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# INTEGRAL EQUATION INSTEAD OF HEAT CONDUCTION EQUATION FOR MEDIUM AND HIGH FREQUENCIES

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## INTRODUCTION

For few years, various approaches have been attempted to generalise the Statistical Energy Analysis (SEA) beyond its limits of application. One of them called Energy Flow Analysis or Power Flow Finite Element Analysis is an energy model for medium and high frequencies. It uses the same quantities as in SEA: energy and energy flow and is based on an equation similar to the heat conduction equation in steady condition with a convection term. This is local equation leading to a continuous analysis of structures whereas SEA is based on a discrete analysis. The particular case of one dimensional systems such as rod or beam had been well studied [1,2]. But the multi dimensional case was investigated by way of a direct generalisation of the thermal analogy valid for the one dimensional case. This generalisation was criticized by Langley [3] who remarked that, for an infinite structure, the decreasing in farfield of the solution of the heat conduction equation is in contradiction with those of the energy density deduced from the Helmoltz equation. I suggested [4] an explanation of this paradox which raises up a limitation of the thermal analogy. But this explanation uses the particular symmetry of infinite systems and no solution has been proposed for the general case without symmetry.

This paper proposes an alternative of the heat conduction equation for multi dimensional systems such as plate or acoustical enclosure. This formulation is based on an integral equation deduced from the Huygens's principle [5]. No demonstration is performed in this paper for reason of space but two numerical illustrations have been exposed: the first with a square plate and the second with a parallelepiped acoustical enclosure.

## THEORETICAL FORMULATION

Two energy quantities are involved in this formulation: the energy density $W$ which is defined by the sum of the kinetic energy and the potential energy (or deformation energy in case of structures) and the energy flow vector **I** which supports the motion of energy inside systems. Moreover the group velocity $c_g$ is needed as a characteristic of the system. A first set of assumptions required to derive the energy model is summarised as follows:

(H1) linear, isotropic system in steady state conditions harmonically excited with pulsation $\omega$,
(H2) light damping loss factor,
(H3) evanescent waves and nearfield are neglected,
(H4) interferences between propagative waves are not taken into account.
Another assumption will be add later.

The first step to establish this energy formulation is the well known power balance for an unloaded region:

$$\mathbf{div.I} + p_{diss} = 0 \text{ with } p_{diss} = \eta\omega W \text{ or } p_{diss} = mc_g W \tag{1}$$

where $p_{diss}$ is the power density dissipated. The first relationship of this dissipative term is related to a hysteretic damping model (for structures) whereas the second one is related to an atmospheric absorption (for acoustical enclosures).

Now let study the pure propagative waves. These waves are defined as the wave produced by a single point $S$ and propagating in an infinite system. They are characterised by an especially simple relationship between the energy flow and the energy density:

$$\mathbf{I}(M) = c_g W(M)\mathbf{u}(S,M) \text{ for } M \in \Omega \tag{2}$$

where $\Omega$ is the domain under study and $\mathbf{u}(S,M)$ the unity vector from the source $S$ to the current point $M$. This relationship is valid in farfield for outgoing travelling wave (and not for evanescent wave) for undamped system. For light damped system, this relationship is assumed to remain valid taking into account the dissipation by way of the dissipative term in the power balance. So, substituting (2) into the power balance (1) and solving the obtained equation gives the energy quantities $W$ and $\mathbf{I}$ of propagative waves which are respectively proportional to the following functions:

$$G(S,M) = \frac{\mathrm{e}^{-\frac{n\omega}{c_g}r}}{r^{n-1}} \text{ and } \mathbf{H}(S,M) = c_g \frac{\mathrm{e}^{\frac{n\omega}{c_g}r}}{r^{n-1}} \mathbf{u}(S,M) \tag{3}$$

where $n$ is the dimension of the space $\Omega$, $r$ is the distance between $S$ and $M$ and the factor $\frac{n\omega}{c_g}$ is replaced by $m$ in case of acoustical enclosure.

In general, many propagative waves travel in a given system. So, we build up the fields $W$ and $\mathbf{I}$ by applying two principles. First of all, we said that the interferences between propagative waves are not taken into account in our model. Therefore, it results that the energy quantities are simply the sum of those of the propagative waves. This is a linear superposition principle. Secondly, following the Huygens's principle, the most general field comes from the superposition of a direct field created by some primary sources (or real sources) inside $\Omega$ and a diffracted field created by some secondary sources (or fictive sources) on the boundary $\partial\Omega$. These considerations are summarised into the following relationships:

$$W(M) = \int_\Omega \rho(S)G(S,M)dS + \int_{\partial\Omega} \sigma(P)f(\mathbf{u}.\mathbf{n})G(P,M)dP \tag{4}$$

$$\mathbf{I}(M) = \int_\Omega \rho(S)\mathbf{H}(S,M)dS + \int_{\partial\Omega} \sigma(P)f(\mathbf{u}.\mathbf{n})\mathbf{H}(P,M)dP \tag{5}$$

where $\rho$ and $\sigma$ are respectively the magnitudes of the primary and the secondary sources, $f$ is the directivity diagram of the secondary sources and $\mathbf{n}$ is the outward normal vector at point $P$. Another assumption concerning $f$ is necessary:

(H5) The directivity diagram $f$ of fictive sources is known and does not depend on point $P$.

So, the function $f$ has to be chosen. In what follows, we study the choices $1, \cos(\mathbf{u}.\mathbf{n})$ (Lambert's law) or $\cos^2(\mathbf{u}.\mathbf{n})$. Indeed, the primary density $\rho$ is assumed to be known because real sources constitute a data of the problem. But the secondary density $\sigma$ is unknown and an equation which determines its value has to be exhibited.

This expected equation on $\sigma$ is obtained by applying the power balance on the boundary $\partial\Omega$. No detail of the demonstration is given there but it is based on first, the incident power at point $P$ which may come from both primary and secondary sources and secondly, the reflected power produced by the secondary source $\sigma(P)$ at point $P$. The boundary is characterised by an absorption coefficient $\alpha$ between 0 and 1 which is the part of absorbed power from incident power. This coefficient $\alpha$ may depend on the point $P$. After this reasoning, one obtains the following equation:

$$\sigma(P) = \frac{1-\alpha(P)}{\gamma c_g}\left\{\int_\Omega \rho(S)\mathbf{H}(S,P)dS + \int_{\partial\Omega}^{*} \sigma(P')f(\mathbf{u}.\mathbf{n}')\mathbf{H}(P',P)dP'\right\}.\mathbf{n}(P) \tag{6}$$

where $\mathbf{n}'$ is the outward normal vector at point $P'$, $\mathbf{u}$ the unity vector from $P'$ to $P$, $*$ designates the principal value in Cauchy sense and $\gamma$ has a value depending on the choice of the directivity $f$. The following table summarises these values:

| directivity $f$ | $1$ | $\cos(\mathbf{u}.\mathbf{n})$ | $\cos^2(\mathbf{u}.\mathbf{n})$ |
|---|---|---|---|
| $n$=2 | $\pi$ | $2$ | $\pi/2$ |
| $n$=3 | $2\pi$ | $\pi$ | $2\pi$ |

Thus, the equation (6) is a Fredholm integral equation of second kind on the layer $\sigma$.

Equations (4), (5) and (6) are the basic relationships of our model. The properties of such a formulation are really interesting but are not studied here. It could be done in another paper. In what follows, we first develop a numerical implementation and then study two examples.

## NUMERICAL IMPLEMENTATION

To solve the system of equations (4,5,6), the boundary $\partial\Omega$ is discretized into segments $S_i$ $i$=1,$n$ (figure 1).

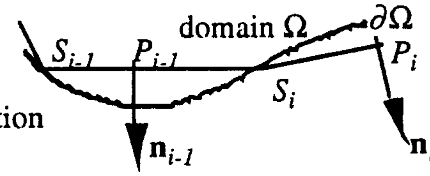



fig. 1 Boundary discretization

If $P_i$ denotes the middle of the segment $S_i$., $\mathbf{n}_i$ the outward normal vector at point $P_i$ , $\alpha_i$ the absorption coefficient at $P_i$. and assume that the layer $\sigma$ is constant over $S_i$. and has the value $\sigma_i$., then the Fredholm equation (6) with a single point source of magnitude $A$ at $S$, becomes:

$$\sigma_i = \frac{1-\alpha_i}{\gamma c_g}\left\{A\mathbf{H}(S,P_i).\mathbf{n}_i + \sum_{j=1, j\neq i}^{n} \sigma_j \int_{S_j} f(\mathbf{u}.\mathbf{n})\mathbf{H}(Q,P_i).\mathbf{n}_i dQ\right\} \text{ for } i=1,n \tag{7}$$

The $n$ unknowns $\sigma_i$ are determined by solving the above equation. Then, the fields $W$ and $\mathbf{I}$ are calculated with:

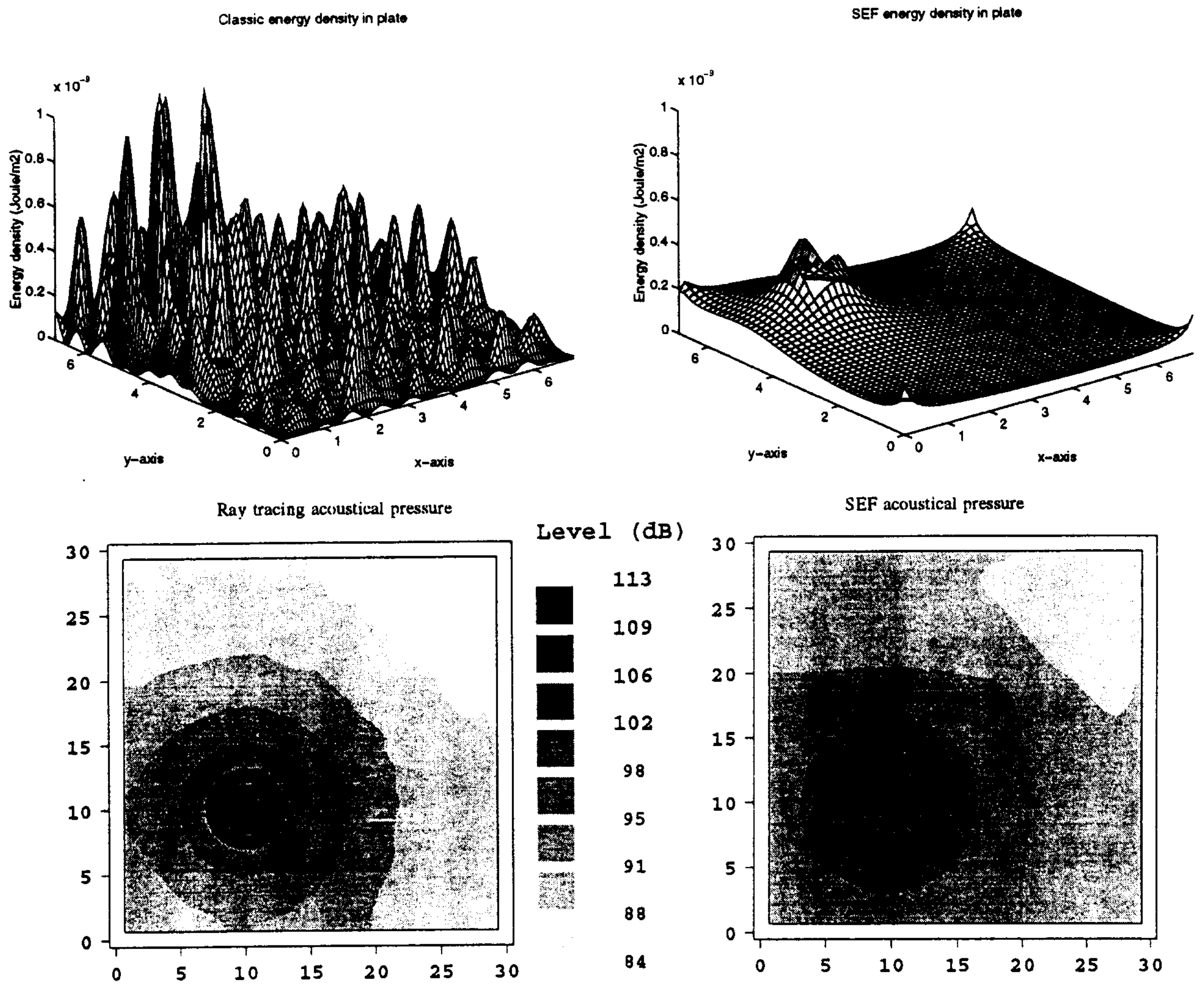
$$W(M) = AG(S,M) + \sum_{j=1}^{n} \sigma_j \int_{S_j} fG(Q,M)dQ, \quad \mathbf{I}(M) = A\mathbf{H}(S,M) + \sum_{j=1}^{n} \sigma_j \int_{S_j} f\mathbf{H}(Q,M)dQ \tag{8,9}$$

where $f$ is the directivity at $Q$ into $M$. These equations can be solved with an appropriate software.

## NUMERICAL SIMULATION

The first example concerns a square plate. A hysteretic damping factor is introduced but there is no dissipation at the boundaries. The boundaries are simple supported and then the reference calculation is based on a modal development of the solution of the Love plate equation. The above formulation (SEF) is implemented for the second calculation with Lambert's law. Results are presented on the two first figures.

The second example concerns an acoustical enclosure. The room under study is parallelepiped and absorption of walls is introduced. A point source is defined. Two calculations are performed. The first is realised with a ray tracing software called RAYON2.0 and developed at EDF. The second is based on the above formulation (SEF) with Lambert's law which is a classical model of diffuse reflection. The two last figures show the acoustical pressure in dB inside the reception plane. No more than 2 dB difference is done.




## CONCLUSIONS

The energy formulation presented in this paper is well suited for medium and high frequencies domain because of the smooth response predicted. In one dimension case, it reduces to the well known equation based on a thermal analogy. But, significant differences appear for multi dimensional case between this formulation and the direct generalisation of the thermal analogy. The numerical results obtained indicate that this method is closed to the ray tracing technique.

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