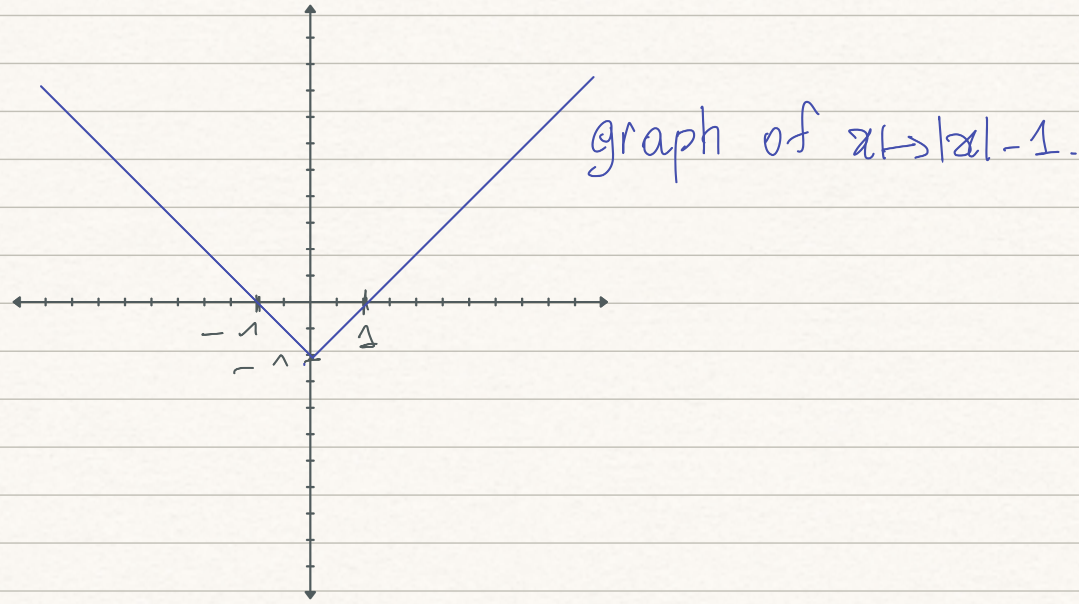
First math test

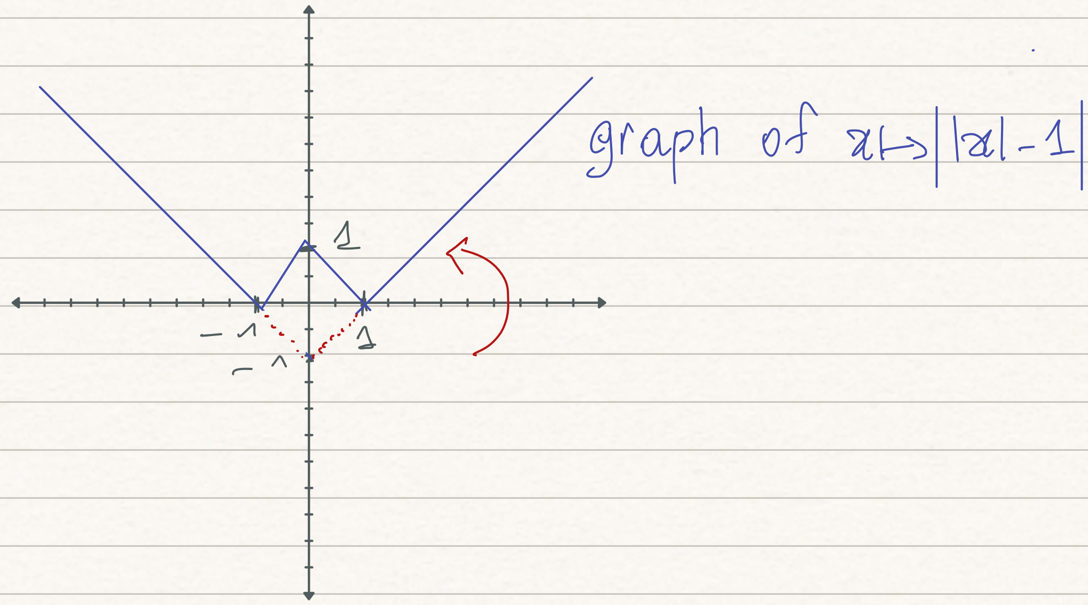
1)

(a)



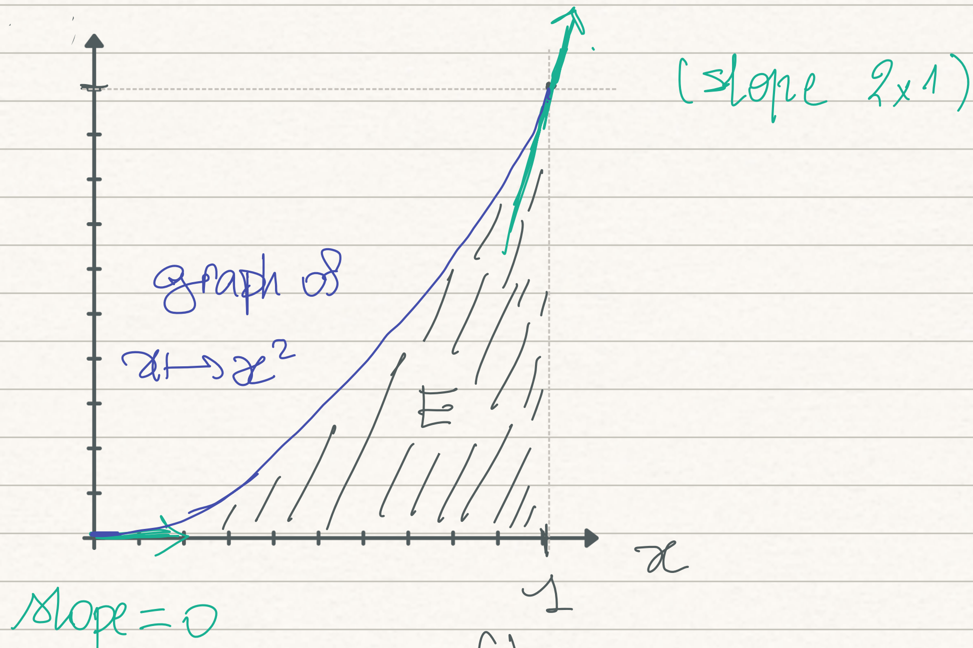
(b)

In order to obtain the absolute value, we change $f(x)$ into $-f(x)$ when $f(x) < 0$ i.e.



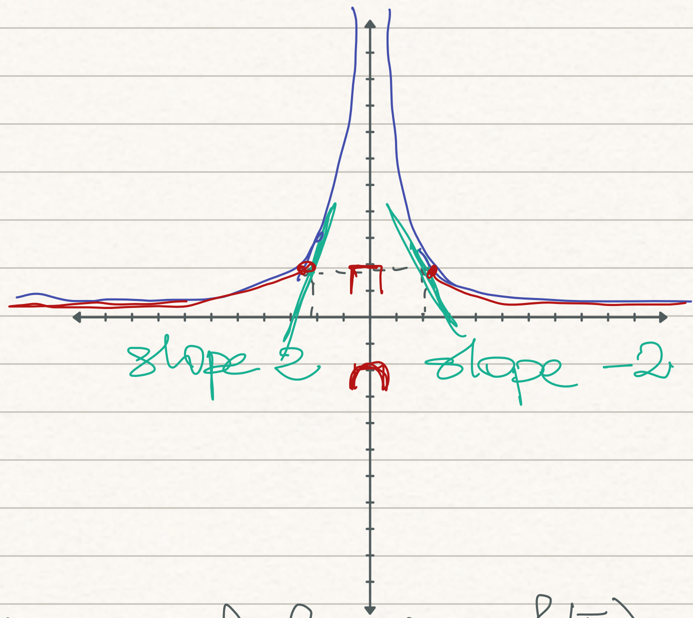
2)

(a)



(b) area of E : $|E| = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1-0}{3} = \frac{1}{3}$

3) a)



b) The image of f is $f(E) = (0, +\infty)$

c) The inverse image of $(-1, 1]$ by f is

$$f^{-1}((-1,1]) = (-\infty, -1] \cup [1, +\infty)$$

d) $y = \frac{1}{x^2}$ iff $x^2 = \frac{1}{y}$ i.e. $x = \pm\sqrt{\frac{1}{y}}$ if $y > 0$

so $f^{-1} : (0, +\infty) \to (0, +\infty)$, $f^{-1}(y) = \sqrt{\frac{1}{y}}$ for $y > 0$.

4] a)

$$u_1 = u_0^2, \quad u_2 = u_1^2 = u_0^4, \quad u_3 = u_2^2 = u_0^8$$

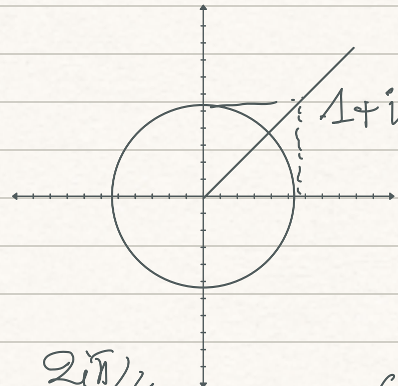
then obviously $u_n = u_0^{2^n}$

b) if $|u_0| > 1$ $u_n \to +\infty$.

if $|u_0| < 1$ $u_n \to 0$

if $u_0 = \pm 1$ $u_n = 1 \ \forall n \geq 1$.

5] a)



$\mathrm{Arg}(1+i) = \frac{\pi}{4}$

b) $z = \sqrt{2}\, e^{2i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$

We recall that $\mathrm{Arg}(zz') = \mathrm{Arg}(z)\ \mathrm{Arg}(z')$

Therefore $z^n = \sqrt{2}^n e^{in\pi/4}$

or $\mathrm{Arg}(z^n) = n\pi/4$. and $|z|^n = \sqrt{2}^n$

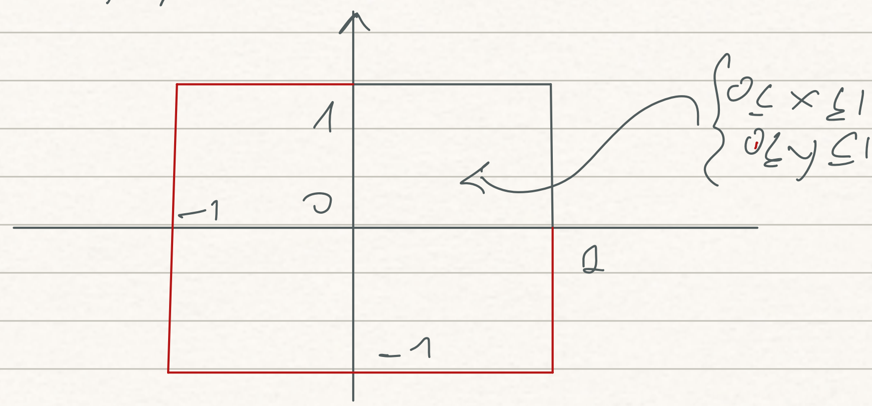
c) $z^n \in \mathbb{R}^+$ when $\mathrm{Arg}(z^n)$ is a multiple of 2π

i.e. for $n = 0, 8, 16, \dots$ i.e. $n = 8k$, $k \in \mathbb{N}$

6] We can make the sketch for $x, y \geq 0$ and then use symmetries over both axis ($x=0$ and $y=0$) ($x \mapsto -x$ and $y \mapsto -y$ do not change the value of $\max(|x|, |y|)$.

Hence, let $x \geq 0$, $y \geq 0$

$\max(x,y) \leq 1$ means $x \leq 1$ and $y \leq 1$



$E = [-1,1] \times [-1,1]$.

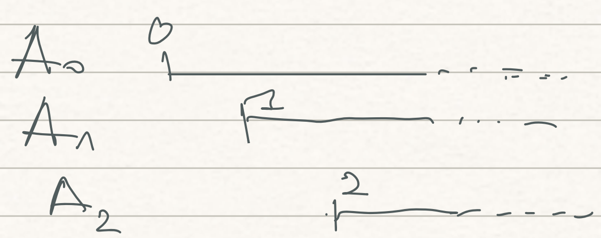
7. Sorry for the notation of this exercise which may have confuse you (but which shouldn't). I replaced the names here and keep $A_0, A_1, A_2, \dots$ as in the lecture

a) $E = \bigcup_{n \geq 0} A_n$.

$$E = [0,0] \cup [0,1] \cup [0,2] \cup \dots \cup [0,n] \cup \dots,$$

$$\Rightarrow \quad E = [0, +\infty)$$

b) $F = [0,+\infty) \cap [1,+\infty) \cap \dots \cap [n,+\infty) \cap \dots$



$\Rightarrow F = \emptyset$

Proofs $\rightarrow$
(if not convinced or curious)

□ $E = \bigcup_n A_n \qquad A_n = [0, n]$

$A_n \subset [0, +\infty) \quad \forall n \geq 0$

$\Rightarrow \bigcup_n A_n \subset [0, +\infty)$ i.e. $E \subset [0, +\infty)$

Conversely, if $x \in [0, +\infty)$ then for $n \geq x$ (large enough) $x \in [0, n] = A_n$

$\Rightarrow x \in \bigcup_n A_n$. ■

□□ $F = \bigcap_n B_n \qquad B_n = [n, +\infty)$

Let $x \in \mathbb{R}$. Then for $n > x$, $x \notin [n, +\infty)$

$\Rightarrow x \notin \bigcap_{n \geq 0} B_n = F$.

Since we found no x in F, $F = \emptyset$.

8. a) $\mathcal{R}$ is an order relation, it is total and it is the alphabetical order.

b) $\mathcal{R}'$ is also an order but it is not total. We cannot compare $(0, 1)$ and $(1, 0)$ for instance.

Proofs for curious $\longrightarrow$

□ $(x_1, x_2) \mathcal{R} (x_1, x_2)$ since $x_1 = x_1$ and $x_2 \leq x_2$

Hence the relation is reflexive

□□ If $(x_1, x_2) \mathcal{R} (y_1, y_2)$ and $(y_1, y_2) \mathcal{R} (x_1, x_2)$ then

$$\text{and} \begin{cases} x_1 < y_1 \text{ or } x_1 = y_1 \text{ and } x_2 \leq y_2 & (1) \\ y_1 < x_1 \text{ or } y_1 = x_1 \text{ and } y_2 \leq x_2 & (2) \end{cases}$$

We cannot have $x_1 < y_1$ in (1) because otherwise (2) wouldn't be possible.

Similarly, we cannot have $y_1 < x_1$ in (2).

Therefore, we have $x_1 = y_1$ an $x_2 \leq y_2$ and $y_2 \leq x_2$ i.e. $x_1 = y_1$ and $x_2 = y_2$: the relation is antisymmetric.

□□□ if $(x_1, x_2) \mathcal{R} (y_1, y_2)$ and $(y_1, y_2) \mathcal{R} (z_1, z_2)$ then $(x_1 < y_1$ or $(x_1 = y_1$ and $x_2 \leq y_2))$ and $(y_1 < z_1$ or $(y_1 = z_1$ and $y_2 \leq z_2))$. Using classical distributivity, this implies either

$$\begin{cases} x_1 < y_1 < z_1 \text{ or } x_1 < y_1 = z_1 \text{ or } x_1 = y_1 < z_1 \\ (\text{i.e. } x_1 < z_1) \end{cases}$$

or

$$\begin{cases} x_1 = y_1 = z_1 \text{ and } x_2 \leq y_2 \leq z_2 \\ (\text{i.e. } x_1 = z_1 \text{ and } x_2 \leq z_2) \end{cases}$$

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