Grandes Transformations en Mécanique des Matériaux

1: Fluide vs Solide: mémoire et/ou symètrie.

François SIDOROFF
Université de Lyon / Ecole Centrale de Lyon
Laboratoire de Tribologie et Dynamique des Systèmes
http://sitasido.ec-lyon.fr & francois.sidoroff@free.fr

Introduction / Rappels

- Contraintes et Déformations : baliser la jungle!
- Pour le fun: Le Cercloïde de Mohr.

Equations constitutives

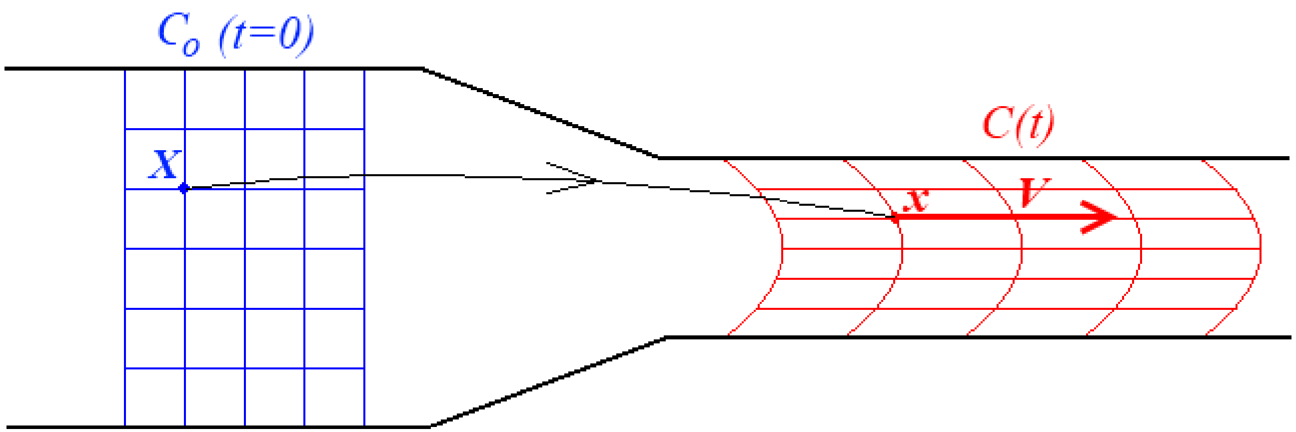
- Les deux mémoires: l'inné et l'acquis
- Symètries matérielles: fluide, solide isotrope ou anisotrope

Les grands classiques

- Hyperélasticité
- Fluides visqueux, viscoplastiques et plastiques
- Pour mémoire, quelques autres

Conclusion : plasticité => isotropie (Aristoteles dixit)

Lagrange – Euler : un faux départ



D. Lagrangienne

$x = x(X, t)$

<===>

D. Eulérienne

$V = V(x, t)$

M. Solides HPP

$$x = X + u(X,t)$$

$$u \ll l \quad \nabla u \ll 1 \quad \Rightarrow \quad x \sim X$$

$\varepsilon = (\nabla u)^S$ loi de Hooke

$$\sigma = \lambda\,(tr\,\varepsilon)\,\mathbf{1} + 2\,\mu\,\varepsilon$$

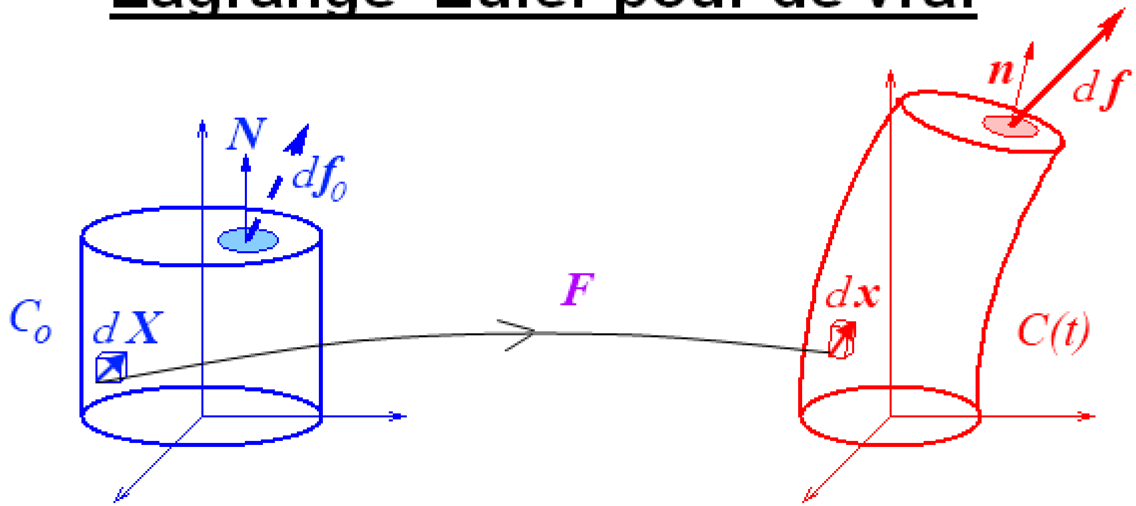
M. des Fluides

$$\sigma = -p\,\mathbf{1} + 2\,\mu\,D \qquad D = (\nabla V)^S$$

$\Rightarrow$ Navier-Stokes

$$\rho\left(\frac{\partial V}{\partial t} + V.\nabla V\right) = -\nabla p + \mu\,\Delta V$$

Lagrange–Euler pour de vrai



Déformations locales $\quad d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X} \qquad dv = J\, dv_0 \qquad J = \det \boldsymbol{F}$

Contraintes

$$d\boldsymbol{f} = \boldsymbol{T}\, \boldsymbol{n}\, dS$$ Cauchy *contraintes vraies*

$$= \boldsymbol{\Pi}\, \boldsymbol{N}\, dS_0$$ PK1 *contraintes nominales mais non symètrique*

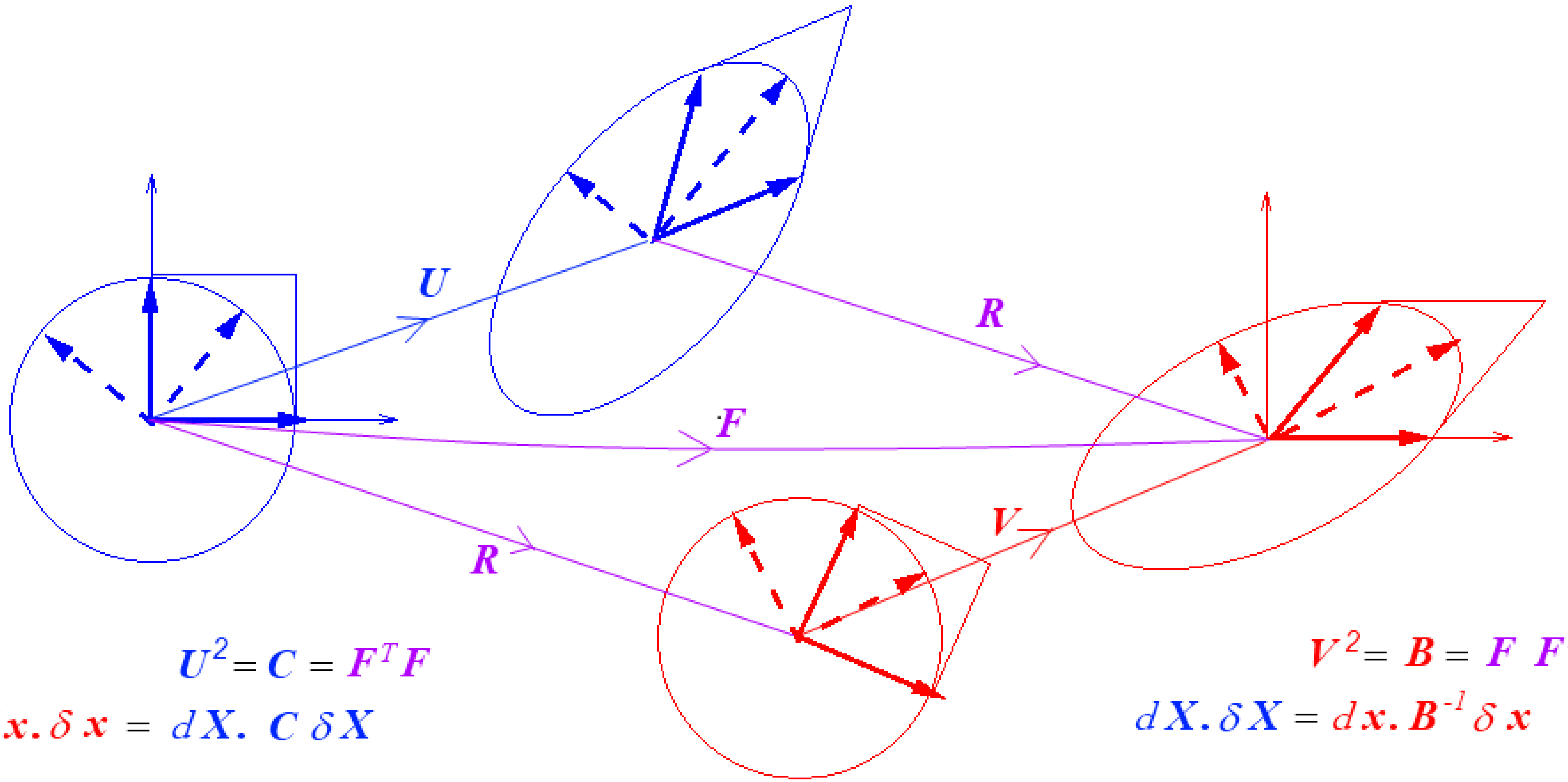
$$d\boldsymbol{f}_0 = \boldsymbol{F}^{-1}\, d\boldsymbol{f} = \boldsymbol{S}\, \boldsymbol{N}\, dS_0$$ PK2 *???*

Kirchhoff $\quad \boldsymbol{\tau} = J\, \boldsymbol{T} = \boldsymbol{\Pi}\, \boldsymbol{F}^T = \boldsymbol{F}\, \boldsymbol{S}\, \boldsymbol{F}^T \qquad \boldsymbol{\sigma}$ en HPP

$$p_{int} = \boldsymbol{\tau} : \boldsymbol{D} = \boldsymbol{\Pi} : \dot{\boldsymbol{F}} = \boldsymbol{S} : \dot{\boldsymbol{C}}/2$$

$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$ en HPP

Déformations - Décomposition polaire



$$U^2 = C = F^T F$$

$$d\boldsymbol{x}.\delta \boldsymbol{x} = d\boldsymbol{X}.\ \boldsymbol{C}\, \delta \boldsymbol{X}$$

$$V^2 = B = F\ F^T$$

$$d\boldsymbol{X}.\delta \boldsymbol{X} = d\boldsymbol{x}.\ \boldsymbol{B}^{-1}\, \delta \boldsymbol{x}$$

=> déformations
Green-lagrange
etc....

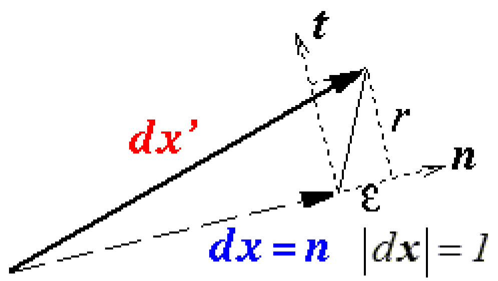
$$F = R\ U = V\ R$$

=> déformations
Euler-Almansi
etc....
Hencky $\quad h = ln(V)$

$$B = R\ C\ R^T \qquad V = R\ U\ R^T$$

taux de déformations : $\quad (d\boldsymbol{x}.\delta \boldsymbol{x})^{\bullet} = d\boldsymbol{X}.\ \dot{\boldsymbol{C}}\, \delta \boldsymbol{X} = d\boldsymbol{x}.\ 2\boldsymbol{D}\, \delta \boldsymbol{x}$

Cercle de Mohr des transformations

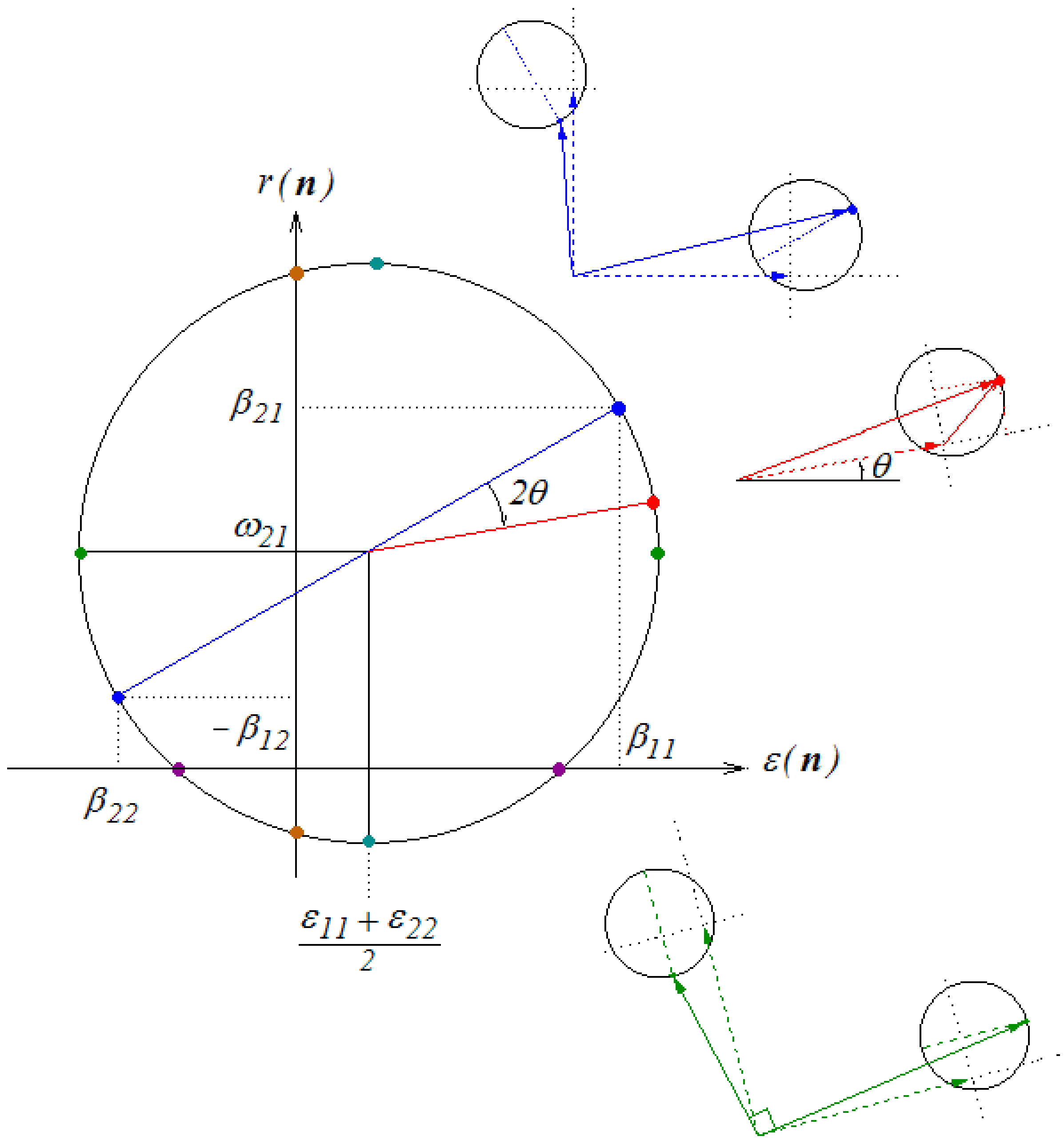


$$dx' = (1 + \beta)\,dx$$

$$\beta = \nabla\,u = \varepsilon + \omega$$

$$\varepsilon(n) = \varepsilon_{ij}\,n_i\,n_j = \beta_{ij}\,n_i\,n_j$$

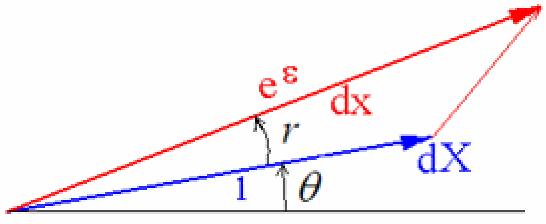
$$r(n) \sim \sin r = \frac{dx'.t}{|dx'|} = \beta_{ij}\,n_i\,t_j$$



Cercloïde de Mohr

$$\varepsilon(\boldsymbol{n}) = ln(d\boldsymbol{x}/d\boldsymbol{X})$$

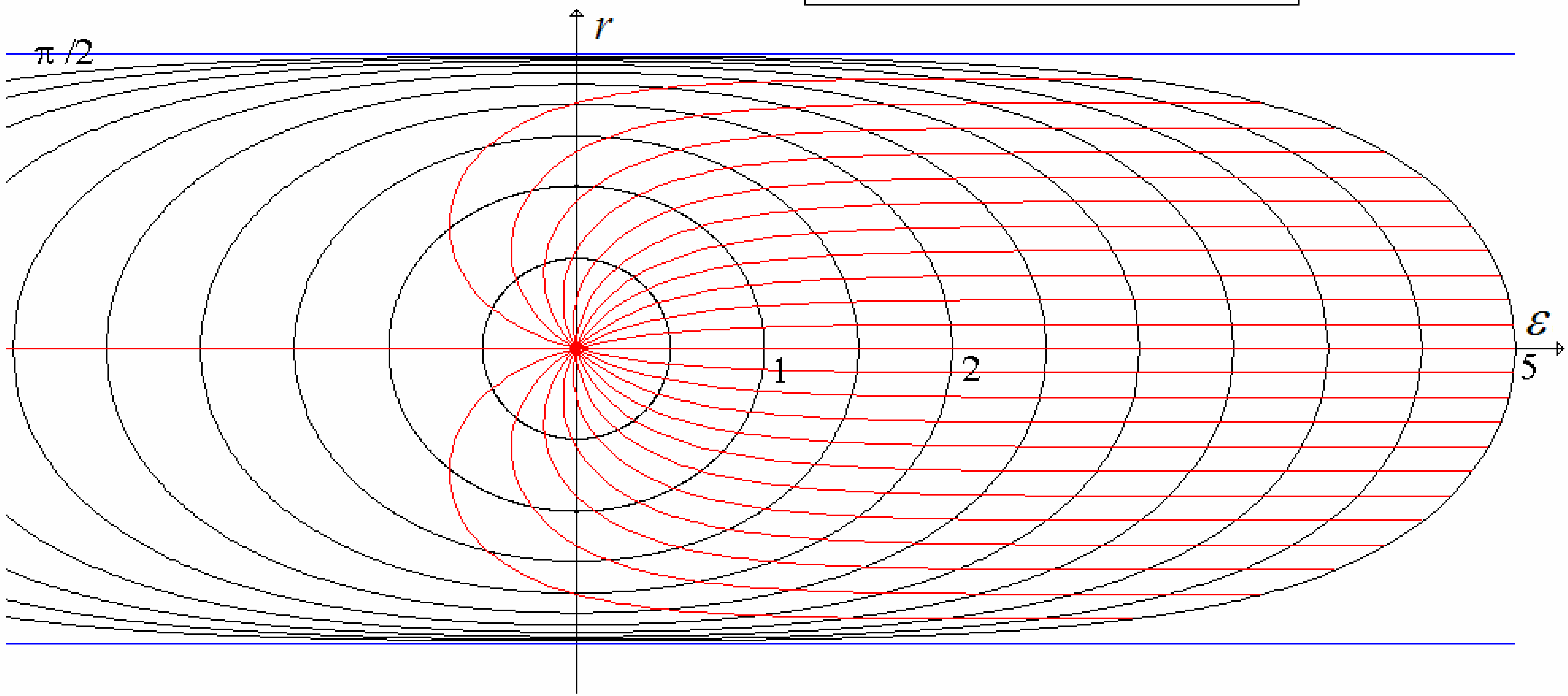
$$r(\boldsymbol{n}) = (d\boldsymbol{x}, d\boldsymbol{X})$$



$$\boldsymbol{F} = \boldsymbol{RU} = rot(\psi)\, rot(\theta_1) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} rot(-\theta_1)$$

$$\lambda_1 = e^{a+d} \qquad \lambda_2 = e^{a-d}$$

$$\boxed{r = \psi \pm Arc\cos \frac{\cosh(\varepsilon - a)}{\cosh(d)}}$$



Equation constitutive

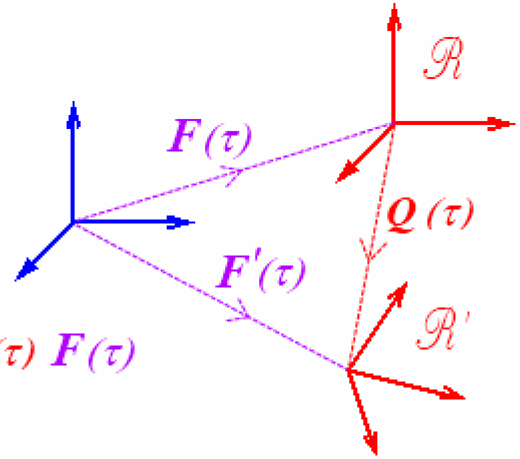
$$\boldsymbol{T}(t) = \underset{\tau \le t}{\mathfrak{F}}\{\boldsymbol{F}(\tau)\}$$

mémoire

Indifférence matérielle

$$\mathfrak{F}\{\boldsymbol{Q}(\tau)\boldsymbol{F}(\tau)\} = \boldsymbol{Q}(t)\ \mathfrak{F}\{\boldsymbol{F}(\tau)\}\ \boldsymbol{Q}^T(t)$$

$$\boxed{\boldsymbol{S}(t) = \mathfrak{S}\{\boldsymbol{C}(\tau)\}}$$

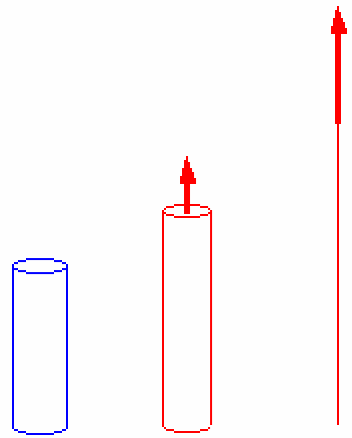


$$\boldsymbol{F}'(\tau) = \boldsymbol{Q}(\tau)\ \boldsymbol{F}(\tau)$$

exemple $\quad \boldsymbol{S} = \lambda\,(tr\ \boldsymbol{E})\boldsymbol{1} + 2\,\mu\,\boldsymbol{E}$

$(2\,\boldsymbol{E} = \boldsymbol{C} - \boldsymbol{1})$

pseudo-Hooke
=> gag !



$$\lambda = \sqrt{\frac{1+\nu}{\nu}}$$

Incompressibilité (liaison interne)

$$\boxed{\boldsymbol{T}(t) = -p\,\boldsymbol{1} + \underset{\tau \le t}{\mathfrak{F}}\{\boldsymbol{F}(\tau)\} \qquad (\det \boldsymbol{F}(\tau) = 1)}$$

exemple $\quad \boldsymbol{S} = -\,p\,\boldsymbol{C}^{-1} + \mu\,\boldsymbol{C}^{-1}\dot{\boldsymbol{C}}\,\boldsymbol{C}^{-1}$ $\quad$? ? ?

Les deux mémoires

mémoire du passé acquis

mémoire absolue inné

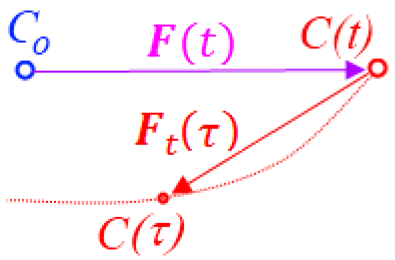
$$\boldsymbol{T}(t) = \underset{\tau \le t}{\mathfrak{F}}\{\boldsymbol{F}_t(\tau), \boldsymbol{F}(t)\}$$

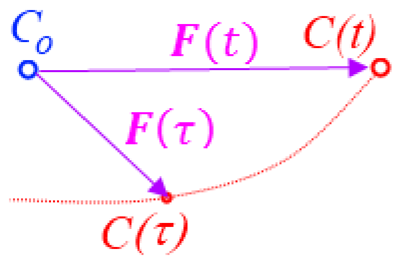
mémoire

$$\boldsymbol{T}(t) = \underset{\tau \le t}{\mathfrak{F}}\{\boldsymbol{F}(\tau)\}$$

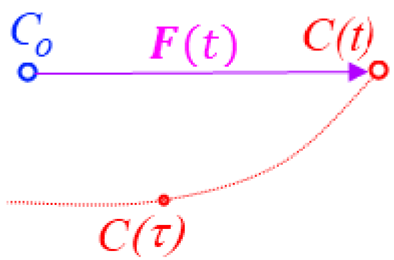
déformation relative

$$\boldsymbol{F}(\tau) = \boldsymbol{F}_t(\tau)\, \boldsymbol{F}(t)$$

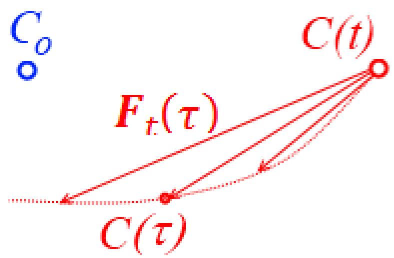




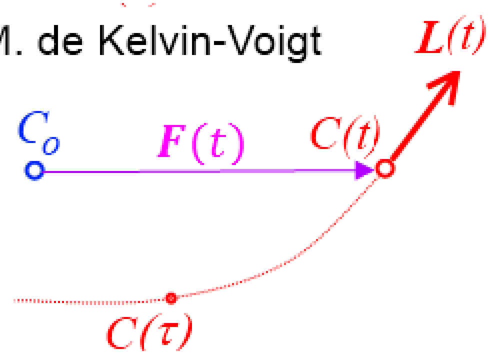
M. élastique



Fluide simple

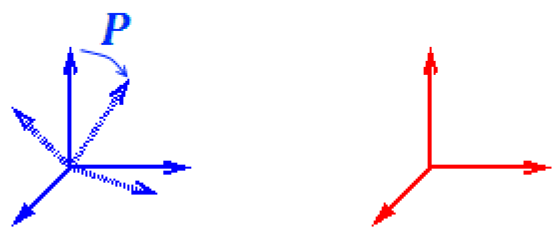


M. de Kelvin-Voigt



Isotropie

Matériau isotrope



$$\mathfrak{F}\{F(\tau)\ P^T\} = \mathfrak{F}\{F(\tau)\}$$ pour toute rotation P

$$F^*(\tau) = F(\tau)\ P^T$$

$$\mathfrak{S}\{P\ C(\tau)\ P^T\} = P\ \mathfrak{S}\{C(\tau)\}\ P^T$$

$$\boxed{T = \mathfrak{F}\{B(t); C_t(\tau)\}}$$

fonctionnelles isotropes

mémoire { inné ; acquis}

- Plus généralement

$$\mathcal{G} = \left\{ P \quad , \ \mathfrak{F}\{F(\tau)\ P^T\} = \mathfrak{F}\{F(\tau)\} \quad \right\}$$

Groupe d'isotropie

$\Longrightarrow$ Caractérisation des symétries matérielles

Symétries matérielles

$\mathcal{G} = \mathcal{O}$ groupe orthogonal $(\boldsymbol{P}\,\boldsymbol{P}^T = \mathbf{1})$

$\rightarrow$ matériau isotrope

$\mathcal{G} \subset \mathcal{O}$ sous-groupe de $\mathcal{O}$

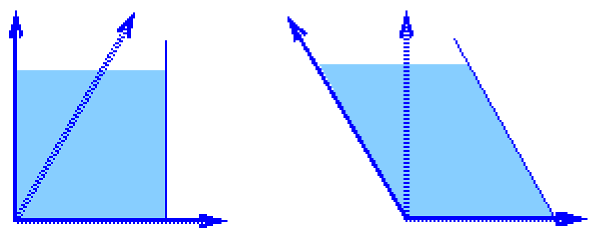
$\rightarrow$ matériau anisotrope (orthotropie, par exemple)

$\mathcal{G} = \mathcal{U}$ groupe unimodulaire $det\,\boldsymbol{P} = 1$

$\rightarrow$ fluide

$$\boldsymbol{T} = \mathfrak{F}\{J(t); \boldsymbol{C}_t(\tau)\}$$

fluide incompressible $\boldsymbol{T} = -p\,\mathbf{1} + \mathfrak{F}\{\boldsymbol{C}_t(\tau)\}$



<u>Théorème</u>. Aucun groupe entre $\mathcal{O}$ et $\mathcal{U}$

$\mathcal{G}$ sous-groupe de $\mathcal{U}$, mais $\mathcal{G} \not\subset \mathcal{O}$

$\rightarrow$ sous-fluide (une curiosité)

Hyper élasticité

$$\boldsymbol{T}:\boldsymbol{D} = \rho\,\dot{u} + \cancel{\Phi} \qquad \frac{1}{2}\boldsymbol{S}:\dot{\boldsymbol{C}} = \boldsymbol{\tau}:\boldsymbol{D} = -\dot{w} + \cancel{\Phi_o} \qquad w = \rho_o u$$

en général $\quad w = w(\boldsymbol{C}) \quad \boxed{\boldsymbol{S} = 2\,\frac{\partial\, w}{\partial\, \boldsymbol{C}}}$

si isotrope $\quad w(\boldsymbol{C}) = w(\boldsymbol{B}) = w(B_I, B_{II}, B_{III}) \quad \boxed{\boldsymbol{T} = \frac{2}{J}\,\boldsymbol{B}\,\frac{\partial w}{\partial \boldsymbol{B}}}$

hyperélasticité compressible

$$\boldsymbol{B} = \underbrace{J^{2/3}}_{\text{dilatation}}\underbrace{\overline{\boldsymbol{B}}}_{\text{distorsion}} \qquad (det\ \overline{\boldsymbol{B}} = 1)$$

$$\boldsymbol{T} = \frac{\partial w}{\partial J}\,\mathbf{1} + \frac{2}{J}\left(\overline{\boldsymbol{B}}\,\frac{\partial w}{\partial \overline{\boldsymbol{B}}}\right)^{D}$$

fluide parfait

$$\boldsymbol{T} = -\,p(J)\,\mathbf{1}$$

$$p = -\frac{\partial w}{\partial J}$$

classique

incompressible

$$\boldsymbol{T} = -p\,\mathbf{1} + 2\,\boldsymbol{B}\,\frac{\partial w}{\partial \boldsymbol{B}}$$

facile

Fluides visqueux (incompressible)

$$\boldsymbol{T} : \boldsymbol{D} = \cancel{\rho \dot{u}} + \Phi \qquad\qquad \boldsymbol{T} = -p\,\mathbf{1} + \frac{\partial\,\Omega}{\partial\,\boldsymbol{D}}$$

Un cas particulier important $\Omega(d) \quad d = |\boldsymbol{D}|$

$$\boldsymbol{T} = -p\,\mathbf{1} + \frac{\Omega'(d)}{d}\boldsymbol{D}$$

$$\boxed{\Omega = \mu\, d^2} \qquad \boldsymbol{T} = -p\,\mathbf{1} + 2\,\mu\,\boldsymbol{D}$$

fluide visqueux newtonien

$$\boxed{\Omega = k\,d + \mu\,d^2}$$

Bingham

$$\boxed{\Omega = \frac{A}{n+1}d^{n+1}}$$

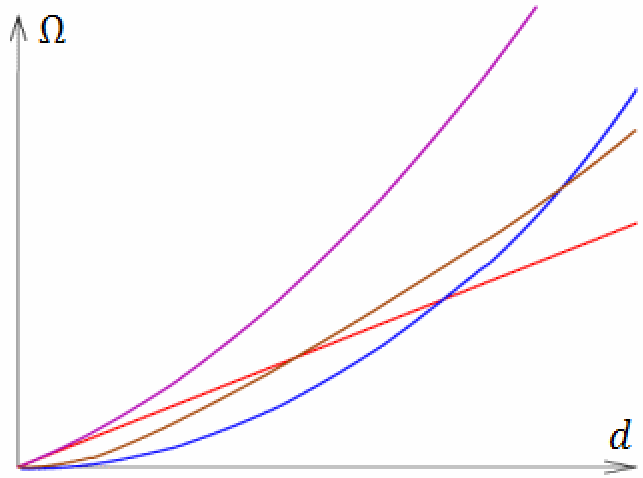
Norton-Hoff

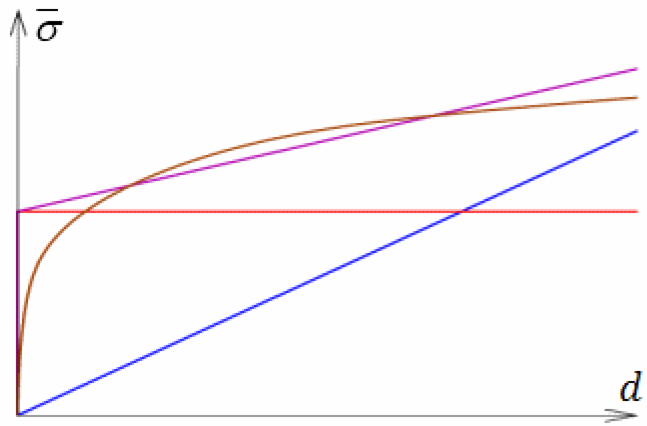
$$\boxed{\Omega = k\,d} \qquad \boldsymbol{T} = -p\,\mathbf{1} + k\,\frac{\boldsymbol{D}}{d}$$

$$|\boldsymbol{T}^D| = k$$

fluide plastique de von Mises

(rigide & parfaitement plastique)





Lois différentielles

$\sigma = \sigma(\varepsilon)$	$\longrightarrow$	$\boldsymbol{T} = t(\boldsymbol{B})$	élasticité
$\sigma = \sigma(\varepsilon, \dot{\varepsilon})$	$\longrightarrow$	$\boldsymbol{T} = t(\boldsymbol{B}, \boldsymbol{D})$	Kelvin-Voigt
$\dot{\sigma} = f(\sigma, \varepsilon, \dot{\varepsilon})$	$\longrightarrow$	$\dot{\boldsymbol{T}} = f(\boldsymbol{T}, \boldsymbol{B}, \boldsymbol{D})$	? NON !

Dérivée objective : transport $\boldsymbol{G}(t)$: $C_o \longrightarrow C(t)$

$\boldsymbol{T}_o \longrightarrow \boldsymbol{G}\,\boldsymbol{T}_o\boldsymbol{G}^T = \boldsymbol{T}$

$$\longrightarrow \quad \frac{D\boldsymbol{T}}{Dt} = \boldsymbol{G}\,(\boldsymbol{G}^{-1}\boldsymbol{T}\,\boldsymbol{G}^{-T})^{\bullet}\,\boldsymbol{G}^T$$

dérivées convectives (Lie, Oldroyd, ...)

$\boldsymbol{G} = \boldsymbol{F}$ $\quad \boldsymbol{T}^C = \dot{\boldsymbol{T}} - \boldsymbol{L}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T$ $\quad$ mais $\quad \boldsymbol{T}^{DC} \neq \boldsymbol{T}^{CD}$

$\boldsymbol{G} = \boldsymbol{F}^{-T}$ $\quad \boldsymbol{T}_C = \dot{\boldsymbol{T}} + \boldsymbol{T}\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{T}$

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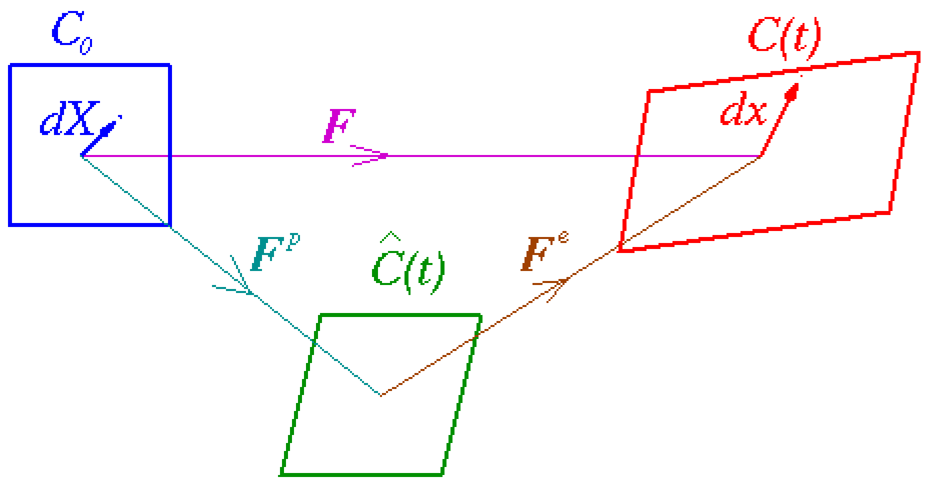
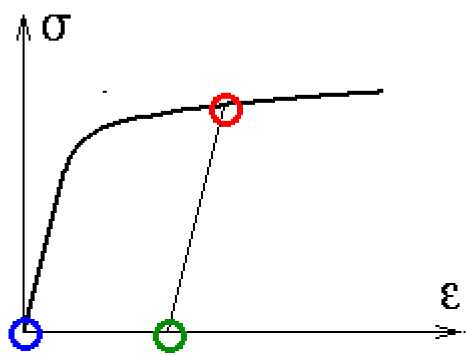
dérivées en rotation $\quad \boldsymbol{G} = \boldsymbol{R}$ $\quad \boldsymbol{T}^R = \boldsymbol{R}\,(\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R})^{\bullet}\,\boldsymbol{R}^T$ $\quad$ mais dépend du choix de C_o

en choisissant $\quad C_o = C(t)$ $\quad \boldsymbol{T}^J = \dot{\boldsymbol{T}} - \boldsymbol{W}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{W}$ $\quad$ Jaumann

mais oscillations (tourne trop vite)

Elasto-plasticité





$$\boldsymbol{F} = \boldsymbol{F}^e\ \boldsymbol{F}^p$$

Modèle élasto-plastique

- élastique par rapport à une configuration "relâchée" $\hat{C}$ $\qquad w = w(\boldsymbol{C}^e)$
- $\hat{C}$ qui évolue par une loi de type plastique. $\qquad \Omega = \Omega(\widehat{\boldsymbol{D}})$

Conclusions (provisoires)

1 Les cadres sont en place

Reste à les remplir

2 a Le comportement plastique est un comportement fluide

b Un fluide est isotrope $\mathcal{O} \subset \mathcal{G}$

donc Plasticité $\Longrightarrow$ Isotropie

et pourtant la plasticité anisotrope existe !