Energy exchange in uncorrelated ray fields of vibroacoustics

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This paper is concerned with the basic equations governing energy and intensity in incoherent ray fields. Some fictitious sources are distributed on the boundary of the domain but also on diffracting wedges and peaks. Their powers are determined by some appropriate boundary integral equations. Once these powers are known, energy and intensity inside the domain are given by a simple superposition of contributions of these sources. All paths of propagation are taken into account including direct, reflected, refracted, transmitted, and diffracted rays, but also, radiation by surface, edge or corner modes, and the reciprocal paths for structural response. This theory unifies several fields from the "radiosity method" in room acoustics which determines the reverberation time to the "radiative transfer method" in structural dynamics which gives the repartition of vibrational energy inside subsystems of built-up structures. This is therefore a candidate for an alternative to statistical energy analysis when fields are nondiffuse. © *2006 Acoustical Society of America.*
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List of Symbols

$\gamma_0=2,\ 2\pi$ or 4π	solid angle of space of n-dimensional space $n=1$, 2, or 3
$\gamma=\int\cos\theta du=1$, 2 or π	hemispherical integral of $\cos\theta$ for $n=1$, 2 or 3
Ω	domain
Γ	regular boundary
Δ	edge of boundary
Υ	vertices of boundary
$\Gamma^0,\Delta^0,\Upsilon^0$	set of directions from Γ, Δ or Υ to $\mathbf{p}$
$\mathbf{s},\mathbf{r}$	source point, receiver point
$\mathbf{p},\mathbf{q}$	reflection or diffraction points
$\mathbf{u},\theta,\alpha$	emission direction, emission angles
$\mathbf{v},\varphi,\beta$	incidence direction, incidence angles
$\delta(\mathbf{r}),\delta(t),\delta_{\Delta^0}(\mathbf{v})$	Dirac function of space, time, unit sphere with support Δ^0
c,c',c_0	group speed, phase speed, sound speed
m	attenuation factor
$I(\mathbf{r},\mathbf{u},t)$	radiative intensity
$W(\mathbf{r},t)$	energy density
$\mathbf{I}(\mathbf{r},t)$	intensity vector
G	energy density of direct field
$\mathbf{H},H$	intensity of direct field, magnitude
$R(\mathbf{v},\mathbf{u}),R(\mathbf{v})$	bidirectional reflectivity, hemispherical reflectivity
$R_{ji}(\mathbf{v},\mathbf{u}),R_{ji}(\mathbf{v})$	bidirectional transmittivity, hemispherical transmittivity
$D(\mathbf{v},\mathbf{u})$	bidirectional diffractivity
A_{sa},A_0	radiation coefficient, forcing coefficient

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D_{sa}, D_{as}	diffractivity for radiation, for structural response
ρ	power density of volume sources
σ	power density of surface sources
λ	power density of line sources
μ	power of point sources

I. INTRODUCTION

The use of energy as a primary variable to describe vibrational fields is an idea widely spread in high frequency modeling. When waves are uncorrelated, the linear superposition applies on energy allowing the summation of energy of individual waves. In addition, if fields are assumed to be diffuse, the application of power balance leads to some simple equations.

In room acoustics, Sabine's formula and other related relationships from Millington and Eyring[1] give the reverberation time in terms of absorption of walls. This is quite a convincing example of the power of the energy approach. When the diffuse field assumption is no longer valid, geometrical acoustics is an alternative. The ray-tracing technique allows us to compute impulse responses in any room and therefore gives the reverberation time as well as many other acoustical criteria. The ray-tracing technique is however a numerical method (very efficient in practice) which is sometimes not convenient for a theoretical purpose. In case of diffusely reflecting walls and applying the factor view method. Kuttruff[2] has derived an integral equation on reverberation time valid for rooms of arbitrary shape. As Sabine's formula, this equation stems from the power balance, rays are again uncorrelated but now the field may be not diffuse.

In vibroacoustics, statistical energy analysis (SEA) (Ref. 3) also starts from diffuse fields. Systems are divided into subsystems in which the diffuse field assumption holds. SEA gives the steady-state energy levels whereas transient SEA predicts the decrease of energy after sources have been switched off. SEA is entirely based on the application of power balance and does not express anything else than power balance. (A very interesting attempt to introduce the second principle of thermodynamics in SEA is given in Ref. 4.) A simple linear equation is obtained which relates injected power and energy levels and whose parameters are damping and coupling loss factors. In this context, the present study aims to avoid the diffuse field assumption in SEA. We attempt to generalize SEA in the same way than Kuttruff's integral equation on reverberation time generalizes Sabine's formula. Rays are always assumed to be uncorrelated and the exchange of energy between two points of the boundary requires a factor view but also some energy conversion factors which generalize the coupling loss factors in SEA.

The outline of the paper is as follows: In Sec. II, fictitious sources for reflection and diffraction are introduced. Energy density and intensity of the ray field are then given in terms of these sources. In Sec. III, the general equation for the reflection sources is derived and then, the particular cases of diffuse and specular reflections are presented. In Sec. IV, these results are generalized to refraction and transmission. In Sec. V, the equations for diffraction sources are derived in the cases of wedge and peak. In Sec. VI, all previous equations are applied to sound radiated by surface, edge, and corner modes. Finally, structural response is tackled in Sec. VII.

II. ENERGY OF RAY FIELDS

Geometrical acoustics and its straightforward generalization, Geometrical Theory of Diffraction[5] are the natural framework for defining and describing rays. Many vibrational fields may be described in terms of rays including optics indeed but also acoustics, vibration of structures such as beams, plates, shells, and so on. For transient problems, the concept of wave packet or sound particle is more appropriate. For instance, it is usual in room acoustics to compute the impulse response with a ray-tracing algorithm by following the trajectory of sound particles emitted at an initial time. For the steady-state problem considered as being a particular case of transient problem, rays may be viewed as stationary flow of sound particles. In what follows, the term ray is employed for both steady-state and transient cases.

The energy density of the ray field is noted as W, whereas intensity, defined as the power per unit surface normal to the ray, is noted as $\mathbf{I}$. Energy density and intensity are always the sum of energies and intensities of individual rays. Interference effects are neglected considering that rays are uncorrelated. This is the main difference with Geometrical Theory of Diffraction where a phase is attached to rays.

Let consider a source point $\mathbf{s}$ in an n-dimensional space ($n=1$, 2, or 3). After an impulse at time τ, sound particles are provided and move away from the source $\mathbf{s}$. Energy density of this spherical ($n=3$), cylindrical ($n=2$), or plane ($n=1$) wave is noted as G while intensity is noted as $\mathbf{H}$. Expressions for this direct field are,

$$G(\mathbf{s},\tau;\mathbf{r},t) = G(\mathbf{s},\mathbf{r})\,\delta(t-\tau-r/c), \tag{1}$$

$$\mathbf{H}(\mathbf{s},\tau;\mathbf{r},t) = \mathbf{H}(\mathbf{s},\mathbf{r})\,\delta(t-\tau-r/c), \tag{2}$$

where $r=|\mathbf{s}-\mathbf{r}|$ is the source-receiver distance. $G(\mathbf{s};\mathbf{r})$ and $\mathbf{H}(\mathbf{s};\mathbf{r})$ are the stationary fields given by

$$G(\mathbf{s},\mathbf{r}) = \frac{e^{-mr}}{\gamma_0 c r^{n-1}} V(\mathbf{s},\mathbf{r}), \tag{3}$$

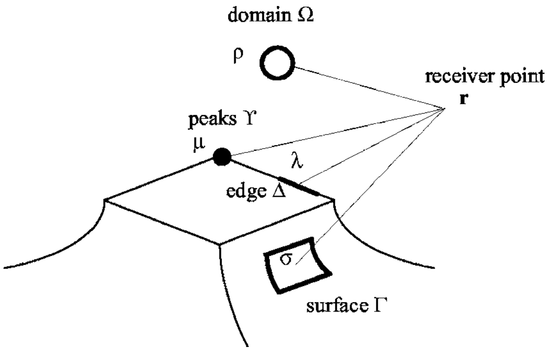

FIG. 1. Actual and fictive sources contributing to the energy at point **r**. Volume sources ρ in the domain Ω stand for actual noise sources, surface sources σ on the regular boundary Γ stand for reflection, line sources λ on edges Δ of boundary and point sources μ on singular points Υ of boundary stand for diffraction.

$$\mathbf{H}(\mathbf{s},\mathbf{r}) = cG(\mathbf{s},\mathbf{r})\mathbf{u}, \tag{4}$$

where $\gamma_0=2$, 2π or 4π is the solid angle of n-dimensional space, c is the group speed, m is the attenuation factor of the medium, and $\mathbf{u}$ is the unit vector from $\mathbf{s}$ to $\mathbf{r}$. $V(\mathbf{s},\mathbf{r})$ is the visibility function whose value is zero if an obstacle blocks the path between the source point $\mathbf{s}$ and the receiver point $\mathbf{r}$, and one otherwise. In some equations, $H=cG$ will denote the magnitude of vector $\mathbf{H}$.

Equations (1) and (2) verify the power balance with an impulse excitation,

$$\mathbf{div}_{\mathbf{r}}\,\mathbf{H} + mcG + \frac{\partial G}{\partial t} = \delta(\mathbf{r}-\mathbf{s})\delta(t-\tau). \tag{5}$$

Equations (1) and (2) are found to be the unique outgoing solution of Eq. (5).[6] Similarly, the power balance in the steady state condition for Eqs. (3) and (4) is[7]

$$\mathbf{div}_{\mathbf{r}}\,\mathbf{H} + mcG = \delta(\mathbf{r}-\mathbf{s}). \tag{6}$$

When considering a ray field in a domain Ω of three-dimensional space, Ω may be bounded or not, energy fields W and $\mathbf{I}$ result from a linear superposition of direct fields stemming from volume sources with power density ρ (W/m^3) located inside domain Ω, from surface sources with power density σ(W/m^2) located on the regular boundary Γ, from some line sources λ(W/m) on the set Δ of diffracting edges of the boundary and also from some point sources with power μ (W) on the set Υ of singular points of the boundary (Fig. 1). ρ are generally physical noise sources, whereas σ is associated with reflection on boundary, radiation by structure or transmission through walls, λ is related to diffraction by wedges, and μ to diffraction by peaks. In two-dimensional space (structure), ρ (W/m^2) is the power per unit surface of driving forces or acoustical pressure, σ is the power per unit length (W/m) of reflection sources or sources transmitted through edges common with other structural elements, and μ is the power (W) of diffracting points such as corners, driving points, small holes, rivets, bolts or any other singular points. No line sources λ are considered in structures. By summing the energy contributions of all these sources, energy density and intensity at any receiver point $\mathbf{r}$ are found to be,

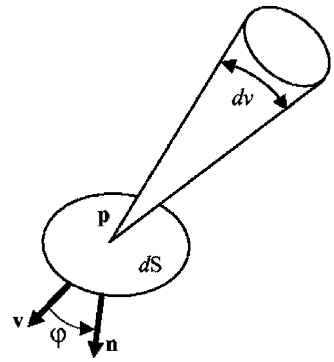

FIG. 2. Radiative intensity at point **p** in direction **v** crossing the surface dS. **n** is the normal to dS and φ the incidence angle.

$$\begin{aligned} W(\mathbf{r},t) &= \int_\Omega \rho(\mathbf{s},t-r/c)G(\mathbf{s},\mathbf{r})d\Omega_{\mathbf{s}} \\ &\quad + \int_\Gamma \sigma(\mathbf{p},\mathbf{v},t-r/c)G(\mathbf{p},\mathbf{r})d\Gamma_{\mathbf{p}} \\ &\quad + \int_\Delta \lambda(\mathbf{p},\mathbf{v},t-r/c)G(\mathbf{p},\mathbf{r})d\Delta_{\mathbf{p}} \\ &\quad + \sum_{\mathbf{p}\in\Upsilon} \mu(\mathbf{p},\mathbf{v},t-r/c)G(\mathbf{p},\mathbf{r}), \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbf{I}(\mathbf{r},t) &= \int_\Omega \rho(\mathbf{s},t-r/c)\mathbf{H}(\mathbf{s},\mathbf{r})d\Omega_{\mathbf{s}} \\ &\quad + \int_\Gamma \sigma(\mathbf{p},\mathbf{v},t-r/c)\mathbf{H}(\mathbf{p},\mathbf{r})d\Gamma_{\mathbf{p}} + \int_\Delta \lambda(\mathbf{p},\mathbf{v},t \\ &\quad - r/c)\mathbf{H}(\mathbf{p},\mathbf{r})d\Delta_{\mathbf{p}} + \sum_{\mathbf{p}\in\Upsilon} \mu(\mathbf{p},\mathbf{v},t-r/c)\mathbf{H}(\mathbf{p},\mathbf{r}), \end{aligned} \tag{8}$$

where $r=|\mathbf{s}-\mathbf{r}|$ or $|\mathbf{p}-\mathbf{r}|$ is the source-receiver distance and $\mathbf{v}$ the unit vector from $\mathbf{p}$ to $\mathbf{r}$. In these integrals and all subsequent ones, the term $\int\lambda G d\Delta$ is cancelled in dimension two. A local power balance applies inside Ω for fields W and $\mathbf{I}$,

$$\mathbf{div}\,\mathbf{I} + mcW + \frac{\partial W}{\partial t} = \rho, \tag{9}$$

where mcW is the power density being dissipated and ρ is the power density being injected.

Another concept useful to describe ray fields is the *radiative intensity*[8] $I(\mathbf{p},\mathbf{v},t)$ also called *specific intensity*.[9] Consider a point $\mathbf{p}$ on the infinitesimal surface dS and an infinitesimal solid angle dv about $\mathbf{v}$. The angle between $\mathbf{n}$ and $\mathbf{v}$ is noted φ (Fig. 2). The radiative intensity is the power per unit solid angle and per unit area normal to the ray,

$$I(\mathbf{p},\mathbf{v},t) = \frac{1}{\cos\varphi}\frac{d\mathcal{P}}{dSdv}. \tag{10}$$

Integration of radiative intensity over all directions gives the intensity vector,

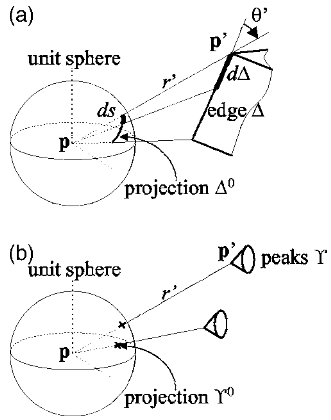

FIG. 3. (a) The set Δ^0 is the projection of Δ onto the unit sphere centered on $\mathbf{p}$. It is a curve with curvilinear abscissa s. The emission angle θ' is measured with the tangent to Δ. (b) The set Υ^0 is the projection of the diffracting peaks Υ onto the unit sphere. This is a discrete set.

$$\mathbf{I}(\mathbf{p},t) = \int I(\mathbf{p},\mathbf{v},t)\mathbf{v}dv. \tag{11}$$

All volume, surface, line, and point sources located inside the incident cone dv contribute to the radiative intensity. Therefore,

$$I(\mathbf{p},\mathbf{v},t)dv = \int \rho H d\Omega + \sigma H d\Gamma + \lambda H d\Delta + \sum \mu H, \tag{12}$$

where the sources ρ, σ, λ, μ of the right-hand side are those located inside the incident cone.

Let us begin by developing the first integral of Eq. (12). In spherical coordinates $(r,\mathbf{v})$ centered on $\mathbf{p}$, the infinitesimal volume is $d\Omega = r^{n-1}drdv$. With Eqs. (3) and (4), the first integral of Eq. (12) then reads $dv\int \rho e^{-mr}dr/\gamma_0$ where the integration is performed over the line beginning at $\mathbf{p}$ and with direction $-\mathbf{v}$. When Ω, is unbounded in direction $-\mathbf{v}$, no boundary sources σ,λ,μ contribute to the radiative intensity and the right-hand side of Eq. (12) reduces to its first integral. The radiative intensity is therefore,

$$I(\mathbf{p},\mathbf{v},t) = \frac{1}{\gamma_0}\int_{\mathbf{p}}^{\mathbf{p}-\infty.\mathbf{v}} \rho(\mathbf{s},t-r/c)e^{-mr}dr. \tag{13}$$

But when Ω is bounded in the direction $-\mathbf{v}$, the line beginning at $\mathbf{p}$ encounters the boundary at point $\mathbf{p}'$. Depending on the position of $\mathbf{p}'$ on Γ, Δ or Υ, a single term among the last three terms of Eq. (12) survives.

Before expanding them, let us introduce the set Γ^0 (resp. Δ^0 and Υ^0) defined as the set of unit vectors pointing from Γ (resp. Δ and Υ) to $\mathbf{p}$. These are subsets of the unit sphere centered in $\mathbf{p}$. The characteristic function χ_{Γ^0} is defined by $\chi_{\Gamma^0}(\mathbf{v})=1$ if $\mathbf{v}\in\Gamma^0$ and $\chi_{\Gamma^0}(\mathbf{v})=0$ otherwise. The set Δ^0 is the projection of Δ onto the unit sphere. Δ^0 is a curve, or union of curves, plotted on the unit sphere. Its curvilinear abscissa is noted as s [Fig. 3(a)]. The Dirac function δ_{Δ^0} is defined such as $\int\delta_{\Delta^0}fdv=\int_{\Delta^0}fds$ for any function f of the unit sphere. It is said to have the support is Δ^0 because $\int\delta_{\Delta^0}fdv=0$ if f does not encounter Δ^0, that is $f(\mathbf{v})=0$ when $\mathbf{v}\in\Delta^0$. Similarly, Υ^0 is the projection of Υ onto the unit sphere. Υ^0 is a discrete set [Fig. 3(b)]. The Dirac function δ_{Υ^0} with support Υ^0 is defined such as $\int\delta_{\Upsilon^0}fdv=\Sigma_{\Upsilon^0}f$ any function f of the unit sphere. The support is Υ^0 meaning that $\int\delta_{\Upsilon^0}fdv=0$ whenever $f(\mathbf{v})=0$ if $\mathbf{v}\in\Upsilon^0$.

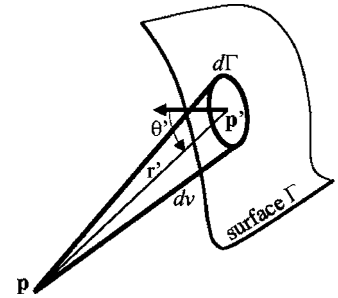

FIG. 4. Relation between the cone angle dv and the part $d\Gamma$ of the boundary enclosed in the cone. θ' is the emission angle measured with the normal to the surface and r' is the source-receiver distance.

First, let us assume that $\mathbf{p}'\in\Gamma$ and let develop the second term of the right-hand side of Eq. (12). The condition $\mathbf{p}'\in\Gamma$ is equivalent to $\mathbf{v}\in\Gamma^0$ and therefore $\chi_{\Gamma^0}(\mathbf{v})=1$. The infinitesimal surface $d\Gamma$ enclosed in the cone is $d\Gamma = r'^{n-1}\chi_{\Gamma^0}dv/\cos\theta'$ (Fig. 4), where θ' is the emission angle at $d\Gamma$ toward $\mathbf{p}$ and $r'=|\mathbf{p}'-\mathbf{p}|\cdot$ θ' is measured between the emission direction and the normal to the surface $d\Gamma$. This normal makes sense since $\mathbf{p}'$ is assumed to be regular. This relationship is trivial when $n=1$. The second term of the right-hand side of Eq. (12) then reads $dv\sigma e^{-mr'}\chi_{\Gamma^0}/(\cos\theta'\gamma_0)$.

Secondly, when $\mathbf{p}'\in\Delta(n=3)$, the length $d\Delta$ is related to its projection ds on the unit sphere by $d\Delta=r'ds/\sin\theta'$ where $r'=|\mathbf{p}'-\mathbf{p}|$ and θ' is the emission angle now measured between the tangent to Δ and $\mathbf{v}$ [Fig. 3(a)]. But the Dirac function δ_{Δ^0} verifies $\int\delta_{\Delta^0}dv=\int_{\Delta^0}ds$ and then $ds=\delta_{\Delta^0}dv$. The infinitesimal length is therefore $d\Delta=r'\delta_{\Delta^0}dv/\sin\theta'$. The third term of the right-hand side of Eq. (12), existing in the only case $n=3$, becomes $dv\lambda e^{-mr'}\delta_{\Delta^0}/(r'\sin\theta'\gamma_0)$.

Finally, when $\mathbf{p}'\in\Upsilon$, the last sum of Eq. (12) may be written $\Sigma\mu H=\delta_{\Upsilon^0}\mu H dv$. This is just a particular case of the definition equation of the function δ_{Υ^0}. $\delta_{\Upsilon^0}dv$ is the number of points of Υ enclosed in cone dv [Fig. 3(b)]. The fourth term of the right-hand side of Eq. (12) is therefore $dv\mu e^{-mr'}\delta_{\Upsilon^0}/(r'^{n-1}\gamma_0)$.

When Ω is bounded in direction $-\mathbf{v}$, Eq. (12) reads,

$$\begin{aligned} I(\mathbf{p},\mathbf{v},t) = \frac{1}{\gamma_0}\Bigg[&\int_{\mathbf{p}}^{\mathbf{p}'} \rho(\mathbf{s},t')e^{-mr}dr \\ &+ \frac{\sigma(\mathbf{p}',\mathbf{v},t')}{\cos\theta'}e^{-mr'}\chi_{\Gamma^0}(\mathbf{v}) \\ &+ \frac{\lambda(\mathbf{p}',\mathbf{v}',t')}{\sin\theta'}\frac{e^{-mr'}}{r'}\delta_{\Delta^0}(\mathbf{v}) \\ &+ \mu(\mathbf{p}',\mathbf{v},t')\frac{e^{-mr'}}{r'^{n-1}}\delta_{\Upsilon^0}(\mathbf{v})\Bigg], \end{aligned} \tag{14}$$

where $t'=t-r/c$ is the time delayed by the flight duration from the source to the receiver. In this equation, the functions χ_{Γ^0}, δ_{Δ^0}, and δ_{Υ^0} have disjoint supports. It means that depending on the position of the point $\mathbf{p}'$ on the boundary,

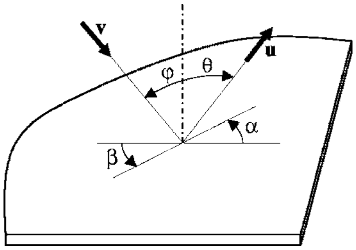

FIG. 5. Reflection on plane surface for incidence direction $\mathbf{v}=\varphi,\beta$ and reflection direction $\mathbf{u}=\theta,\alpha$. Elevation angles φ, θ are measured with the normal to the surface and azimuthal angles β, α are measured in the plane of the surface.

only one function among $\chi_{\Gamma^0}(\mathbf{v})$, $\delta_{\Delta^0}(\mathbf{v})$, $\delta_{\Upsilon^0}(\mathbf{v})$ is not null. For instance, if $\mathbf{p}'\in\Delta$ or equivalently $\mathbf{v}\in\Delta^0$, then $\mathbf{p}'\notin\Gamma$ and $\mathbf{p}'\notin\Upsilon$ and therefore $\chi_{\Gamma^0}(\mathbf{v})=\delta_{\Upsilon^0}(\mathbf{v})=0$. The right-hand side of Eq. (14), can just have one nonvanishing term among the last three terms (the third term is always canceled in dimension two). For this reason, it is not disturbing that θ' has different definitions in Γ and Δ. Equation (14) gives the radiative intensity in direction $\mathbf{v}$ in terms of all sources, ρ,σ,λ, and μ located on the path with direction $-\mathbf{v}$.

III. REFLECTION

Reflection of rays on the boundary may occur in different manners. Two extreme situations are of interest: diffuse reflection and specular reflection. The former is often encountered in acoustics when walls are rough and the second situation stands for perfect mirror for instance a plane hard wall or a straight free edge for structural rays. The case of general reflectivity is first considered before both particular cases are detailed. In all cases, a fictitious source layer σ being the density of reflected power is introduced on the surface Γ. It is then determined by an appropriate equation deduced from the power balance.

Let the *bidirectional reflectivity*[8] $R(\mathbf{v},\mathbf{u})$ of the boundary be the ratio of radiative intensity $I(\mathbf{p},\mathbf{u},t)$ reflected in direction $\mathbf{u}$ and the incident flux $I(\mathbf{p},\mathbf{v},t)\cos\varphi$ from direction $\mathbf{v}$ with incidence φ. At any point $\mathbf{p}$ of the boundary, the leaving radiative intensity, $I=d\mathcal{P}/\cos\theta dSdu$ in direction $\mathbf{u}$ with emission angle θ, is the sum of all incident fluxes $I\cos\varphi$ times the reflectivity R. Angles φ, θ and directions $\mathbf{v}$, $\mathbf{u}$ are defined in Fig. 5. It yields

$$I(\mathbf{p},\mathbf{u},t)=\frac{1}{\cos\theta}\frac{d\mathcal{P}}{dSdu}=\int R(\mathbf{v},\mathbf{u})I(\mathbf{p},\mathbf{v},t)\cos\varphi dv. \quad (15)$$

This is the so-called *detailed power balance*[10] which gives the power in any direction $\mathbf{u}$ from contributions of other directions $\mathbf{v}$. Consider an infinitesimal surface source $\mathbf{p}$ of area $d\Gamma$ with power $d\Gamma\sigma$. The flux of intensity $d\Gamma\sigma H$ through the infinitesimal solid angle du is $d\Gamma du\sigma/\gamma_0$. The meaning of σ is now apparent, σ/γ_0 is the reflected power per unit area of boundary and unit solid angle. Since the area normal to the ray is $d\Gamma\cos\theta$, the radiative intensity leaving the source $\mathbf{p}$ in direction $\mathbf{u}$ is from Eq. (10),

$$\frac{1}{\cos\theta}\frac{d\mathcal{P}}{dSdu}=\frac{\sigma(\mathbf{p},\mathbf{u},t)}{\gamma_0\cos\theta}. \quad (16)$$

Substitution of Eqs. (12) and (16) into Eq. (15) gives,

$$\frac{\sigma(\mathbf{p},\mathbf{u},t)}{\gamma_0\cos\theta}=\int_\Omega R(\mathbf{v},\mathbf{u})\rho(\mathbf{s},t')H(\mathbf{s},\mathbf{p})\cos\varphi d\Omega_{\mathbf{s}} + \int_\Gamma R(\mathbf{v},\mathbf{u})\sigma(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\cos\varphi d\Gamma_{\mathbf{q}} + \int_\Delta R(\mathbf{v},\mathbf{u})\lambda(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\cos\varphi d\Delta_{\mathbf{q}} + \sum_{\mathbf{q}\in\Upsilon} R(\mathbf{v},\mathbf{u})\mu(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\cos\varphi. \quad (17)$$

This is an integral equation which gives σ at any point $\mathbf{p}$ and in any direction $\mathbf{u}$ in terms of other volume, surface, line, and point sources.

Let us define the *hemispherical reflectivity*[8] $R(\mathbf{v})$ be the total flux leaving the boundary, for any unit incident flux from $\mathbf{v}$,

$$R(\mathbf{v})=\int R(\mathbf{v},\mathbf{u})\cos\theta du, \quad (18)$$

where the integral runs over the hemisphere of inward directions. By multiplying Eq. (15) by $\cos\theta$ and integrating over du gives,

$$\int I(\mathbf{p},\mathbf{u},t)\cos\theta du=\int R(\mathbf{v})I(\mathbf{p},\mathbf{v},t)\cos\varphi dv. \quad (19)$$

It is then apparent that the hemispherical reflectivity $R(\mathbf{v})$ is the ratio of the reflected power and the incident power from direction $\mathbf{v}$. This is a non-negative number less than 1 sometimes called *reflection efficiency*[11] in structural wave literature. Some surfaces have the property of constant bidirectional reflectivity for any fixed incidence $\mathbf{v}$. In this case, the bidirectional directivity is given by the hemispherical directivity $R(\mathbf{v},\mathbf{u})=R(\mathbf{v})/\gamma$, where $\gamma=\int\cos\theta du=1$, 2 or π for $n=1$, 2 or 3. For such surfaces, the reflection is said to be *diffuse*. When R does not depend on $\mathbf{u}$, the right-hand side of Eq. (17) does not depend on $\mathbf{u}$, too. Thus, the power density $\sigma(\mathbf{p},\mathbf{u},t)$ has the directivity given by Lambert's law,

$$\sigma(\mathbf{p},\mathbf{u},t)=\sigma(\mathbf{p},t)\cos\theta. \quad (20)$$

Multiplying Eq. (17) by $\cos\theta$ and integrating over the hemisphere of all emission directions $\mathbf{u}$ leads to,

$$\frac{\gamma}{\gamma_0}\sigma(\mathbf{p},t)=\int_\Omega R(\mathbf{v})\rho(\mathbf{s},t')H(\mathbf{s},\mathbf{p})\cos\varphi d\Omega_{\mathbf{s}} + \int_\Gamma R(\mathbf{v})\sigma(\mathbf{q},t')\cos\theta' H(\mathbf{q},\mathbf{p})\cos\varphi d\Gamma_{\mathbf{q}} + \int_\Delta R(\mathbf{v})\lambda(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\cos\varphi d\Delta_{\mathbf{q}} + \sum_{\mathbf{q}\in\Upsilon} R(\mathbf{v})\mu(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\cos\varphi, \quad (21)$$

where θ' is the emission angle at point $\mathbf{q}$. This is a Fredholm's integral equation of second kind on σ. The first two terms of this equation were first derived by Kuttruff[2,12] in the context of room acoustics. The equation was originally,

$$B(\mathbf{r},t) = B_0(\mathbf{r},t) + \int_S (1-\alpha) B\left(\mathbf{r}, t - \frac{R}{c}\right) K(\mathbf{r},\mathbf{r}')dS', \tag{22}$$

where $B=\sigma 1/(1-\alpha)$ is the irradiation density used as unknown, S is the enclosure, $1-\alpha$ is the reflection coefficient [noted as R in Eq. (21)], B_0 is the contribution of direct sources is [first integral in Eq. (21)], R the source-receiver distance, and $K=\cos\theta\cos\theta'/\pi R^2$. Assuming that $B(\mathbf{r},t)=B(\mathbf{r})e^{-\lambda t}$, Eq. (22) leads to an integral equation on reverberation time of rooms which applies beyond the validity of Sabine's formula and especially for rooms having atypical shapes. Some algorithms have been proposed for solving this integral equation[13–16] and even an original closed-form solution was found for spherical enclosures.[17] It was also numerically solved for early decaying of sound in Ref. 15, where it is also proved the existence and uniqueness of reverberation time. On the other hand, Eq. (22) is also useful to compute the SPL map in the steady-state condition[18–21] and then Eq. (22) is an alternative to the ray-tracing technique. More generally, Eq. (22) and its generalization to diffracting sources Eq. (21), embody all geometrical acoustics with diffuse reflecting surfaces.

In the case of specular reflection, the bidirectional reflectivity is given by,

$$R(\mathbf{v},\mathbf{u}) = R(\mathbf{v})\frac{\delta(\mathbf{v}-\mathbf{u}')}{\cos\varphi}, \tag{23}$$

where $\mathbf{u}'=\mathbf{u}-\mathbf{2}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$ is the incident direction which specularly reflects in $\mathbf{u}\cdot\mathbf{n}$ is the unit outward normal to the boundary. The hemispherical reflectivity is from Eq. (18), $\int R(\mathbf{v})\delta(\mathbf{v}-\mathbf{u}')\cos\theta/\cos\varphi du$, where $\mathbf{u}=\theta,\alpha$ and $\mathbf{v}=\varphi,\beta$ (Fig. 5). The change of variable $\mathbf{u}=\theta,\alpha\to\mathbf{u}'=\theta,\alpha+\pi$ gives $du'=du$ and the hemispherical reflectivity is therefore $\int R(\mathbf{v})\delta(\mathbf{v}-\mathbf{u}')\cos\theta/\cos\varphi du'=R(\mathbf{v})$. Substitution of Eq. (23) into Eq. (15) gives the equality $I(\mathbf{p},\mathbf{u}',t)=R(\mathbf{u}')I(\mathbf{p},\mathbf{u},t)$. For a perfect mirror $R(\mathbf{v})=1$ and therefore the incident radiative intensity $I(\mathbf{p},\mathbf{u}',t)$ equals the reflected radiative intensity $I(\mathbf{p},\mathbf{u},t)$. But for an absorbing mirror, $R(\mathbf{u}')$ is the ratio of reflected and incident powers. Expanding $I(\mathbf{p},\mathbf{u},t)$ with Eq. (16) and $I(\mathbf{p},\mathbf{u}',t)$ with Eq. (14), the above equality leads to

$$\begin{aligned}\frac{\sigma(\mathbf{p},\mathbf{u},t)}{\cos\theta} = R(\mathbf{u}')\Bigg[&\int_{\mathbf{p}}^{\mathbf{p}'}\rho(\mathbf{s},t')e^{-mr}dr \\ &+ \frac{\sigma(\mathbf{p}',\mathbf{u}',t')}{\cos\theta'}e^{-mr'}\chi_{\Gamma^0}(\mathbf{u}') \\ &+ \frac{\lambda(\mathbf{p}',\mathbf{u}',t')}{\sin\theta'}\frac{e^{-mr'}}{r'}\delta_{\Delta^0}(\mathbf{u}') \\ &+ \mu(\mathbf{p}',\mathbf{u}',t')\frac{e^{-mr'}}{r'^{n-1}}\delta_{\Upsilon^0}(\mathbf{u}')\Bigg],\end{aligned} \tag{24}$$

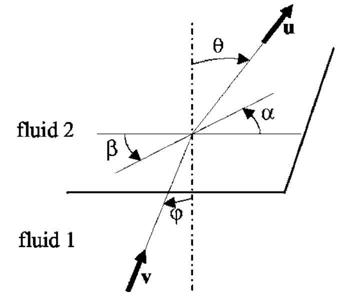

FIG. 6. Refraction at plane interface for incidence direction $\mathbf{v}=\varphi,\beta$ and refraction direction $\mathbf{u}=\theta$, α.

where $\mathbf{p}'$ is the point located on the boundary in direction $-\mathbf{u}'$ from $\mathbf{p}$. This is a functional equation on σ. The first two terms of Eq. (24) were derived in Ref. 22 as an alternative to Eq. (21) for specular reflection. However, in case of partially diffuse reflecting surfaces and then, partially specular reflecting surfaces, the most widely spread solution is rather based on an algorithm which jointly uses the view factor method and the image-source technique[23,24] In Ref. 22, it was pointed out that Eq. (24) can be solved by the image-source technique and that Eq. (24) is more generally equivalent to the ray-tracing technique with specular reflection. But to find an algorithm similar to the collocation method and valid beyond the image-source technique limited in practice to simple polyhedra shape, remains an open question.

IV. TRANSMISSION AND REFRACTION

When waves impinge on the interface separating two media with different phase speeds, they are partially reflected and partially refracted. This is the case for optical or acoustical rays passing through the air-water interface for instance or structural waves in plates at discontinuity of thickness. In some cases, the media are the same on both sides but, the interface is material and then also gives rise to reflection and transmission of waves. Transparency of acoustical waves through walls but also transmission of structural waves through the common edge of right-angled plates are some examples. A fictitious source layer σ_i, where i is a subscript referring to the medium, is introduced on each side of the interface. Equations on σ_i are a simple generalization of previous ones in case of reflection. These equations apply for any transmission and refraction, the details of the particular system at hand, coincidence frequencies, double-leaf panel resonance, air-gap resonance and so on, are rejected in expressions of transmittivity.

Let $R_{ji}(\mathbf{v},\mathbf{u})$ be the *bidirectional transmittivity* from medium j to medium i defined as in Sec. III. R_{ii} is simply the reflectivity of the boundary in medium i. The angles φ and θ are defined in Fig. 6. The leaving radiative intensity $I_i=d\mathcal{P}_i/\cos\theta dSdu$ in medium i is the sum of all fluxes incident from all media. The detailed power balance now reads,

$$I_i(\mathbf{p},\mathbf{u},t) = \frac{1}{\cos\theta}\frac{d\mathcal{P}i}{dSdu} = \sum_i \int R_{ji}(\mathbf{v},\mathbf{u}) I_j(\mathbf{p},\mathbf{v},t)\cos\varphi dv. \tag{25}$$

This equation generalizes Eq. (15). As in Sec. III, the equation on σ_i, is obtained by substituting Eqs. (12) and (16) into Eq. (25),

$$\begin{aligned}\frac{\sigma_i(\mathbf{p},\mathbf{u},t)}{\gamma_0\cos\theta} &= \sum_j \int_{\Omega_j} R_{ji}(\mathbf{v},\mathbf{u})\rho_j(\mathbf{s},t')H_j(\mathbf{s},\mathbf{p})\cos\varphi d\Omega_{\mathbf{s}} \\ &+ \int_{\Gamma_j} R_{ji}(\mathbf{v},\mathbf{u})\sigma_j(\mathbf{q},\mathbf{v},t')H_j(\mathbf{q},\mathbf{p})\cos\varphi d\Gamma_{\mathbf{q}} \\ &+ \int_{\Delta_j} R_{ji}(\mathbf{v},\mathbf{u})\lambda_j(\mathbf{q},\mathbf{v},t')H_j(\mathbf{q},\mathbf{p})\cos\varphi d\Delta_{\mathbf{q}} \\ &+ \sum_{\mathbf{q}\in\Upsilon_j} R_{ji}(\mathbf{v},\mathbf{u})\mu_j(\mathbf{q},\mathbf{v},t')H_j(\mathbf{q},\mathbf{p})\cos\varphi.\end{aligned} \tag{26}$$

This equation gives σ_i in terms of other sources.

When emitted energy is diffuse, the bidirectional transmittivity does not depend on the emission direction, $R_{ji}(\mathbf{v},\mathbf{u})=R_{ji}(\mathbf{v})/\gamma$ for all $\mathbf{u}$ where $R_{ji}(\mathbf{v})$ is the *hemispherical transmittivity or transmission efficiency* defined as in Eq. (18). The reflection and transmission sources follow Lambert's law (20) and Eq. (26) on unknown σ_i becomes,

$$\begin{aligned}\frac{\gamma}{\gamma_0}\sigma_i(\mathbf{p},t) &= \sum_j \int_{\Omega_j} R_{ji}(\mathbf{v})\rho_j(\mathbf{s},t')H_j(\mathbf{s},\mathbf{p})\cos\varphi d\Omega_{\mathbf{s}} \\ &+ \int_{\Gamma_j} R_{ji}(\mathbf{v})\sigma_j(\mathbf{q},t')\cos\theta' H_j(\mathbf{q},\mathbf{p})\cos\varphi d\Gamma_{\mathbf{q}} \\ &+ \int_{\Delta_j} R_{ji}(\mathbf{v})\lambda_j(\mathbf{q},\mathbf{v},t')H_j(\mathbf{q},\mathbf{p})\cos\varphi d\Delta_{\mathbf{q}} \\ &+ \sum_{\mathbf{q}\in\Upsilon_j} R_{ji}(\mathbf{v})\mu_j(\mathbf{q},\mathbf{v},t')H_j(\mathbf{q},\mathbf{p})\cos\varphi,\end{aligned} \tag{27}$$

θ' is the emission angle at $\mathbf{q}$ and $t'=t-r/c$. This is a set of Fredholm's integral equations of the second kind. This set of equations turns out to be a powerful tool to predict the repartition of energy in assembled plates in high frequency range.[6,7,25] The solving of this set of equations in steady-state condition is done by a collocation method. It then possible to compute the vibrational energy of each component of built-up structures but also to get the repartition of energy inside each subsystem. This is an improvement of (SEA) which just provides the total vibrational energy of subsystems. Indeed, this method is more time-computation consuming than SEA but significantly less time-consuming than FEM or BEM applied to the classical governing equations.

In case of perfect refraction, incoming rays $\mathbf{u}_j'$ with incidence φ_j in medium j which are refracted into a single ray $\mathbf{u}$ with angle θ in medium i are linked by Snell-Descartes' law of refraction, $\sin\varphi_j/c_j'=\sin\theta/c_i'$, where c_j' the phase speed in medium j. Rays $\mathbf{u}_j'$ and $\mathbf{u}$ are coplanar. The bidirectional transmittivity must read in such a manner that incident power from direction $\mathbf{u}_j'$ is refracted in the only direction $\mathbf{u}$ of medium i,

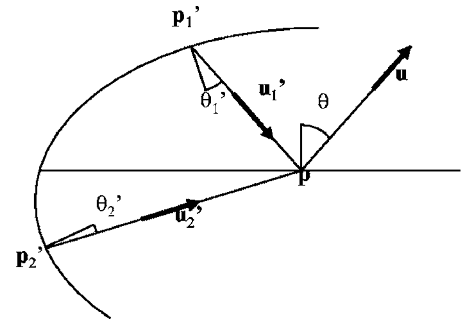

FIG. 7. The power emitted in direction $\mathbf{u}=\theta$, α is the sum of the reflection from direction $\mathbf{u}_1'$ and the transmission from direction $\mathbf{u}_2'$. $\mathbf{p}_i'$ is the point of the boundary which emits in the direction $\mathbf{u}_i'$ with emission angle θ_i'.

$$R_{ji}(\mathbf{v},\mathbf{u}) = \left(\frac{c_j'}{c_i'}\right)^{n-1} R_{ji}(\mathbf{v})\frac{\delta(\mathbf{v}-\mathbf{u}_j')}{\cos\varphi}. \tag{28}$$

To show that $R_{ji}(\mathbf{v})$ is the hemispherical transmittivity, we substitute Eq. (28) into Eq. (18), $\int(c_j'/c_i')^{n-1}R_{ji}(\mathbf{v})\delta(\mathbf{v}-\mathbf{u}_j')\cos\theta/\cos\varphi du$. The solid angles attached to the change of variables $\mathbf{u}=\theta,\ \alpha\to\mathbf{u}_j'=\varphi_j,\ \alpha+\pi$ are $du=\sin\theta d\theta d\alpha$ and $du_j'=\sin\theta_j' d\varphi_j' d\alpha$. They are related by $\cos\theta du/c_i'^{n-1} =\cos\varphi_j' du_j'/c_j'^{n-1}$ (derive the square of Snell-Descartes' equality). The hemispherical transmittivity is therefore $\int R_{ji}(\mathbf{v})\delta(\mathbf{v}-\mathbf{u}_j')\cos\varphi_j/\cos\varphi du_j'=R_{ji}(\mathbf{v})$ which justifies the form of Eq. (28).

Substitution of Eq. (28) into Eq. (25) leads to $I_i(\mathbf{p},\mathbf{u},t)=\Sigma_j(c_j'/c_i')^{n-1}R_{ji}(\mathbf{u}_j')I_j(\mathbf{p},\mathbf{u}_j',t)$. Thus, further substitution of Eqs. (16) and (14) into this equality gives some functional equations for σ_i,

$$\begin{aligned}\frac{\sigma_i(\mathbf{p},\mathbf{u},t)}{\cos\theta} = \sum_j \left(\frac{c_j'}{c_i'}\right)^{n-1} R_{ji}(\mathbf{u}_j') \Bigg[&\int_{\mathbf{p}}^{\mathbf{p}_j'} \rho_j(\mathbf{s},t')e^{-m_j r}dr \\ &+ \frac{\sigma_j(\mathbf{p}_j',\mathbf{u}_j',t')}{\cos\theta_j'} e^{m_j r'}\chi_{\Gamma_j^0}(\mathbf{u}_j') \\ &+ \frac{\lambda_j(\mathbf{p}_j',\mathbf{u}_j',t')}{\sin\theta_j'}\frac{e^{-m_j r'}}{r'}\delta_{\Delta_j^0}(\mathbf{u}_j') \\ &+ \mu_j(\mathbf{p}_j',\mathbf{u}_j',t')\frac{e^{-m_j r'}}{r'^{n-1}}\delta_{\Upsilon_j^0}(\mathbf{u}_j')\Bigg],\end{aligned} \tag{29}$$

where $\mathbf{p}_j'$ is the first point of the boundary Γ_j encountered in direction $-\mathbf{u}_j'$ from $\mathbf{p}$ and θ_j' is the emission angle at $\mathbf{p}_j'$ (Fig. 7). This is a set of functional equations on unknowns σ_i. Equation (29) has been solved for a couple of plates in Ref. 26, whereas some features of Eq. (29) have been discussed in Ref. 22. In particular, it has been shown that Eq. (29) is not symmetrical under time reversing. This is due to the underlying assumption that rays are uncorrelated. To neglect the phase between incoming rays is not equivalent that to neglect the phase between outcoming rays.

Two expressions of transmission efficiency are given in the Appendix for the cases of refraction between two acoustical media and transmission through single walls. Many other cases are tackled in the literature, for instance, transmission through double walls is studied in Refs. 27 and 28, transmission of structural waves at joint of assembled beams in Refs. 29 and 30, and at a joint of assembled plates in Refs. 31 and 32.

V. DIFFRACTION

Geometrical Theory of Diffraction (GTD) introduces diffraction effects in geometrical acoustics. Whereas the condition for existence of classical rays is given by Fermat's principle expressing the stationarity of ray path, existence of diffracted rays follows from the generalized Fermat's principle which states that ray paths with constraints have an extremum optical length.[5] It leads to the existence of a new class of rays diffracted by wedges, peaks, and corners, but also creeping rays in case of diffraction by smooth obstacles. In this section, we just consider diffraction by wedges and corners but not diffraction by smooth obstacles. A fictitious source layer λ is introduced along diffracting edges and some fictitious sources μ are laid on vertices of peaks and corners of plates.

Let define the *bidirectional diffractivity* $D(\mathbf{v},\mathbf{u})$ with incidence $\mathbf{v}$ and emission direction $\mathbf{u}$, by analogy with the bidirectional reflectivity. For diffraction by corners and peaks, $D(\mathbf{v},\mathbf{u})$ is defined as the ratio of the emitted power $d\mathcal{P}$ per unit solid angle du about $\mathbf{u}$ and the incident radiative intensity I in direction $\mathbf{v}$. The detailed power balance is thus,

$$\frac{d\mathcal{P}}{du} = \int D(\mathbf{v},\mathbf{u}) I(\mathbf{p},\mathbf{v},t)dv. \tag{30}$$

The infinitesimal emitted power $d\mathcal{P}$ for a point source μ is the flux of $\mu\mathbf{H}$ through a small area $\epsilon^{n-1}du$. of the sphere of radius ϵ, $d\mathcal{P}=\mu e^{-m\epsilon}/\gamma_0\epsilon^{n-1}\times\epsilon^{n-1}du$. When ϵ goes to zero,

$$\frac{d\mathcal{P}}{du} = \frac{\mu(\mathbf{p},\mathbf{u},t)}{\gamma_0}. \tag{31}$$

The equation on μ is then derived by substituting Eqs. (12) and (31) into Eq. (30),

$$\begin{aligned}\frac{\mu(\mathbf{p},\mathbf{u},t)}{\gamma_0} &= \int_\Omega D(\mathbf{v},\mathbf{u})\rho(\mathbf{s},t')H(\mathbf{s},\mathbf{p})d\Omega_{\mathbf{s}} \\ &+ \int_\Gamma D(\mathbf{v},\mathbf{u})\sigma(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})d\Gamma_{\mathbf{q}} \\ &+ \int_\Delta D(\mathbf{v},\mathbf{u})\lambda(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})d\Delta_{\mathbf{q}} \\ &+ \sum_{\mathbf{q}\in\Upsilon} D(\mathbf{v},\mathbf{u})'\mu(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p}).\end{aligned} \tag{32}$$

This equation gives the power μ in terms of powers of other sources.

For diffraction by wedge whose edge has a length measure noted ν, the *bidirectional diffractivity* $D(\mathbf{v},\mathbf{u})$ is defined as the ratio of the emitted power $d\mathcal{P}$ per unit solid angle du about $\mathbf{u}$ and per unit length $\sin\theta d\nu$ normal to the ray, and the power density $I(\mathbf{p},\mathbf{v},t)\sin\varphi$ incident on the edge,

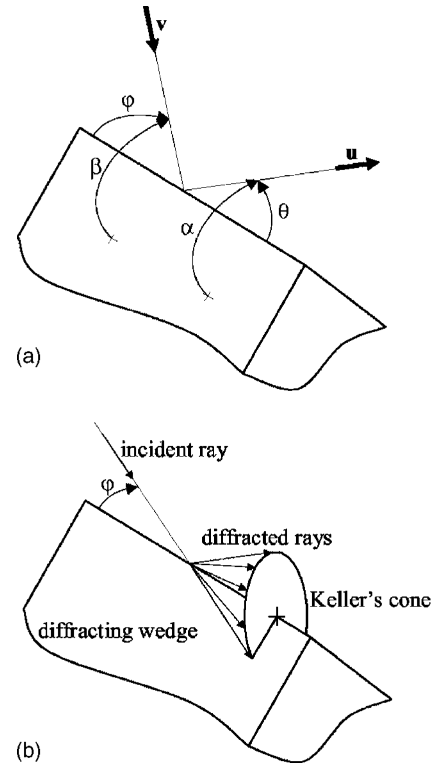

FIG. 8. Diffraction by wedge. (a) Elevation angles and φ, θ are measured with the tangent to the edge and azimuthal angles and β, α are measured in the plane normal to the edge of the wedge. (b) Keller's cone is the set of emission directions for which $\theta=\varphi$.

$$\frac{1}{\sin\theta}\frac{d\mathcal{P}}{d\nu du} = \int D(\mathbf{v},\mathbf{u})I(\mathbf{p},\mathbf{v},t)\sin\varphi dv. \tag{33}$$

Directions $\mathbf{v}=\varphi,\beta,\mathbf{u}=\theta,\alpha$ are defined in Fig. 8(a). For a line source λ, the infinitesimal emitted power per unit length $d\mathcal{P}/d\nu$ is the flux of $\lambda\mathbf{H}$ through a part $\epsilon^2 du$ of sphere and therefore, $d\mathcal{P}/d\nu=\lambda e^{-m\epsilon}/4\pi\epsilon^2\times\epsilon^2du$. When ϵ goes to zero,

$$\frac{1}{\sin\theta}\frac{d\mathcal{P}}{d\nu du} = \frac{\lambda(\mathbf{p},\mathbf{u},t)}{4\pi\sin\theta}. \tag{34}$$

The equation on λ is obtained by substitution of Eqs. (12) and (34) into Eq. (33),

$$\begin{aligned}\frac{\lambda(\mathbf{p},\mathbf{u},t)}{4\pi\sin\theta} &= \int_\Omega D(\mathbf{v},\mathbf{u})\rho(\mathbf{s},t')H(\mathbf{s},\mathbf{p})\sin\varphi d\Omega_{\mathbf{s}} \\ &+ \int_\Gamma D(\mathbf{v},\mathbf{u})\sigma(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\sin\varphi d\Gamma_{\mathbf{q}} \\ &+ \int_\Delta D(\mathbf{v},\mathbf{u})\lambda(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\sin\varphi d\Delta_{\mathbf{q}} \\ &+ \sum_{\mathbf{q}\in\Upsilon} D(\mathbf{v},\mathbf{u})\mu(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\sin\varphi.\end{aligned} \tag{35}$$

This is an integral equation on the unknown λ.

Until now, it has been tacitly assumed that rays impinging on wedges, peaks, and corners may be diffracted in any direction. However, the generalized Fermat's principle specifies which ray paths are admissible for diffraction. It is found that corners and peaks diffract in all directions, whereas wedges only diffract in the so-called *Keller's cone*. For an incident ray with direction $\mathbf{v}=\varphi,\beta$, Keller's cone is the set of all emission directions $\mathbf{u}=\theta,\ \alpha$ verifying $\theta=\varphi$ [Fig. 8(b)]. The equality of incidence and emission angles is known as Keller's law of diffraction. Let us introduce the *reciprocal Keller's cone K* of direction $\mathbf{u}$ as being the set of all incidence directions $\mathbf{v}$ whose incidence angle φ is equal to the emission angle θ. All energy emerging from the wedge in direction $\mathbf{u}$ stems from the reciprocal Keller's cone. The following form of the bidirectional diffractivity satisfies this condition,

$$D(\mathbf{v},\mathbf{u}) = D(\beta,\alpha)\frac{\delta(\varphi-\theta)}{\sin\varphi\sin\theta}. \tag{36}$$

$D(\beta,\alpha)$ is related to the classical diffraction coefficient d at normal incidence used in GTD by $D(\beta,\alpha)=|d(\beta,\alpha)|^2$. Some expressions for the diffractivity $D(\beta,\alpha)$ are given in the Appendix.

For the particular diffractivity given in Eq. (36), the equation on the unknown λ is obtained by multiplying Eq. (35) by $\sin\theta$ and by substituting Eq. (36). Four integrals then appear in right-hand side which must be carefully evaluated. Since the diffractivity of Eq. (36) contains a Dirac function, the integrands $d\Omega$, $d\Gamma$, $d\Delta$ of these integrals reduces to the surface of reciprocal Keller's cone. Then, let introduce the notation K for the reciprocal Keller's cone, $L=K\cap\Gamma$ for the trace of the reciprocal Keller's cone on the regular boundary and $M=K\cap\Delta$ the discret set of points of Δ lying on the reciprocal Keller's cone. The surface measure on the reciprocal Keller's cone is $dK=r\sin\varphi d\beta dr$ [Fig. 9(a)]. Since the infinitesimal volume is $d\Omega=r^2\sin\varphi d\varphi d\beta dr$ in spherical coordinates, it related to dK by $d\Omega=rd\varphi dK$. The first integral of Eq. (35) multiplied by $\sin\theta$ becomes $\int\delta(\varphi-\theta)D\rho H\times rd\varphi dK=\int_K D\rho HrdK$, where D of the right-hand side designates $D(\beta,\alpha)$ of Eq. (36). For the second integral, the infinitesimal surface is $d\Gamma=r^2\sin\varphi d\beta/\cos\theta'$, where as usual θ' is the emission angle (Fig. (4)). The length measure is $dL=r\sin\varphi d\beta/\sin\chi$, where χ is the angle between L and the generating line of K [Fig. 9(b)]. Thus, $d\Gamma=r\sin\chi/\cos\theta'\times d\varphi dL$. The second integral is therefore $\int_L D\sigma Hr\sin\chi/\cos\theta' dL$. The third integral reads $\int\delta(\varphi-\theta)D\lambda Hd\Delta=\Sigma D\lambda H\times d\Delta/d\varphi$ where the sum runs over the set $M=K\cap\Delta$. But $d\Delta/d\varphi=r\cos\psi$, where ψ is the angle between Δ and the normal to Keller's cone [Fig. 9(c)]. Finally, the fourth term is $\Sigma\delta(\varphi-\theta)D\mu H$ and the equation on λ for diffraction by wedges is

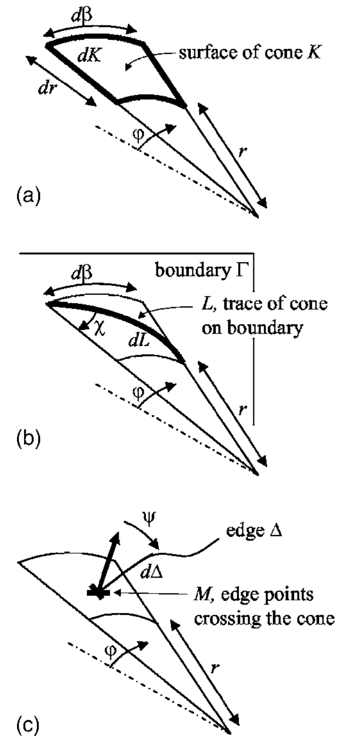

FIG. 9. (a) Surface measure on the reciprocal Keller's cone K. (b) Length measure on the curve $L=K\cap\Gamma$, χ is the angle between the generating line of the cone and the line L. (c) Discrete set $M=K\cap\Delta$, ψ is the angle measured between the tangent to Δ and the normal to K.

$$\begin{aligned}\lambda(\mathbf{p},\mathbf{u},t) &= \int_K D(\beta,\alpha)\rho(\mathbf{s},t')\frac{e^{-mr}}{r}dK_{\mathrm{s}} \\ &\quad + \int_L D(\beta,\alpha)\sigma(\mathbf{q},\mathbf{v},t')\frac{\sin\chi}{\cos\theta'}\frac{e^{-mr}}{r}dL_{\mathbf{q}} \\ &\quad + \sum_{\mathbf{q}\in M} D(\beta,\alpha)\frac{\lambda(\mathbf{q},\mathbf{v},t')}{\cos\psi}\frac{e^{-mr}}{r} \\ &\quad + \sum_{\mathbf{q}\in Y} D(\beta,\alpha)\mu(\mathbf{q},\mathbf{v},t')\frac{e^{-mr}}{r^2}\delta(\varphi-\theta). \end{aligned} \tag{37}$$

This is a functional equation on λ.

Diffraction by the top of noise barriers or around buildings are two common examples of multidiffraction where rays can be diffracted more than once. These problems are usually solved by BEM for low frequencies and by GTD for higher frequencies. For instance, Pierce[33] gives an approximate expression for double-edge diffraction by thick three-sided barriers using GTD. This solution is assessed for wide barriers by Kurze[34] while Medwin *et al.*[35] generalize this result in the time domain. However, all these methods require a finite number of rays between the diffracting edges and then are limited to a finite order of diffraction. Equations (32) and (37) are an alternative ray method for multiple diffraction. They lead to a finite set of linear equations where the unknowns are the diffracted powers. Solving these equations allows to account the infinite number of diffractions in a single step[36,37] but does not account for interference effects between edges.

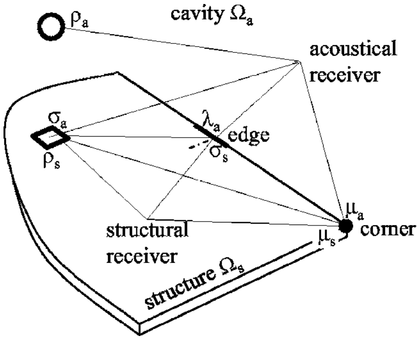

FIG. 10. Actual and fictitious sources contributing to the energy in structure and in acoustics. Structural sources are noted as ρ_s, reflection sources on edge are σ_s, and diffraction sources of corners are μ_s. Acoustical sources are noted as ρ_a, surface radiation sources are σ_a (surface mode radiation), line radiation sources are λ_a (edge mode radiation), and point radiation sources are μ_a (corner mode radiation).

VI. RADIATION OF SOUND

Radiation of sound is usually described using two different approaches. The modal approach[38,39] introduces a radiation factor for any mode of the radiating structure. It is well adapted to identify which modes are responsible of the radiation. The wave approach is more appropriate to ray theories. From this point of view, radiation occurs in three situations.[40] First, structural rays continuously loss their energy when traveling and then give rise to radiation of acoustical rays. This is the so-called surface mode radiation. Second, structural rays when impinging on edges are partially reflected and partially diffracted into acoustics. This is the edge mode radiation. Finally, singular points of structures are also responsible of diffraction of structural rays. This is the corner mode radiation. The subscript s is introduced for quantities related to structure and a for acoustics. The sound speed is noted c_0. Then, a fictitious source layer σ_a, density of radiated power, is introduced at any point of the radiating surface. For edge mode radiation, a fictitious source σ_s is introduced on the edge for reflection into structure and a source λ_a for radiation. Finally, some point sources μ_s for structure and μ_a a for acoustics are laid on each singular point of the structure (Fig. 10).

Radiation by surface mode only occurs beyond the coincidence frequency when structural waves are supersonic. Let us introduce a *bidirectional radiation coefficient* $A_{sa}(\mathbf{v},\mathbf{u})$ as the acoustical radiative intensity in direction $\mathbf{u}$ for a unit incident structural radiative intensity in direction $\mathbf{v}$. The directions $\mathbf{v}=\beta$ and $\mathbf{u}=\theta$, α and their related angles are drawn in Fig. 11. The detailed power balance then states that the radiated intensity is the sum of all contributions of incident structural rays,

$$I(\mathbf{p},\mathbf{u},t)=\frac{1}{\cos\theta}\frac{d\mathcal{P}}{dSdu}=\int A_{sa}(\mathbf{v},\mathbf{u})I(\mathbf{p},\mathbf{v},t)dv. \tag{38}$$

In the mean time, the attenuation factor m_s for structure is the sum of a term for internal losses $\eta\omega/c_s$, where η is the damping loss factor and ω is the circular frequency, and an additional term for radiation. Consider a structural plane wave with radiative intensity $I=I_0\delta(\mathbf{v}-\mathbf{v}_0)$. The decrease of intensity thickness dx is $-dI_0=m_sI_0dx$. Internal losses are $\eta\omega/c_s\times I_0dx$, whereas radiation losses are $2d\mathcal{P}/dS\times dx$. The factor 2 stems from the presence of fluid on both sides of the structure. From Eq. (38), $d\mathcal{P}/dS\times dx=\int A_{sa}\cos\theta du\times I_0dx$ and therefore the power balance states,

$$m_s=\frac{\eta\omega}{c_s}+2\int A_{sa}(\mathbf{v}_0,\mathbf{u})\cos\theta du. \tag{39}$$

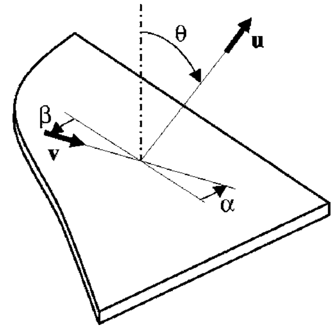

FIG. 11. Surface mode radiation. Elevation angle θ is measured with the normal to the surface and azimuthal angles β, α are measured in the plane of the surface.

The integral does not depend on direction $\mathbf{v}_0$ for isotropic structure. Now, the acoustical surface sources σ_a distributed over the structure provide this energy lost by the structure. Equation (16) gives the left-hand side of Eq. (38), whereas the right-hand side is obtained by substituting Eq. (12),

$$\begin{aligned}\frac{\sigma_a(\mathbf{p},\mathbf{u},t)}{4\pi\cos\theta}&=\int_{\Omega_s}A_{sa}(\mathbf{v},\mathbf{u})\rho_s(\mathbf{s},t')H(\mathbf{s},\mathbf{p})d\Omega_{\mathbf{s}}\\&\quad+\int_{\Gamma_s}A_{sa}(\mathbf{v},\mathbf{u})\sigma_s(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})d\Gamma_{\mathbf{q}}\\&\quad+\sum_{\mathbf{q}\in\Upsilon_s}A_{sa}(\mathbf{v},\mathbf{u})\mu_s(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p}).\end{aligned} \tag{40}$$

In this equation, Ω_s denotes the structural domain, Γ_s its boundary, and Υ_s the set of diffracting points. Snell-Descartes' law states that the emission direction $\mathbf{u}$ has a polar angle θ_0, measured with the normal to the surface, such as $1/c_s'=\sin\theta_0/c_0$. The radiated ray $\mathbf{u}$ has also an azimuthal angle β, measured in the plane of structure, equal to the azimuthal angle α of structural ray $\mathbf{v}$. Then, the bidirectional radiation coefficient A_{sa} reduces to the particular form,

$$A_{sa}(\mathbf{v},\mathbf{u})=A_{sa}\delta(\alpha-\beta)\frac{\delta(\theta-\theta_0)}{\sin\theta_0}, \tag{41}$$

where A_{sa} in the second-hand side is a parameter depending on structure and fluid properties. This parameter and the related attenuation factor m_s are given in the Appendix for lightly loaded thin plates while the most general case of fluid-loaded thick plates is discussed in Ref. 41. Now substitution of Eq. (41) into Eq. (40) leads to the equation on σ_a,

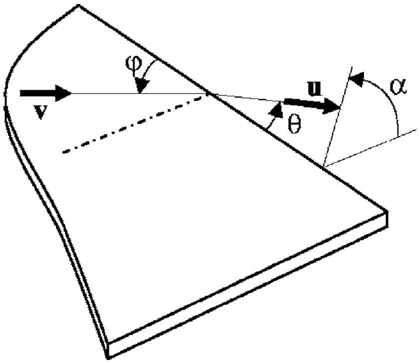

FIG. 12. Edge mode radiation. Elevation angle φ, θ is measured with the tangent to the edge and azimuthal angle α is measured in the plane of the edge.

$$\frac{\sigma_a(\mathbf{p},\mathbf{u},t)}{2\cot\theta_0} = A_{sa}\delta(\theta-\theta_0)\times\left[\int_{\mathbf{p}}^{\mathbf{p}'}\rho_s(\mathbf{s},t')e^{-m_s r}dr + \frac{\sigma_s(\mathbf{p}',\mathbf{u}',t')}{\cos\theta'}e^{-m_s r'}\chi_{\Gamma_s^0(\mathbf{u}')} + \mu_s(\mathbf{p}',\mathbf{u}',t')\frac{e^{-m_s r'}}{r'}\delta_{\Upsilon_s^0}(\mathbf{u}')\right], \tag{42}$$

where $\mathbf{u}'=\alpha$ and $\mathbf{u}=\theta,\alpha$. $\mathbf{p}'$ is as usual the first point of the boundary in direction $-\mathbf{u}'$. This equation gives the fictitious sources σ_a for surface mode radiation in terms of structural sources ρ_s, σ_s, and μ_s. It is solved in Ref. 42 for the case of a beam radiating in a two-dimensional acoustic medium. Radiation only occurs in direction θ_0 and therefore, sound is emitted in two strips. The intersection of these strips is a double zone with a higher sound pressure level and outside of the strips is a shadow zone.

Radiation by edge mode occurs at any frequency. This is a particular case of diffraction with a structural ray diffracted into fluid. The structure to acoustic bidirectional diffractivity for edge is defined as in Eq. (33),

$$\frac{1}{\sin\theta}\frac{d\mathcal{P}}{d\nu du} = \int D_{sa}^e(\mathbf{v},\mathbf{u})I(\mathbf{p},\mathbf{v},t)\sin\varphi dv, \tag{43}$$

where the incident direction $\mathbf{v}=\varphi$ belongs to the plane of structure and the radiation direction $\mathbf{u}=\theta$, α is in the fluid (Fig. 12). Structural waves are partially reflected into structure itself and partially diffracted into acoustics. Thus, the edge has a reflection efficiency R_s less than unity. Consider a structural plane wave with radiative intensity $I=I_0\delta(\mathbf{v}-\mathbf{v}_0)$ incident upon the edge. The diffracted power per unit length is given by Eq. (43), $d\mathcal{P}/d\nu=I_0\sin\varphi_0\int D_{sa}\sin\theta du$. This power is not reflected into structure and therefore the power balance imposes,

$$R_s(\mathbf{v}_0) = 1-\int D_{sa}^e(\mathbf{v}_0,\mathbf{u})\sin\theta du. \tag{44}$$

This reflection efficiency R_s depends on the incident direction $\mathbf{v}_0$. The exact expression for the reflection coefficient R_s of membranes is derived in Ref. 43 while the case of baffled and nonbaffled plates is solved in Ref. 44. The unknown σ_s is determined by applying Eq. (17) with the reflection efficiency R_s. Indeed Eqs. (21) and (24) must be preferred in case of diffuse and specular reflection. The energy converted into acoustical waves is emanated by some acoustical sources distributed along the edge of the structure. Their power per unit length λ_a is determined by,

$$\frac{\lambda_a(\mathbf{p},\mathbf{u},t)}{4\pi\sin\theta} = \int_{\Omega_s} D_{sa}^e(\mathbf{v},\mathbf{u})\rho_s(\mathbf{s},t')H(\mathbf{s},\mathbf{p})\sin\varphi d\Omega_{\mathbf{s}} + \int_{\Gamma_s} D_{sa}^e(\mathbf{v},\mathbf{u})\sigma_s(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\sin\varphi d\Gamma_{\mathbf{q}} + \sum_{\mathbf{q}\in\Upsilon_s} D_{sa}^e(\mathbf{v},\mathbf{u})\mu_s(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\sin\varphi. \tag{45}$$

This is the most general equation governing the line source λ_a. But as in diffraction by wedges, acoustical rays are radiated in Keller's cone. However, incidence and emission angles measured with the tangent to the edge are now related by $\cos\varphi/c_s'=\cos\theta/c_0$. Since the incident direction is restricted to be in the plane of structure, the reciprocal Keller's cone of a radiated ray $\mathbf{u}$ now reduces to a single direction noted $\mathbf{u}'=\varphi'$. The bidirectional diffractivity takes the form,

$$D_{sa}^e(\mathbf{v},\mathbf{u}) = \frac{c_s'}{c_0}D_{sa}^e(\alpha)\frac{\delta(\varphi-\varphi')}{\sin^2\varphi}, \tag{46}$$

where $D_{sa}^e(\alpha)$ is the square of the classical diffraction coefficient at normal incidence. When introducing Eq. (46) into Eq. (45) and, as usual, developing the infinitesimal surface $d\Omega=rdrd\varphi$ and the infinitesimal boundary length $d\Gamma=rd\varphi/\cos\theta'$, the equation on λ_a is obtained,

$$\frac{\lambda_a(\mathbf{p},\mathbf{u},t)}{2\sin\theta} = \frac{c_s'}{c_0}D_{sa}^e(\alpha)\frac{1}{\sin\varphi'}\left[\int_{\mathbf{p}}^{\mathbf{p}'}\rho_s(\mathbf{s},t')e^{-mr}dr + \frac{\sigma_s(\mathbf{p}',\mathbf{u}',t')}{\cos\theta'}e^{-mr'}\chi_{\Gamma_s^0(\mathbf{u}')} + \mu_s(\mathbf{p}',\mathbf{u}',t')\frac{e^{-mr'}}{r'}\delta_{\Upsilon_s^0}(\mathbf{u}')\right]. \tag{47}$$

The power density λ_a is then related to structural sources ρ_s, σ_s, and μ_s.

Radiation by corner mode also occurs at any frequency. The bidirectional diffractivity, as in Eq. (30), is defined by the following detailed power balance:

$$\frac{d\mathcal{P}}{du} = \int D_{sa}^c(\mathbf{v},\mathbf{u})I(\mathbf{p},\mathbf{v},t)dv, \tag{48}$$

where $\mathbf{v}$ is in the structure and $\mathbf{u}$ is any direction in the fluid (Fig. 13). The left-hand side of Eq. (48) is given by Eq. (31) and the right-hand side by Eq. (12),

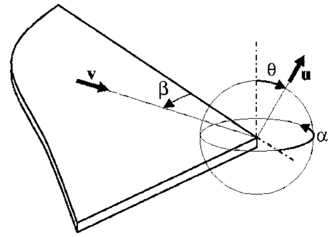

FIG. 13. Corner mode radiation. Elevation angle θ is measured with the normal to the surface and azimuthal angles β, α are measured in the plane of the surface.

$$\frac{\mu_a(\mathbf{p},\mathbf{u},t)}{4\pi} = \int_{\Omega_s} D^c_{sa}(\mathbf{v},\mathbf{u})\rho_s(\mathbf{s},t')H(\mathbf{s},\mathbf{p})d\Omega_s + \int_{\Gamma_s} D^c_{sa}(\mathbf{v},\mathbf{u})\sigma_s(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})d\Gamma_{\mathbf{q}} + \sum_{\mathbf{q}\in\Upsilon_s} D^c_{sa}(\mathbf{v},\mathbf{u})\mu_s(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p}). \tag{49}$$

The fictitious acoustical sources μ_a are related to structural sources ρ_s, σ_s, and μ_s. Concerning the sources μ_s at the same corner, this is a structural diffraction problem. Thus, μ_s is given by Eq. (32) with the appropriate diffraction coefficient.

VII. STRUCTURAL RESPONSE

Reflection, transmission, and sound absorption by walls has been addressed in Secs. III and IV. But in some cases, the acoustical energy lost during reflection is converted into structural energy. This phenomenon followed by radiation of sound is named *resonant transparency* in SEA literature meaning that structural modes are involved. An overview of sound transmission through infinite and finite structures is available in Ref. 45. Acoustic to structure conversion mode is the reciprocal problem of sound radiation and since all ray paths can be traveled in both directions, any radiation mode *a priori* gives rise to a conversion mode and conversely. The case of rays impinging on the surface of the structure is first considered. Diffraction of acoustical rays into structure by edges is modeled by introducing some fictitious sources σ_s for the structure and λ_a for acoustics while diffraction by edges or any singular point requires some point sources μ_s and μ_a.

This is a well-known result that a plane wave impinging on an infinite structure is reflected, transmitted, and that a forced term appears in the structure. Fictitious sources σ_a for reflection and transmission are given by Eqs. (26), (27), and (29) depending on the type of transmittivity. The incident power is totally reflected and transmitted and no energy is supplied to the structure. This result holds even at θ_0 incidence and thus, seems to state that the reciprocal path of surface radiated rays does not exist. However it does exist, but for an incident wave having a complex wave number. In the presence of fluid on both sides of the structure, a further requirement is that two rays simultaneously strike the structure at θ_0 incidence and at the same point. This phenomenon is possible but improbable. Anyway, within the framework of the present theory, correlation of rays have been neglected and thus, this type of conversion is assumed to be never realized.

Equations (7) and (8) give the contribution of the free waves propagating inside the structure. But, the presence of a forced term in the structure can lead to a significant increase of vibrational level and therefore cannot be neglected. Let define the forcing coefficient A_0 as the ratio of the vibrational energy of the forced term and the incident acoustical power of a plane wave (Appendix . The total vibrational energy of the forced term W_0 is then the sum,

$$W_0(\mathbf{r},t) = \int A_0(\varphi)I(\mathbf{r},\mathbf{v},t)\cos\varphi dv, \tag{50}$$

for all incident waves. Introducing Eq. (12), the forced term becomes,

$$W_0(\mathbf{r},t) = \int_{\Omega_a} A_0(\varphi)\rho(\mathbf{s},t')H(\mathbf{s},\mathbf{r})\cos\varphi d\Omega_{\mathbf{s}} + \int_{\Gamma_a} A_0(\varphi)\sigma(\mathbf{p},\mathbf{v},t')H(\mathbf{p},\mathbf{r})\cos\varphi d\Gamma_{\mathbf{p}} + \int_{\Delta_a} A_0(\varphi)\lambda(\mathbf{p},\mathbf{v},t')H(\mathbf{p},\mathbf{r})\cos\varphi d\Delta_{\mathbf{p}} + \sum_{\mathbf{p}\in\Upsilon_a} A_0(\varphi)\mu(\mathbf{p},\mathbf{v},t')H(\mathbf{p},\mathbf{r})\cos\varphi. \tag{51}$$

The vibrational energy in the structure is therefore the sum $W(\mathbf{r},t)+\mathbf{W}_0(\mathbf{r},t)$, where W is the energy of the free term given by Eq. (7) and W_0 the energy of the forced term given by Eq. (51).

Conversion by edge is once again a diffraction problem. The bidirectional diffractivity $D_{as}(\mathbf{v},\mathbf{u})$ is defined as in Eq. (33),

$$\frac{1}{\sin\theta}\frac{d\mathcal{P}}{dvdu} = \int D^e_{as}(\mathbf{v},\mathbf{u})I(\mathbf{p},\mathbf{v},t)\sin\varphi dv. \tag{52}$$

All the energy impinging on the edge is either diffracted into structural wave or diffracted into acoustical wave. The acoustical diffraction sources λ_a have ever been found in Eq. (37) and the structural diffraction sources σ_s distributed along the edge are given by

$$\frac{\sigma_s(\mathbf{p},\mathbf{u},t)}{2\pi\sin\theta} = \int_{\Omega_a} D^e_{as}(\mathbf{v},\mathbf{u})\rho_a(\mathbf{s},t')H(\mathbf{s},\mathbf{p})\sin\varphi d\Omega_{\mathbf{s}} + \int_{\Gamma_a} D^e_{as}(\mathbf{v},\mathbf{u})\sigma_a(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\sin\varphi d\Gamma_{\mathbf{q}} + \int_{\Delta_a} D^e_{as}(\mathbf{v},\mathbf{u})\lambda_a(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\sin\varphi d\Delta_{\mathbf{q}} + \sum_{\mathbf{q}\in\Upsilon_a} D^e_{as}(\mathbf{v},\mathbf{u})\mu_a(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p})\sin\varphi. \tag{53}$$

The power densities σ_s are then related to acoustical sources $\rho_a, \sigma_a, \lambda_a$, and μ_a. For any ray $\mathbf{u}$ in the structure is attached a reciprocal Keller's cone whose axis is the edge and angle φ'

is defined by the law of diffraction $\cos \varphi'/c_0 = \cos \theta / c'_s$. The bidirectional diffractivity taking into account this law of diffraction is,

$$D^e_{as}(\mathbf{v},\mathbf{u}) = \frac{c_0}{c'_s} D^e_{as}(\beta) \frac{\delta(\varphi - \varphi')}{\sin^2 \varphi}. \tag{54}$$

Substitution of Eq. (54) into Eq. (53) is done in same condition as for wedge diffraction. The measures $d\Omega$, $d\Gamma$, and $d\Delta$ are developed in spherical coordinates with the result,

$$\begin{aligned} 2\frac{\sigma_s(\mathbf{p},\mathbf{u},t)}{\sin \theta} = \frac{c_0}{c'_s}\frac{1}{\sin \varphi'}\Bigg[& \int_K D^e_{as}(\beta)\rho_a(\mathbf{s},t')\frac{e^{-mr}}{r} dK_{\mathbf{s}} \\ & + \int_L D^e_{as}(\beta)\sigma_a(\mathbf{q},\mathbf{v},t')\frac{\sin \chi}{\cos \theta'}\frac{e^{-mr}}{r} dL_{\mathbf{q}} \\ & + \sum_{\mathbf{q}\in M} D^e_{as}(\beta)\frac{\lambda_a(\mathbf{q},\mathbf{v},t')}{\cos \psi}\frac{e^{-mr}}{r} \\ & + \sum_{\mathbf{q}\in \Upsilon_a} D^e_{as}(\beta)\mu_a(\mathbf{q},\mathbf{v},t')\frac{e^{-mr}}{r^2}\delta(\varphi - \varphi')\Bigg], \end{aligned} \tag{55}$$

where as for wedged diffraction, dK is the surface measure on the reciprocal Keller's cone K whose semiangle is φ', dL is the length measure on $L = K \cap \Gamma_a$, θ' is the emission angle, and χ is the angle between L and the emission direction, $M = K \cap \Delta_a$ is the set of points where the line Δ_a crosses the cone K and ψ is the related angle between the tangent to Δ_a and the normal to K. The bidimensional Eq. (55) jointly with Eq. (47) is solved in Ref. 46 for the case of a finite baffled plate with simply supported edges. Results of this method are compared with some reference results from BEM.

Finally, conversion by corner is also a diffraction problem. The detailed power balance similar to Eq. (30) reads

$$\frac{d\mathcal{P}}{du} = \int D^c_{as}(\mathbf{v},\mathbf{u}) I(\mathbf{p},\mathbf{v},t) dv, \tag{56}$$

for any direction $\mathbf{u}$ into the structure. The structural fictitious source μ_s emits a power per unit angle $d\mathcal{P}/du = \mu_s/2\pi$ and, thus, once again μ_s is determined by an integral equation,

$$\begin{aligned} \frac{\mu_s(\mathbf{p},\mathbf{u},t)}{2\pi} = & \int_{\Omega_a} D^c_{as}(\mathbf{v},\mathbf{u})\rho_a(\mathbf{s},t')H(\mathbf{s},\mathbf{p}) d\Omega_{\mathbf{s}} \\ & + \int_{\Gamma_a} D^c_{as}(\mathbf{v},\mathbf{u})\sigma_a(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p}) d\Gamma_{\mathbf{q}} \\ & + \int_{\Delta_a} D^c_{as}(\mathbf{v},\mathbf{u})\lambda_a(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p}) d\Delta_{\mathbf{q}} \\ & + \sum_{\mathbf{q}\in \Upsilon_a} D^c_{as}(\mathbf{v},\mathbf{u})\mu_a(\mathbf{q},\mathbf{v},t')H(\mathbf{q},\mathbf{p}), \end{aligned} \tag{57}$$

where μ_s is related to acoustical sources ρ_a, σ_a, λ_a, and μ_a. The sources μ_a for the same problem are given by Eq. (32) with the acoustics to acoustics diffraction coefficient.

It is remarkable that the intensity of the forced term does not appear in Eq. (17) giving σ_s the power being reflected neither in Eq. (47) giving λ_a the power being radiated by edges. Its seems that this forced intensity is not reflected neither radiated and thus the power balance seems to be violated. However, it is not. The diffraction coefficients D_{as} and the related one D_{aa} for the diffraction of acoustical waves interacting with the edge of the structure take into account the presence of the forced vibration. And therefore, the energy of the forced term is ever accounted for in the fictitious source σ_s of Eq. (53) (same for λ_a).

VIII. CONCLUSION

In this paper, the basic equations governing energy of uncorrelated ray fields have been presented. All classical phenomena of vibroacoustics are accounted for, that is reflection, refraction, transmission, diffraction, radiation by surface, edge or corner modes and structural response. These equations are derived within the framework of the Geometrical Theory of Diffraction which is the natural theory to describe these rays. The method is based on the use of some fictitious sources each time rays are reflected, refracted, transmitted or diffracted. The power of a fictitious source is the sum of powers of all individual rays being deviated at this point. This is the main difference with the ray-tracing technique which requires that all ray paths from a source to a receiver are determined.

Besides, the proposed formalism is entirely based on energy variables. Indeed, the underlying assumption is that all rays are uncorrelated. In general, this is not true in low frequency range but this is a relevant assumption in high frequency range especially if rms values of the field in wide-band are expected. The spirit of Statistical Energy Analysis is preserved: a description of vibrational fields in terms of energy well suited for high frequencies. However, the present equations do not assume that fields are diffuse in all subsystems and even, fields can be largely nondiffuse. The present equations can thus be considered as an extension of SEA.

Energy equations of vibrational fields are useful for several purposes. The "radiosity method" in room acoustics is an efficient method to determine the reverberation-time beyond the validity of Sabine's formula. But similar equations in steady-state condition can further give the repartition of vibrational energy inside largely nondiffuse fields. The present theory embodies these two methods with, in addition, the contribution of diffraction.

APPENDIX

The reflection and transmission efficiencies at the interface between two fluids with different acoustical impedances $Z_i = \rho_i c_i$, $i=1,2$, where ρ_i is the mass per unit volume and c_i is the sound speed are[47]

$$R_{11}(\varphi) = \left| \frac{Z_2 \sec \theta - Z_1 \sec \varphi}{Z_2 \sec \theta + Z_1 \sec \varphi} \right|^2, \tag{A1}$$

$$R_{12}(\varphi) = \frac{c_1 \sec \varphi}{c_2 \sec \theta} \times \left| \frac{2 Z_2 \sec \theta}{Z_2 \sec \theta + Z_1 \sec \varphi} \right|^2, \tag{A2}$$

where φ is the incidence angle and θ is the refracted angle measured with the normal to the interface in Figs. 5 and 6,

$\sec\varphi = 1/\cos\varphi$ is the secant function. These angles are related by Snell-Descartes' law of refraction $\sin\varphi/c_1' = \sin\theta/c_2'$. These equations apply for instance at the water-air interface.

Reflection and transmission of acoustical waves through walls with same fluid of impedance $Z_0 = \rho_0 c_0$ on both sides are given by the following relationships:[11]

$$R_{11}(\varphi) = \left| \frac{Z_s}{Z_s + 2Z_0 \sec\varphi} \right|^2, \tag{A3}$$

$$R_{12}(\varphi) = \left| \frac{2Z_0 \sec\varphi}{Z_s + 2Z_0 \sec\varphi} \right|^2, \tag{A4}$$

where $Z_s = [B(\omega \sin\varphi/c_s')^4 - m\omega^2]/i\omega$ is the mechanical impedance of the wall at incidence φ, B being the bending stiffness and m the mass per unit area of the wall. The forced term in the wall is not a structural ray. Its wave number matches with the trace of the acoustical wave number and the forcing coefficient is,

$$A_0(\varphi) = \frac{8B\omega^2 Z_0 \sec\varphi \sin^3\varphi}{c_0^3 |Z_s + 2Z_0 \sec\varphi|^2}. \tag{A5}$$

The diffractivity of a wedge at normal incidence $\varphi = \theta = \pi/2$ is[5,48,49]

$$D(\beta,\alpha) = \frac{\nu^2 \sin^2(\nu\pi)}{2\pi k} \left[\frac{1}{\cos(\nu\pi) - \cos[\nu(\alpha+\beta)]} + \frac{1}{\cos(\nu\pi) - \cos[\nu(\alpha-\beta)]} \right]^2, \tag{A6}$$

where $\nu = \pi/\phi$ is the wedge index, ϕ being the outer angle of the wedge, and k is the acoustical wave number. The angles α and β are measured in the normal plane to the wedge as defined in Fig. 8. This expression is singular for $\alpha = \pm\beta$ and does not predict the correct values near these angles. An alternative expression for D valid both within and outside the transition regions is given by the Uniform Theory of Diffraction,[49]

$$\begin{aligned} D(\beta,\alpha) = \frac{\nu^2}{8\pi k} \Bigg| & \cot\left[\frac{\nu(\pi+\alpha-\beta)}{2} F(kLa^+(\alpha-\beta)) \right] \\ & + \cot\left[\frac{\nu(\pi-\alpha+\beta)}{2} F(kLa^-(\alpha-\beta)) \right] \\ & + \cot\left[\frac{\nu(\pi+\alpha+\beta)}{2} F(kLa^+(\alpha+\beta)) \right] \\ & + \cot\left[\frac{\nu(\pi-\alpha-\beta)}{2} F(kLa^-(\alpha+\beta)) \right] \Bigg|^2, \end{aligned} \tag{A7}$$

$L = r\sin^2\varphi$, r being the source-receiver distance and φ the incidence angle and $a^{\pm}(\zeta) = 2\cos^2(\pi N^{\pm}/\nu - \zeta/2)$, $N^{\pm}$ is an integer which more nearly satisfies the equality $2\pi N^{\pm}/\nu - \zeta = \pm\pi$. $F(X) = 2i\sqrt{X}e^{iX}\int_{\sqrt{X}}^{\infty} e^{-i\tau^2} d\tau$ is a transition function which involves a Fresnel integral, and $\cot x$ is the cotangent function. It can be checked that Eqs. (A6) and (A7) agree well for large X.[49]

The radiation coefficient A_{sa} introduced in Eq. (41) is found by solving the classical governing equation of fluid-loaded Love's plate, with an incident structural plane wave. For a light fluid (air),

$$A_{sa} = \frac{\rho_0}{2m} \frac{M}{M^2 - 1} \quad M > 1, \tag{A8}$$

where $M = c_s'/c_0$ is the Mach number of the structural wave. When the structural wave is substance ($M < 1$) there is no radiation, i.e., $A_{sa} = 0$. The attenuation factor m_s is calculated with Eq. (39),

$$m_s = \frac{\eta\omega}{c_s} + \frac{\rho_0}{m} \frac{1}{\sqrt{M^2 - 1}} \quad M > 1. \tag{A9}$$

This last equality is also twice the imaginary part of the structural wave number given by the dispersion of the fluid-loaded plate.

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