Entropy in statistical energy analysis

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In this paper, the second principle of thermodynamics is discussed in the framework of statistical energy analysis (SEA). It is shown that the "vibrational entropy" and the "vibrational temperature" of sub-systems only depend on the vibrational energy and the number of resonant modes. A SEA system can be described as a thermodynamic system slightly out of equilibrium. In steady-state condition, the entropy exchanged with exterior by sources and dissipation exactly balances the production of entropy by irreversible processes at interface between SEA sub-systems.
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I. INTRODUCTION

Statistical energy analysis[1] (SEA) is a statistical theory of vibroacoustics. When modes are too numerous and when waves are uncorrelated, the vibrational energy tends to be uniformly distributed inside the system. This is the state of diffuse field[2] or equipartition of modal energy.[3] Furthermore, when two sub-systems with diffuse field are connected together, the power flow between them is proportional to the difference of their modal energies.[4]

It has been recognized for a long time[5] that this proportionality relationship for power flow is the foundation of an analogy with thermodynamics provided that the "vibrational temperature" is defined as the modal energy. The "heat" or vibrational energy then flows from hot sub-system to cool sub-system.

SEA is entirely based on the application of the energy balance that is the first principle of thermodynamics. And the natural question that arises is the following: What about the second principle of thermodynamics? This question was raised by Carcaterra in Ref. 6. In Ref. 7, it has been shown that the mechanical statistics approach leads to a strict definition of the vibrational temperature and gives an argument in favor of the proportionality relationship of power flow with the difference of modal energies.

This aim of this study is to explicit the notion of entropy in SEA for systems at equilibrium but also for systems in non-equilibrium, which is the only case of interest in SEA.

II. ENTROPY OF A SINGLE SEA SUB-SYSTEM

The first question to solve is the following: How can one define the entropy of a single SEA sub-system?

A. Thermodynamic entropy

Before giving a rigorous answer to this question, let us develop some simple arguments based on classical thermodynamics. Clausius' principle states that thermal energy always flows from high temperature to low temperature. In SEA, the net power exchanged by two sub-systems i and j is given by

$$P_{ij} = \omega \eta_{ij} n_i \left(\frac{E_i}{n_i} - \frac{E_j}{n_j} \right), \tag{1}$$

where E_i is the total vibrational energy contained in sub-system i, n_i is the modal density in mode per rad/s, ω is the central frequency, and η_{ij} is the so-called coupling loss factor in which all the physics of the problem at hand is included. It is clear that the relevant quantity, which governs the flow of vibrational energy, is the modal energy E_i/N_i where $N_i = n_i \Delta \omega$ is the number of modes contained in the frequency band of width $\Delta \omega$. The modal energy in SEA is therefore analogous to the temperature in thermodynamics and for this reason, it can be called vibrational temperature,

$$T \propto \frac{E}{N}. \tag{2}$$

In thermodynamics, at least two kinds of energy can be exchanged with a system: heat and work. Work is the energy carried by external forces. This is the case when a pressure p is applied to the boundary and leads to a change in volume dV, the work is $\delta W = -p\,dV$ or when the system of mass m is raised to height dh and therefore increases its potential energy by $\delta W = mg\,dh$. In all cases, it is tacitly assumed that the force is static or at least slowly varying. This means that the period of the variation is very large compared with the period of thermal vibrations. In vibroacoustics, it is irrelevant to consider the deformation of the sub-system or to take into account the gravitational potential energy. The only way to provide energy to sub-systems is by driving forces for structures and noise sources in rooms. The forces are then time-varying and the energy they provide must be considered as the equivalent of heat. The "vibrational heat" provided by a source of power P^i during dt is then

$$\delta Q = P^i dt. \tag{3}$$

The "vibrational entropy" can now be introduced in the same way Clausius did a century and half ago in thermodynamics. Entropy is a state function that depends on the ex-

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tensive variables of the sub-system. These are the total vibrational energy E and the number of modes N. For a reversible process, Clausius definition of entropy is

$$dS = \frac{\delta Q}{T}. \tag{4}$$

Since the vibrational sources do not provide work, $\delta W = 0$ and the first principle of thermodynamics reads

$$dE = \delta Q. \tag{5}$$

By substituting Eqs. (2) and (5) into Eq. (4), it yields

$$dS \propto N \frac{dE}{E}. \tag{6}$$

Finally, the variation of entropy between two states of energy E_2 and E_1 is therefore

$$\Delta S \propto N \log\left(\frac{E_2}{E_1}\right). \tag{7}$$

This relationship gives the vibrational entropy of SEA subsystems. It has been established for reversible processes. But, the entropy is a function of state and, therefore, the same relationship must also hold for irreversible processes.

It could be inferred from Eq. (7) that the absolute entropy $S(E,N)$ is $N \log E$. This expression is compatible with the variation of entropy derived here, but, it is inconsistent with the requirement that the entropy is an extensive quantity in classical thermodynamics. An extensive quantity must verify the equality $S(2E,2N) = 2S(E,N)$, which is obviously not the case with $N \log E$.

Equation (7) has been derived by considering a small variation dE of internal energy. The number of modes N has been tacitly assumed to be constant during this transformation. Thus, if one fixes a reference energy E_0, the absolute entropy $S(E,N) = \Delta S(E,E_0) + S(E_0,N)$ is known from Eq. (7) apart from a function $S(E_0,N)$, which now only depends on the number of modes N. But, it is not possible to go beyond and, in particular, to fully derive the function $S(E,N)$ with Clausius' approach. This is what is done in Sec. II B by introducing the statistical entropy.

B. Statistical entropy

In statistical mechanics,[8] the microscopic properties of a system are known as soon as the Hamiltonian of the system is established. A SEA sub-system is a set of N modes of frequencies ω_i $i = 1, \ldots, N$ in the band $[\omega - \Delta\omega/2, \omega + \Delta\omega/2]$. Since the modes are orthogonal, the vibrational energy of the sub-system can be decomposed on the modes. If q_i is the modal factor and $p_i = m_i \dot{q}_i$ is its conjugate momentum, the Hamiltonian of the sub-system is

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{m_i \omega_i^2}{2} q_i^2 + \frac{1}{2m_i} p_i^2, \tag{8}$$

where m_i is the modal mass of the mode i. For normalized modes, $m_i = 1$ or m the mass of the sub-system depending on the choice of the norm.

In the phase space Γ defined by all positions and momenta $q_1, \ldots, q_N, p_1, \ldots, p_N$ of the sub-system, the hypersurface of constant energy $H(q_1, \ldots, q_N, p_1, \ldots, p_N) = E$ is an ellipsoid of dimension $2N-1$ with semi-axes $(2m_i E)^{1/2}$ and $(2E/m_i \omega_i^2)^{1/2}$. Its volume $V(E)$ is

$$V(E) = \int_{H \leq E} d\Gamma = \frac{(2\pi)^N}{N! \prod_{i=1}^{N} \omega_i} E^N. \tag{9}$$

In SEA, the set of modes is considered to be a random population of modes. The frequencies ω_i have a uniform probability density over the band $[\omega - \Delta\omega/2, \omega + \Delta\omega/2]$. The product $\prod_i \omega_i$ can therefore be substituted by its expected value ω^N. The volume becomes

$$V(E) = \left(\frac{2\pi}{\omega}\right)^N \frac{E^N}{N!} \tag{10}$$

and the structure function $\Omega(E)$ is

$$\Omega(E) = \frac{dV}{dE} = \left(\frac{2\pi}{\omega}\right)^N \frac{E^{N-1}}{N-1!}. \tag{11}$$

All the information useful for a macroscopic description of the sub-system is contained in the structure function Ω.

The general method of statistical mechanics is to consider that the position of the system in Γ-space can never be known exactly. The only available information on the system is the probability density function of presence at any point in Γ-space. At this stage, it is common in statistical mechanics to distinguish two statistical ensembles. The microcanonical ensemble is the statistical distribution of positions in the Γ-space of an isolated system for which the energy is constant. And the canonical ensemble is the statistical distribution of a system in equilibrium with a thermal bath. Whatever the distribution of probability ρ_i of microstates (positions in Γ-space), the statistical entropy is always defined as

$$S = \sum_i -k\rho_i \log \rho_i, \tag{12}$$

where the sum runs over all microstates. This is the expectation of information $I_i = -k \log \rho_i$ associated with an event of probability ρ_i.

Indeed, the two statistical ensembles do not lead to the same probability distribution ρ_i. And the resulting entropy that is the mean loss of information of system whose the only macroscopic state is known is *a priori* not the same for both ensembles. But, this is a well-known result in statistical mechanics that the microcanonical entropy and the canonical entropy only differ with a negligible term.[8] For this reason, in the following, the SEA sub-system is assumed to be isolated and the microcanonical approach is adopted.

In the microcanonical ensemble, all the positions in the Γ-space around the surface $H(q_1, \ldots, q_N, p_1, \ldots, p_N) = E$ are equally probable. This is the so-called fundamental assumption of statistical mechanics. If the position of modes q_i and p_i is known with the uncertainties δq_i and δp_i such that $\delta q_i \delta p_i = h$ and the energy of the system is known with the

uncertainty δE, the number of microstates W is the volume of the shell of thickness δE and surface $\Omega(E)$ divided the volume of a microstate h^N,

$$W = \frac{\Omega(E)}{h^N} \delta E. \tag{13}$$

The entropy is then defined by substituting the uniform probability $\rho_i = 1/W$ in Eq. (12), which gives the Boltzmann's equation $S = k \log W$,

$$S = kN \log \frac{2\pi E}{\omega h} - k \log(N-1)! + k \log \frac{\delta E}{E}. \tag{14}$$

For a large population $N \simeq N-1$ and Stirling's approximation gives $\log N! \simeq N \log N - N$. Furthermore, the term $\log \delta E/E$ is usually discarded since it is related to the choice of the relative uncertainty on the energy, which is not a physical process. The final result is

$$S(E,N) = kN \left[1 + \log \left(\frac{2\pi}{h\omega} \frac{E}{N} \right) \right]. \tag{15}$$

This is the microcanonical entropy of a SEA sub-system. It can be remarked that this relationship well agrees with Eq. (7), which stems from the difference of $S(E_2,N) - S(E_1,N)$. Equation (15) gives the complete expression of the vibrational entropy of an isolated SEA sub-system.

The temperature is obtained as for any thermodynamic system with

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_N, \tag{16}$$

with the result

$$T = \frac{E}{kN}. \tag{17}$$

As it was expected from Eq. (1) giving the power flow between two SEA sub-systems, the vibrational temperature is well defined by the modal energy. This result was obtained from an analogy between SEA and thermodynamics. This is now a logical consequence of the expression of the entropy obtained from Boltzmann's definition. The same result has been derived in Ref. 7 with Khinchin's definition[9] of entropy.

III. SECOND PRINCIPLE OF THERMODYNAMICS IN SEA

The second principle of thermodynamics states that an isolated system (with no exchange of energy) always evolves in such a way that its entropy increases. To illustrate what is the second principle of thermodynamics in the particular framework of SEA, let us consider two isolated acoustical cavities. Isolated means that the cavities do not exchange energy and, in particular, that there is no noise source inside and that they do not dissipate the sound. As usual, it is also assumed that the thermal equilibrium is realized within the cavities, that is, the sound is diffuse. Therefore, these cavities are perfect reverberation chambers with infinite reverberation times; the sound pressure level is constant, but indeed, may have different values in the two cavities. The entropy of the

acoustical cavities $i=1,2$ are $S(E_i,N_i)$ where S is given in Eq. (15). For an acoustical cavity, the modal density is

$$n_i = \frac{V_i \omega^2}{2\pi^2 c_0^3}, \tag{18}$$

where c_0 is the sound speed and V_i is the volume of the cavity.

Now, let us consider that the two rooms are adjacent and that they are separated by a small closed window. The problem is to observe the evolution of the coupled rooms after the window has been opened. By opening the aperture between the two cavities, the energy is mixed and, since there is no cause of dissipation, the global system has energy $E = E_1 + E_2$. The total volume is also $V = V_1 + V_2$ so that the modal density of the total system is

$$n = \frac{(V_1 + V_2)\omega^2}{2\pi^2 c_0^3} = n_1 + n_2. \tag{19}$$

Thus, the number of modes of the global system $N = n\Delta\omega = (n_1+n_2)\Delta\omega$ is the sum of the numbers of modes of the two cavities $N = N_1 + N_2$.

When the equilibrium is reached, the entropy of the entire system becomes

$$S(E,N) = S(E_1 + E_2, N_1 + N_2). \tag{20}$$

The difference between the final entropy and the sum of initial entropy is therefore the entropy created during the mixing process.

$$\Delta S = S(E,N) - [S(E_1,N_1) + S(E_2,N_2)]. \tag{21}$$

By substituting Eq. (15), it yields

$$\Delta S = k(N_1 + N_2)\log \frac{E_1 + E_2}{N_1 + N_2} - kN_1 \log \frac{E_1}{N_2} - kN_2 \log \frac{E_2}{N_2}. \tag{22}$$

This is the entropy created by mixing the energy of two acoustical cavities. It is expected to be non-negative.

To prove this statement, it is necessary to introduce the function $f(x,y) = y \log(x/y)$ and to rewrite the entropy of mixing ΔS,

$$\Delta S = kf\left(\frac{E_1 + E_2}{2}, \frac{N_1 + N_2}{2} \right) - \frac{k}{2}[f(E_1,N_1) + f(E_2,N_2)]. \tag{23}$$

The function f, defined for $x > 0$ and $y > 0$, has the following derivatives:

$$\frac{\partial^2 f}{\partial x^2} = -\frac{y}{x^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{x}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{1}{y}, \tag{24}$$

so that the quadratic form tangent to f is

$$q(X,Y) = -\frac{y}{x^2}X^2 + \frac{1}{x}XY - \frac{1}{y}Y^2 = -\left(\frac{\sqrt{y}}{x}X - \frac{1}{\sqrt{y}}Y \right)^2. \tag{25}$$

Since $q(X,Y) < 0$ for all X and Y, $-f$ is convex. The convexity of a function means that its value at the middle point of a

segment is lower than the mean of the values at the extremities. This reads

$$f\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) - \frac{1}{2}f(x_1, y_1) - \frac{1}{2}f(x_2, y_2) > 0. \qquad (26)$$

This proves the inequality $\Delta S > 0$. The entropy of mixing is therefore always non-negative. The evolution of an isolated system (which does not receive neither dissipates energy) always leads to an increasing of the entropy. This is the second principle of thermodynamics in the particular framework of SEA.

IV. ENTROPY PRODUCTION AT INTERFACE

Up to now, we have considered that sub-systems are isolated and that they have reached their equilibrium state, or, in other words, that the equipartition of energy among modes is verified. Indeed, this situation is an ideal case, which has been introduced only to obtain the explicit expression of the entropy of a single sub-system.

But in vibroacoustics, a sub-system can never been considered as isolated. For instance, in Sabine's theory of reverberation chambers, the room is first filled up with noise and, after the source has been switched off, the energy starts to decrease. The rate of dissipation gives the reverberation time of the room. This is therefore not an isolated room. In steady-state SEA, the sub-systems can receive vibrational energy and in the same time, they can dissipate the energy as well as they can exchange energy with neighbor sub-systems. SEA sub-systems are therefore not isolated systems.

A. Single sub-system

Let us first consider a single sub-system submitted to input power and dissipated power. This sub-system is assumed to be in steady-state condition, that is, all macroscopic quantities such as input power, dissipated power, vibrational energy, vibrational temperature, and entropy are constant in time. The sub-system is not in thermodynamic equilibrium in the meaning accepted in the first two sections since a flux of energy can flow through it. This is the simplest problem in non-equilibrium thermodynamics.

If the flux is not too important, the sub-system is in local equilibrium. This means that the relaxation processes inside the sub-system such as the mixing of rays are more rapid than the breakdown of equilibrium imposed by sources and dissipation. This assumption of local equilibrium is important since it allows to define a unique vibrational temperature for the sub-system.

The transfer of vibrational energy by sources and dissipation is therefore a reversible process. At any time,

$$dS = \frac{\delta Q^i + \delta Q^d}{T}, \qquad (27)$$

where δQ^i is the vibrational heat supplied by sources to the sub-system and δQ^d is the vibrational heat extracted from the sub-system by dissipation. This net vibrational heat is the sum of the gain δQ^i and the loss δQ^d. Each term is the product of a power by the infinitesimal time dt,

$$\delta Q^i = P^i dt \qquad (28)$$

for the gain term and

$$\delta Q^d = -P^d dt \qquad (29)$$

for the loss term.

The vibrational entropy may therefore be separated into two terms: the entropy dS^i provided by sources and the entropy dS^d consumed by dissipation.

$$dS = dS^i + dS^d = \frac{P^i - P^d}{T}dt. \qquad (30)$$

The first term dS^i is positive, and the sources provide heat to the sub-system and therefore warm it and increase its entropy. The second term dS^d is negative, and the dissipation extracts heat from the sub-system. This cooling process makes the entropy to decrease.

In steady-state condition, the power balance reduces to

$$P^i = P^d, \qquad (31)$$

where $P^d = \eta \omega E^d$ is the power being dissipated. This simple equation is the SEA equation in the trivial case of a single sub-system. Substitution into Eq. (30) gives

$$\frac{dS}{dt} = 0. \qquad (32)$$

Thus, any SEA system composed of a single sub-system and submitted to sources and dissipation has a null production entropy.

B. Coupled sub-systems

Let us now consider a couple of two adjacent sub-systems. Once again, the sub-systems are assumed to be in local equilibrium allowing to define a vibrational temperature for all sub-systems. But, these vibrational temperatures T_1 and T_2 may be different. The conditions under which a local equilibrium is reached while a global equilibrium is not is discussed in Ref. 10.

By applying Eq. (4) to sub-system 1, it yields

$$dS_1 = \frac{P_1^i - P_1^d - P_{12}}{T_1}dt \qquad (33)$$

and for sub-system 2

$$dS_2 = \frac{P_2^i - P_2^d + P_{12}}{T_2}dt. \qquad (34)$$

Entropy is an extensive quantity. Therefore, for the entire system, the entropy dS is at any time the sum of the entropy of its parts $dS_1 + dS_2$.

$$dS = \frac{P_1^i - P_1^d - P_{12}}{T_1}dt + \frac{P_2^i - P_2^d + P_{12}}{T_2}dt. \qquad (35)$$

This sum may be re-combined into two terms, $dS = dS^{rev} + dS^{irr}$. The first term dS^{rev} is the sum of the net entropy provided by the exchange of vibrational heat with exterior (sources and dissipation).

$$dS^{\text{rev}} = \frac{P_1^i - P_1^d}{T_1}dt + \frac{P_2^i - P_2^d}{T_2}dt. \tag{36}$$

This entropy is supplied to the system by a reversible process since equality (4) applies. The second term,

$$dS^{\text{irr}} = P_{12}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)dt, \tag{37}$$

is the entropy created during the exchange of vibrational heat between the two sub-systems. This term is not related to an exchange of heat with exterior. This is the entropy created inside the system by the mixing of vibrational energy. This entropy is therefore the entropy created by irreversible processes.

By substituting Eqs. (1) and (17)

$$\frac{dS^{\text{irr}}}{dt} = k\eta_{12}\omega N_1\left(\frac{E_1}{N_1} - \frac{E_2}{N_2}\right)\left(\frac{N_2}{E_2} - \frac{N_1}{E_1}\right). \tag{38}$$

This relationship gives the production rate of entropy at the interface between two adjacent SEA sub-systems. The right-hand side can be re-arranged,

$$\frac{dS^{\text{irr}}}{dt} = k\eta_{12}\omega N_1\frac{N_1 N_2}{E_1 E_2}\left(\frac{E_1}{N_1} - \frac{E_2}{N_2}\right)^2. \tag{39}$$

It is now clear that the right-hand side is always positive. Therefore,

$$\frac{dS^{\text{irr}}}{dt} \geq 0. \tag{40}$$

This result generalizes the second principle of thermodynamics to the case of non-equilibrium SEA sub-systems.

The procedure followed to obtain Eq. (38) is similar to the one adopted in thermodynamics of linear irreversible processes[11] except that the power flow has been substituted by the correct SEA power flow given in Eq. (1).

C. Entropy balance

The last question to examine is the following: How to write an entropy balance for the entire SEA system? Since the steady-state condition is assumed to apply, the entropy is constant in time and therefore the production of entropy dS/dt for the entire system must vanish.

Any sub-system i receives an infinitesimal entropy supplied by sources,

$$dS_i^i = \frac{P_i^i}{T_i}dt, \tag{41}$$

and also loses entropy by dissipation,

$$dS_i^d = -\frac{P_i^d}{T_i}dt. \tag{42}$$

At any interface, the entropy created by the irreversible process is

$$dS_{ij}^{\text{irr}} = P_{ij}\left(\frac{1}{T_j} - \frac{1}{T_i}\right)dt. \tag{43}$$

Now, consider the entire system. Since entropy is an extensive variable, the entropy of the whole system is the sum of the entropy exchanged with exterior by sub-systems and the entropy created at interfaces by irreversible processes.

$$dS = \sum_{i=1}^{N} dS_i^i + dS_i^d + \sum_{i>j} dS_{ij}^{\text{irr}}. \tag{44}$$

The last sum runs for $i > j$ since each interface must be counted only one time. By substituting Eqs. (41)–(43) into Eq. (44), it yields

$$\frac{dS}{dt} = \sum_{i=1}^{N} \frac{P_i^i - P_i^d}{T_i} + \sum_{i>j} P_{ij}\left(\frac{1}{T_j} - \frac{1}{T_i}\right). \tag{45}$$

By splitting the last sum and remarking that $P_{ij} = -P_{ji}$,

$$\frac{dS}{dt} = \sum_{i=1}^{N} \frac{1}{T_i}\left(P_i^i - P_i^d + \sum_{j\neq i} P_{ij}\right). \tag{46}$$

But the energy balance of any sub-system reads

$$P_i^i - P_i^d + \sum_{j\neq i} P_{ij} = 0. \tag{47}$$

Finally,

$$\frac{dS}{dt} = 0. \tag{48}$$

This is the expected result. It means that there is no production of entropy for the entire SEA system. But there is an exchange of entropy with exterior, which exactly balances the production of entropy by irreversible processes.

V. CONCLUSION

In this paper, it has been shown that the second principle of thermodynamics can be stated in the particular context of SEA. Explicit relationships have been derived for the vibrational entropy of sub-systems and the production of vibrational entropy by irreversible processes at interfaces of sub-systems. The mixing of vibrational energy of two sub-systems as well as the exchange of power flow between two sub-systems always lead to an increase in the entropy.

Beyond the work of Carcaterra[6] who demonstrated that the proportionality of power flow with the difference of modal energies can be derived from the statistical mechanics, this study highlights the importance of entropy in SEA. The thermodynamical analogy on which SEA is based could not be completed without the introduction of the second principle.

In SEA, the study of perfectly isolated sub-systems in the meaning of classical thermodynamics is of no interest. Systems of interest are always crossed by a flux of energy supplied by sources and lost in dissipation process. These thermodynamic systems are maintained in non-equilibrium condition by external constraints. But, the diffuse field assumption in SEA requires that all sub-systems remain in

equilibrium. From this point of view, SEA is therefore the thermodynamics of linear irreversible processes in vibrational systems.

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