Suite calcul commencé en cours

$$E[L(s(Y),X)] \xrightarrow{\text{Fubini}} E\Big[\underbrace{E[L(s(Y),X)|Y]}_{\phi(y)}\Big]$$

Minimiser la perte moyenne, revient à minimiser $\phi(y)$

$$\phi(y) = \sum_{k=1}^{K} L(\hat{s}(y), k)\, p(X=k|Y)$$

On cherche $\hat{s}(y) = i$ qui minimise $\phi(y)$.

Si i est la solution, alors

$$\hat{s}(y) = i \iff \left[\forall j \in \Omega, \sum_{k=1}^{K} L(i,k)\, p(X=k|Y) \leq \sum_{k=1}^{K} L(j,k)\, p(X=k|Y) \right]$$

$$\iff \left[\forall j \in \Omega, \sum_{\substack{k=1\\ k\neq i}}^{K} \lambda_{ik}\, p(X=k|Y) \leq \sum_{\substack{k=1\\ k\neq j}}^{K} \lambda_{jk}\, p(X=k|Y) \right]$$

et donc $\boxed{\hat{S}_B(y) = \arg\min_j \sum_{\substack{k=1\\ k\neq j}}^{K} \lambda_{jk}\, p(X=k|Y)}$

Si $K=2$ et $L_{01}(i,j) = \begin{cases} 0 & \text{si } i=j \\ 1 & \text{si } i\neq j \end{cases}$

$$\hat{S}_B(y) = \arg\min \Big(\underbrace{\cancel{\lambda_{11}\, p(X=1|Y)}}_{=0} + \underbrace{\lambda_{12}}_{=1}\, p(X=2|Y),$$

$$\underbrace{\lambda_{21}}_{=1}\, p(X=1|Y=y) + \underbrace{\cancel{\lambda_{22}\, p(X=2|Y)}}_{=0} \Big)$$

$$= \arg\min \big(P(X=2|Y), P(X=1|Y) \big)$$

La solution est la classe 1 si $p(X=2|Y) < p(X=1|Y)$

Donc $\boxed{\hat{S}_B(y) = \arg\max_{j\in\{1,2\}} P(X=j|Y)}$