mathcal{E}_n. \quad (3.36)$$

Eventually, we have the following estimate for  $E_{n+1}$  using (3.29), (3.35) and (3.36)

$$E_{n+1} \leq \mathcal{E}_n + 2\mathcal{C}\Delta t^2.$$

This implies that  $\mathcal{E}_{n+1} \leq \mathcal{E}_n + 2\mathcal{C}\Delta t^2$ , which gives, knowing that  $\mathcal{E}_0 = 0$ ,

$$\mathcal{E}_{n+1} \leq 2\mathcal{C}(n+1)\Delta t^2.$$

Denoting  $T$  the final time of the algorithm and  $N$  such that  $\Delta t = \frac{T}{N}$ , we have for all  $0 \leq n \leq N-1$ ,  $n+1 \leq T/\Delta t$ , and this ends the proof of the uniform accuracy.  $\square$

**REMARK 3.** *The previous scheme of Prop. 3.3 can be modified into a numerical scheme which degenerates into a second order time approximation of the asymptotic model. Indeed, we can modify (3.23) to get the following scheme*

$$\begin{aligned}
\hat{\rho}^{n+1}(k) &= A_0(t_{n+1}, k) + b_0 \hat{\rho}^n(k) + \sum_{j=1}^n (c_j \hat{\rho}^{n+1-j}(k) + b_j \hat{\rho}^{n-j}(k)) \\
&+ (c_0 + b_0 - 1) \frac{\hat{\rho}^{n+1}(k) + \hat{\rho}^n(k)}{2} + (1 - b_0) \hat{\rho}(t_{n+1}, k).
\end{aligned}$$

*By construction, this scheme degenerates when  $\varepsilon$  goes to zero, into a Crank-Nicolson numerical scheme for the diffusion or anomalous diffusion equations.*

**REMARK 4.** *The scheme of Prop. 3.3 is an expression closed on  $\rho$ . It implies that it is not necessary to solve the equation for  $f$  to compute the approximated value of  $\rho$  at each time step. Since  $\rho$  does not depend of the velocities, this improves the*

computational time. However, it is still possible to recover the values of the distribution function  $f$  from  $\rho$ . As an example, this can be done using the expression for  $\hat{f}$  of the implicit scheme we described in the beginning of this section.

Note that a very similar approach, based on an integration of (1.2) between the times  $t_n$  and  $t_{n+1}$  provides a one-step first order scheme which enjoys AP property and that is of order 1 uniformly in  $\varepsilon$ . With this scheme, the storage of all the previous values of the density  $\hat{\rho}$  is no longer necessary, but this gain of computational time is counterbalanced by the fact that it is now necessary to solve the equation for  $\hat{f}$ . Hence, the complexity coming from the number of terms in the sum of the expression of  $\hat{\rho}$  disappears but we must now deal with the velocity space and perform numerical velocities integrations at each time step.

**4. Numerical results.** In this part, we present the numerical tests for the implicit scheme, the micro-macro scheme and the scheme based on the Duhamel formulation of the kinetic equation in the case of anomalous diffusion. In the sequel, we will denote by ISD (resp. ISA) the implicit schemes set in (3.5)-(3.8) (resp. in Prop. 3.1) in the diffusion case (resp. anomalous diffusion case); we will call MMSA the micro-macro scheme in Prop. 3.2 in the anomalous diffusion case. Eventually, the acronym DSA will refer to the scheme based on Duhamel Formulation in the Prop. 3.3. Finally, adiff denotes the numerical scheme of the anomalous diffusion equation given in (3.9).

In this part, we will consider  $t > 0, x \in [-1, 1], v \in \mathbb{R}$  and the following initial data

$$f_0(x, v) = (1 + \sin(\pi x))M_\beta(v),$$

and periodic boundary conditions are imposed in  $x$ . In the fractional diffusion case, we will consider the equilibrium

$$M_\beta(v) = \frac{m}{1 + |v|^\beta},$$

with  $m$  chosen such that  $\langle M_\beta(v) \rangle_{N_v} = 1$ . In the sequel, unless other discretizations are mentioned, the following numerical parameters will hold:

- (i) time: the final time will be set  $T = 0.1$  and  $\Delta t = 10^{-3}$ .
- (ii) space: we will consider a uniform mesh in  $x$  considering  $N_x = 64$  points. We will use (3.2) for the Fourier variable and (3.1) for the original space variable.
- (iii) velocity: an uniform velocity grid of the truncated domain  $[-v_{\max}, v_{\max}]$  is considered, with  $2N_v + 1$  points. Here, we will consider  $v_{\max} = 50$  and  $N_v = 100$ . The discretization (3.3) ensures that  $\langle v M_\beta(v) \rangle_{N_v} = 0$ , with  $\langle \cdot \rangle_{N_v}$  defined in (3.4). In the tests for the anomalous diffusion, we will consider  $\alpha = 1.5$ .

As we are testing three different schemes, it is worth checking whether, at least for large  $\varepsilon$ , they all agree to give close numerical solutions of the kinetic equation. Indeed, in the case of fractional diffusion, we plot the densities given by the three schemes at time  $T = 0.1$  for  $\varepsilon = 1$  and  $\varepsilon = 0.1$  with the previous parameters. The results in Fig. 4.1 show that the three scheme give the same solution.

To get more precise results, in the sequel, we will be interested in the relative error. The relative error is defined by the difference between a reference solution (obtained by a scheme using a refined mesh) and the solution obtained by the tested scheme through the following formula

$$\text{Error}(T) = \frac{\|\rho_{\text{reference}}(T) - \rho_{\text{scheme}}(T)\|_2}{\|\rho_{\text{reference}}(T)\|_2}, \quad (4.1)$$

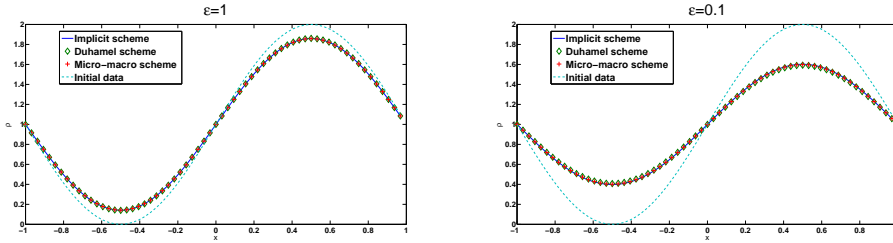


FIG. 4.1. For  $\Delta t = 10^{-3}$ , the densities  $\rho(t = 0.1, x)$  obtained with the three schemes. Left: for  $\varepsilon = 1$ . Right: for  $\varepsilon = 0.1$ .

where  $\|\cdot\|_2$  denotes the discrete  $L^2$  norm and the reference scheme will be precised in each case.

#### 4.1. The implicit scheme in the case of the anomalous diffusion limit.

In this section, we test the ISA scheme. We check that it is of order 1 for a fixed value of  $\varepsilon$  and the AP property when  $\varepsilon$  goes to 0 for a fixed value of  $\Delta t$ .

Firstly, we remark that the scheme (3.5)-(3.8) does not preserve the asymptotic of anomalous diffusion: the left hand side of Fig. 4.2 shows that for  $\varepsilon = 10^{-6}$  the implicit scheme returns a result very different from the result given by (3.9). Indeed, when  $\varepsilon \ll 1$ , (3.5)-(3.8) produces a stationary solution. So it appears that the change of variables in the velocity integrals is necessary to capture the anomalous diffusion limit, as shown in the right hand side of Fig. 4.2.

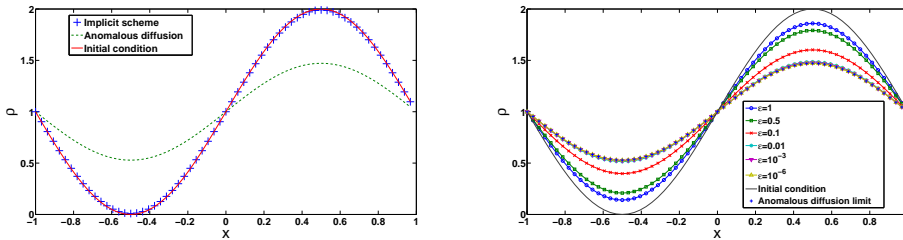


FIG. 4.2. For  $\Delta t = 10^{-3}$ , the densities  $\rho(t = 0.1, x)$ . Left: scheme defined in (3.5)-(3.8) (with  $\varepsilon = 10^{-6}$ ), by the adiff scheme and the initial data. Right: ISA scheme for different values of  $\varepsilon$  and the adiff scheme.

The left hand side of Fig. 4.3 shows the error between the reference scheme (defined as the adiff scheme) and the ISA scheme as a function of  $\varepsilon$ , for  $\alpha = 0.8, 1, 1.5$ . We observe that the convergence of the kinetic equation to the anomalous diffusion when  $\varepsilon$  goes to zero arises with a speed  $\alpha$ . Theoretically, the convergence to the anomalous diffusion equation arises with a speed  $\varepsilon^{\min(\alpha, 2-\alpha)}$ , but as we consider a finite domain for  $v$  in the tests, the convergence rate always appears to be  $\varepsilon^\alpha$ . This result will be proved in [8].

However the convergence in time of the ISA scheme to the DSA scheme is not uniform with respect to  $\varepsilon$ . Setting  $\Delta t = \varepsilon^\alpha$  in the scheme and letting  $\Delta t$  go to 0 shows that the densities does not converge to the density given by the anomalous diffusion equation. On the right hand side of Fig. 4.3, we plot the error in time between the adiff scheme as reference and the ISA scheme computed for different time steps



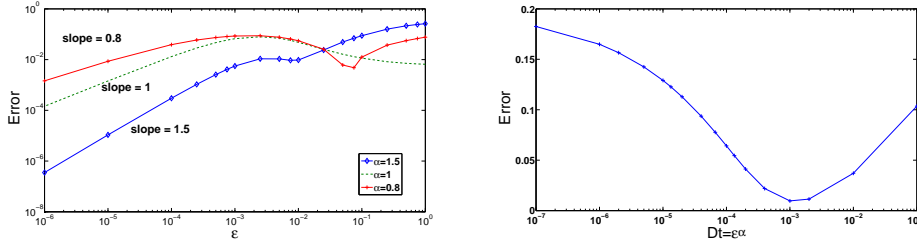


FIG. 4.3. *Left: For  $\Delta t = 10^{-3}$  and  $\alpha = 0.8, 1, 1.5$ , the error as a function of  $\epsilon$  between the reference adiff scheme and the ISA scheme (log scale). Right: The error between the ISA scheme for a range of  $\Delta t = \epsilon^\alpha$  and the adiff scheme as a function of  $\Delta t$ , which is fixed equal to  $\epsilon^\alpha$  (log scale).*

satisfying  $\Delta t = \epsilon^\alpha$ . We observe that the error does not converge to 0 when  $\Delta t$  goes to 0, illustrating the lack of uniformity.

**4.2. The micro-macro scheme.** In this part, we focus on the MMSA scheme in the case of the kinetic equation with anomalous diffusion limit. As it does not use only the Fourier variable, we must consider a grid for the space variable. Here we will take  $x \in [-1, 1]$  discretized with  $N_x = 32$  points. Following Remark 1, we could have used (1.9) to compute the term giving the fractional diffusion for small  $\epsilon$  in the macro part of (3.17) in the original space variable. For a sake of simplicity, we decided to treat it in Fourier variable, using (3.2). All the other space derivatives in this scheme are treated with a classical order 1 upwind scheme, using (3.1). Eventually, we use (3.3) for the velocities. As we solve the transport of the micro part in an explicit way, a CFL condition has to be imposed; then the time step is chosen small enough.

We start by testing the consistency of the scheme. We fix  $\alpha = 1.5$  and for a decreasing range of time steps, we compute the solution given by the MMSA scheme. Then, for  $\epsilon = 1, 0.5$ , we compare it to the reference solution given by MMSA with  $\Delta t = 10^{-6}$ . For  $\epsilon = 10^{-7}, 10^{-8}$ , we compare to the solution given by the adiff scheme. Fig. 4.4 shows the error curves obtained for different values of  $\epsilon$ . It appears that the MMSA scheme is of order 1 in time.

Let us recall that the micro-macro scheme we propose enjoys the AP property but has no uniform order 1 of accuracy. In order to show that the scheme preserves the asymptotic of anomalous diffusion we compute the densities  $\rho(t = 0.1, x)$  obtained by the MMSA scheme for a range of  $\epsilon$ ; we compare them to the density obtained by the adiff scheme computed with the same  $\Delta t$ . In order to respect the CFL condition, we took  $\Delta t = 10^{-4}$  for  $\epsilon = 1, 0.5, 0.1, 0.01$  and  $\Delta t = 10^{-3}$  for the smallest  $\epsilon$ . The left hand side of Fig. 4.5 shows these densities in the case  $\alpha = 1.5$ , we observe that they are very close to the anomalous diffusion limit for small  $\epsilon$ . On the right hand side of Fig. 4.5, the errors associated to this latter study are displayed in the cases  $\alpha = 0.8, \alpha = 1$  and  $\alpha = 1.5$ , the adiff scheme being considered as reference. We observe that the convergence happens with speed  $\alpha$ , as expected.

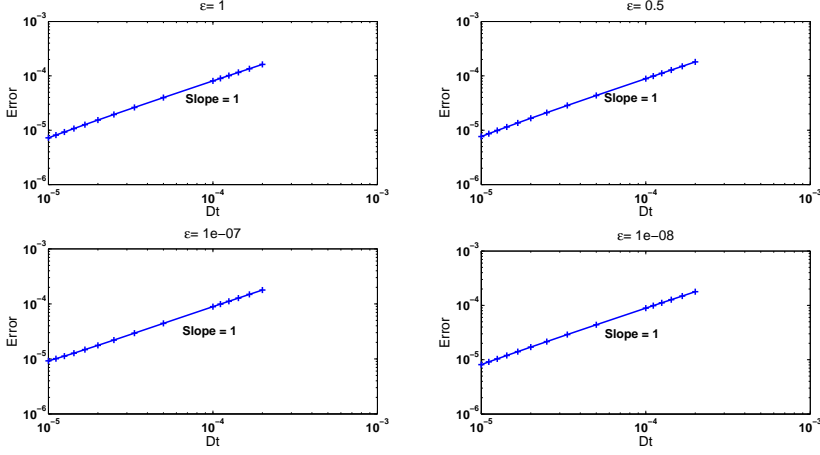


FIG. 4.4. For  $\varepsilon = 1, 0.5, 10^{-7}, 10^{-8}$ , the error in time between the MMSA scheme computed for a range of  $\Delta t$  and the reference scheme computed with  $\Delta t = 10^{-6}$  (log scale).

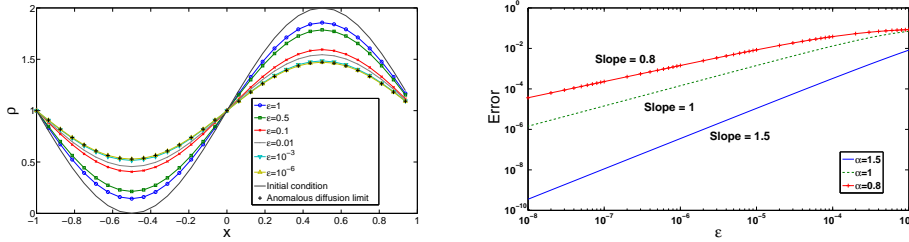


FIG. 4.5. The densities  $\rho(t = 0.1, x)$ . Left: MMSA scheme for different values of  $\varepsilon$  and the diff scheme. Right: For  $\Delta t = 10^{-3}$  and  $\alpha = 0.8, 1, 1.5$ , the error with respect to  $\varepsilon$  between the reference adiff scheme and the MMSA scheme (log scale).

**4.3. The integral formulation based scheme.** In this section, we test the Duhamel formulation based scheme DSA in anomalous diffusion regime. We put its properties of consistency, AP character and uniformity with respect to  $\varepsilon$ , written in Prop. 3.3, in evidence in this case.

The first thing we check is the convergence of the algorithm when  $\Delta t$  goes to zero. We compare the results given by the DSA scheme to the results given by the DSA scheme computed with a smaller time step, to make the consistency of the scheme appear. Considering different values of  $\varepsilon$ , we compute a reference solution with the scheme DSA with  $\Delta t = 10^{-5}$  and we compare it to the results given by the DSA scheme for some larger time steps. Fig. 4.6 show error study of the convergence of the densities obtained. We observe that for  $\varepsilon = 1, 0.5$  the scheme seems to be of order 2 whereas it is of order 1 for smaller values of  $\varepsilon$ .

Then, we can study the AP character of the DSA scheme. The time step being fixed to  $\Delta t = 10^{-3}$ , we check that the results given by the DSA scheme converge to the result given by the adiff scheme when  $\varepsilon$  goes to 0. Fig. 4.7 presents the densities  $\rho(t = 0.1, x)$  obtained by these two schemes for different values of  $\varepsilon$  and then, the error as a function of  $\varepsilon$  is plotted for  $\alpha = 0.8, 1, 1.5$ . As previously, the expected numerical speed of the convergence to the anomalous diffusion is recovered.

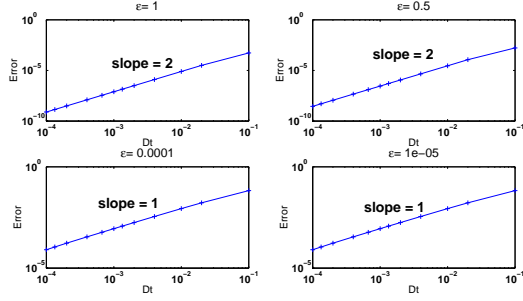


FIG. 4.6. For  $\varepsilon = 1, 0.1, 10^{-4}, 10^{-5}$ , the error as a function of  $\Delta t$  between the DSA scheme and the DSA scheme computed for  $\Delta t = 10^{-5}$  (log scale).

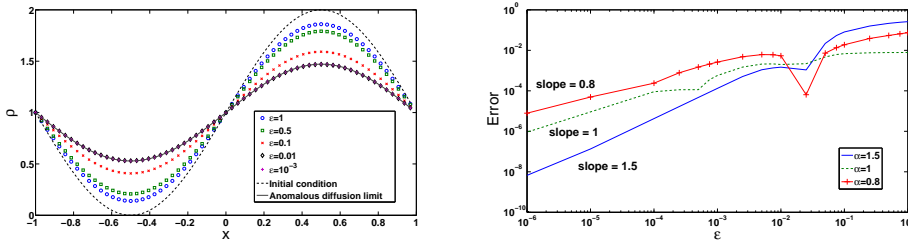


FIG. 4.7. Left: For  $\Delta t = 10^{-3}$  the densities  $\rho(t = 0.1, x)$  given by the DSA scheme for different values of  $\varepsilon$  and the adiff scheme. Right: For  $\Delta t = 10^{-3}$  and  $\alpha = 0.8, 1, 1.5$ , the error in  $\varepsilon$  between the reference adiff scheme and the DSA scheme (log scale).

To highlight the fact that the scheme is first order in time uniformly in  $\varepsilon$ , we compare the results given by the DSA scheme with  $\Delta t = 10^{-5}$  to the results given by the same DSA scheme for a range of  $\Delta t$ . These errors are displayed in the left hand side of Fig. 4.8 as a function of  $\varepsilon$  where we observe that the error curves are stratified with respect to  $\Delta t$ , showing the uniformity of the scheme with respect to  $\varepsilon$ . As for the case of diffusion limit, the right hand side of Fig. 4.8 presents the error between the DSA scheme computed for  $\Delta t = \varepsilon^\alpha$  and the limit adiff scheme computed for  $\Delta t = 10^{-7}$ . Since the DSA scheme is of order 1 uniformly in  $\varepsilon$ , this error tends to zero when  $\Delta t$  and  $\varepsilon$  go to zero.

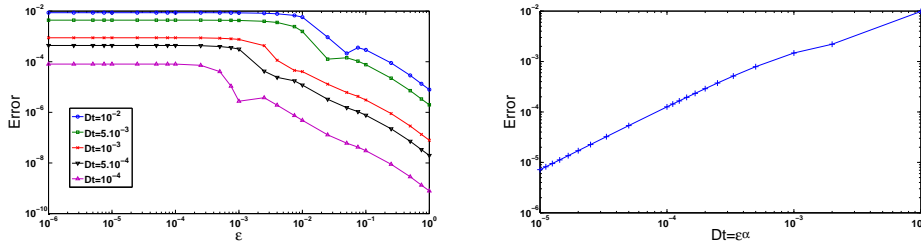


FIG. 4.8. Left: The difference between the DSA scheme computed for  $\Delta t = 10^{-5}$  and the DSA scheme computed for a range of  $\Delta t$  as a function of  $\varepsilon$  (log scale). Right: The difference between the DSA scheme computed for  $\Delta t = \varepsilon^\alpha$  and the DSA scheme computed for  $\Delta t = 10^{-7}$  as a function of  $\Delta t$  (log scale).

**5. Conclusion.** In this paper, we have first presented a formal derivation of anomalous diffusion asymptotics from a BGK kinetic equation. This analysis enables us to understand the role of the large velocities induced by the heavy-tailed equilibrium in the anomalous diffusion asymptotics. Moreover, this formal derivation paves the way to construct three different numerical schemes for the kinetic equation, which all enjoy the asymptotic preserving property in the anomalous diffusion regimes.

The asymptotic preserving character in the case of anomalous diffusion comes from the fact that we managed to take into account the effect of large velocities of the equilibrium. As we saw in the simplest case of a direct fully implicit scheme, the asymptotic is not preserved when these large velocities are truncated. The schemes using the micro-macro formulation as well as the Duhamel formulation of the equations require the use of the same trick on the velocities. Moreover, this last scheme enjoys a uniform accuracy property.

In the near future, we aim at extending this work to more general context, considering higher dimensions, non periodic boundary conditions. The case of singular collision frequency also may generate anomalous diffusion (see [1]) and this also deserves a numerical study that we plan to do in a forthcoming work [8].

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