SMOOTH ENERGY FORMULATION FOR MULTI-DIMENSIONAL PROBLEM

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INTRODUCTION

During the last decade, the so-called Energy Flow Analysis (EFA) [1], [2] has been studied to model the spatial evolution of the energy density in structures and acoustical enclosures in medium and high frequency domain. This method is a generalization of the Statistical Energy Analysis and is based on a thermal analogy. The aim of such a method is to provide results for a smaller computation cost than the classical finite element method and with a more detailled description than in SEA. However, recent investigations in that way [3] and [4], show that the asymptotic behavior of the energy density provided by the thermal analogy does not agree with the one predicted by the equation of motion for an infinite system. Thus, in this paper, an original proof is presented to deduce an alternative method called Smooth Energy Formulation (SEF). Both, Smooth Energy Formulation and Energy Flow Method are identical in the special case of one dimensional sructures. But significant differencies appear in multi dimensional cases.

GENERALITY

The assumptions required to derive the smooth energy model are the following: steady state conditions with pulsation ω, light damping loss factor (η<<1), evanescent waves are neglected and interferences between propagative waves are not taken into account.

Two energy quantities are used: the total energy density W and the active energy flow **P** which is a vector. Since steady state conditions are assumed, those quantities are time-averaged over a period. The first step of the smooth energy formulation is to define a local frame in which the energy equation will be set. Let us consider the vector field **P** that defines a stream lines field. At every point M, the vector **P** is tangential to the stream line. Let s be the curvilinear abscissa along a stream line and **u** a unity vector tangential to this line and oriented in the positive direction of s. **u** is the first vector of the local frame. At every point M in the structure, the vector **P** is such that **P**=P**u** where P is a real number. It is not necessary to specify the second vector **v** (nor the third one **w** in the three dimensional case) of the local frame because it does not appear in the demonstration as we shall see. In the frame R defined by its origin M and the vectors **u**,**v** and **w**, the divergence of vector **P** is:

$$\mathbf{div.P} = \frac{\partial P}{\partial s} + \Gamma P. \tag{1}$$

Factor Γ does not depend on the choice of **v** because P and $\partial P / \partial s$ do not. This factor has a geometrical meaning. It is equal to divergence of **u**: $\Gamma = \mathbf{div.u}$. Thus, by using the Ostrogradski formula, for every volume V surrounded by the closed surface S $\int_V \mathbf{div.u}\, d\tau = \oint_S \mathbf{u.n}\, dS$, where **n** is the outwards unity vector. Let choose an infinitesimal volume V, then **div.u** is constant over V. So Γ is the outwards flow of vector **u** from the closed surface S and divided by the volume V: $\Gamma = \frac{1}{V}\oint_S \mathbf{u.n}\, dS$. Assume this surface to be a tube of stream lines limited by two sections as shown Figure 1. Γ is the relative increasing of section of such a tube around the considered point.

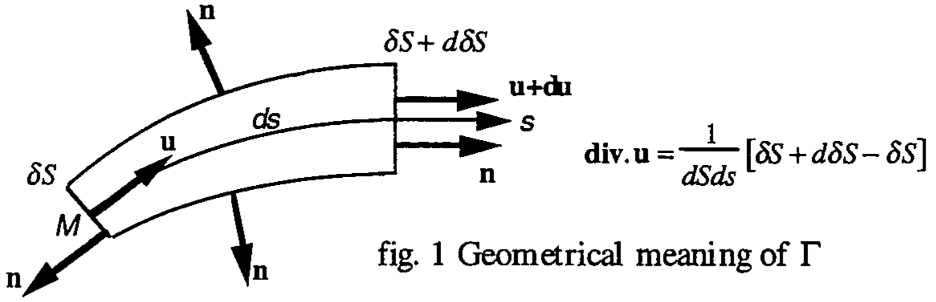


fig. 1 Geometrical meaning of Γ

In what follows, all relationships will be written along a stream line with abscissa s. Moreover partial derivatives respect to s will be replaced by total derivatives.

At each point, energy density and energy flow are provided by the superposition of two propagative waves. Those waves are propagating along the stream line in both orientations. One has the same orientation as **u** and is noted with upperscript + and one has the inverse orientation and is noted with upperscript -. The partial energy densities W^+ and W^- and the partial energy flows with signed values P^+ and P^- are associated to those fields. As mentioned above, interferences between propagative waves are not taken into account. This simplification is the backbone of the smooth energy formulation. So, complete quantities W and P are the sum of partial quantities. Thus,

$$W = W^+ + W^- \quad \text{and} \quad P = P^+ + P^-. \tag{2,3}$$

Let study now the behavior of each wave. The energy balance associated to the partial energies is $\mathbf{div.P}^{+/-} + p_{diss}^{+/-} = 0$, and reffering to (1) it yields:

$$\frac{dP^+}{ds} + \Gamma P^+ + p_{diss}^+ = 0, \qquad \frac{dP^-}{ds} + \Gamma P^- + p_{diss}^- = 0. \tag{4,5}$$

The damping model adopted here is the same as the one used in Statistical Energy Analysis for which dissipated power density is proportional to energy density. Hence:

$$p_{diss}^{+} = \eta\omega W^{+} \text{ and } p_{diss}^{-} = \eta\omega W^{-}. \quad (6,7)$$

Finally, we have to find a relationship between energy flows and energy densities. For a pure propagative wave in an undamped system, partial energy flow is proportional to partial energy density. The proportionality ratio is the group velocity c_g. So, taking into account the sign convention:

$$P^{+} = c_g W^{+} \text{ and } P^{-} = -c_g W^{-}. \quad (8,9)$$

For light damping loss factor η, equations (8,9) remain valid and the damping is taken into account by means of the dissipated power in the energy balance.

It is now possible to derive the smooth energy equation. Let us compute the difference between (4) and (5) and substitute the relationships (6) to (9) into the result, it leads to:

$$P = \frac{-c_g^2}{\eta\omega}\left(\frac{dW}{ds} + \Gamma W\right). \quad (10)$$

Now by sustituting this relationship (10) into the sum (4) and (5), one obtains:

$$\frac{d^2W}{ds^2} + 2\Gamma\frac{dW}{ds} + \left(\frac{d\Gamma}{ds} + \Gamma^2 - \frac{\eta^2\omega^2}{c_g^2}\right)W = 0. \quad (11)$$

Equations (10) and (11) constitute a general form of the Smooth Energy Formulation (SEF) for a multi-dimensional structure.

At this stage, it can be noticed that energy equations (10) and (11) depend on factor Γ which depends upon the geometry of the stream lines. So, equations (10) and (11) cannot be solved without the knowledge of this factor. Thus, these equations require that the solver *a priori* knows the geometry of the stream lines. Equations (10) and (11) contain information about the magnitude of energy density and energy flow but not about the direction of the latter. This situation is exactly the same as in fluid mechanics with Bernoulli's equation. This equation describes the energy balance in terms of energy density solely. But Bernoulli's equation is essentially expressed along a stream line. Each time that such a stream line is known (pipes, emptying of tank and so on), Bernoulli's equation provides a solution to the problem. However except in seldom cases of irrotational motion, Bernoulli's equation cannot be generalized over the whole domain.

PLANE WAVES

In dimension one, system behaves as wave guide. The lines of propagation of energy are parallel curves. So the factor Γ vanishes and the energy equations are:

$$P = \frac{-c_g^2}{\eta\omega}\frac{dW}{ds}, \qquad \frac{d^2W}{ds^2} - \frac{\eta^2\omega^2}{c_g^2}W = 0. \quad (12,13)$$

Both of these equations have been largely studied. For instance, Wohlever & Bernhard [5] or Luzzato [6] compare the numerical solutions of this equation system with the energy quantities deduced from the equation of motion. These authors show that the solutions of the system (12) and (13) are the energy quantities (deduced from the equation of motion) averaged over a wavelenght. Thus, all fluctuations whose order of magnitude is one wavelengh are not taken into account. The second conclusion is that active energy flow is proportional to the gradient of energy density. An analogy with the Fourier's law in thermic is natural. It is explained by Nefske & Sung [1]. For EFA, this is the first step for generalization of system (12) and (13) to membranes and plates.

CYLINDRICAL WAVES AND SPHERICAL WAVES

In the case of multi-dimensional systems, let see first the generalization of system (12) and (13) from Fourier's law. This law is:

$$\mathbf{P} = \frac{-c_g^2}{\eta\omega}\,\mathbf{grad}W. \tag{14}$$

By substituting this relationship into the energy balance, it yields:

$$\mathbf{div.grad}W - \left(\frac{\eta\omega}{c_g}\right)^2 W = 0. \tag{15}$$

Equation (15) is analogous to the heat conduction equation in steady conditions with a convective term. This generalization was proposed by Nefske &Sung [1] and Bouthier & Bernhard [2] and seems to be natural. However Langley [3] remarked that for infinite system in dimension two, the farfield of the solution of equation (15) decreqses as $1/\sqrt{r}$. In opposition the farfield predicted by the equation of motion for plate or membrane decreases as $1/r$. Thus, Langley raised a paradox that we try to explain below.

Let precise the relationships (14) and (15) in the particular case of an axisymetric system in dimension two:

$$\text{n=2} \qquad P = \frac{-c_g^2}{\eta\omega}\frac{dW}{dr}, \quad \frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} - \frac{\eta^2\omega^2}{c_g^2}W = 0. \tag{16,17}$$

where r is the distance between the origin and the considered point. And for three dimensional system with spherical symmetry:

$$\text{n=3} \qquad P = \frac{-c_g^2}{\eta\omega}\frac{dW}{dr}, \quad \frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} - \frac{\eta^2\omega^2}{c_g^2}W = 0. \tag{18,19}$$

Let evaluate energy equations (10) and (11). Factor Γ is equal to $1/r$ in dimension two and $2/r$ in dimension three. The energy equations are:

$$\text{n=2} \qquad P = \frac{-c_g^2}{\eta\omega}\left(\frac{dW}{dr} + \frac{1}{r}W\right), \quad \frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} - \frac{\eta^2\omega^2}{c_g^2}W = 0. \tag{20,21}$$

$$\text{n=3} \qquad P = \frac{-c_g^2}{\eta\omega}\left(\frac{dW}{dr} + \frac{2}{r}W\right), \quad \frac{d^2W}{dr^2} + \frac{4}{r}\frac{dW}{dr} + \left(\frac{2}{r^2} - \frac{\eta^2\omega^2}{c_g^2}\right)W = 0. \tag{22,23}$$

Equation (21) (resp. (23)) is different from equation (17) (resp.(19)) because of factor $2/r$ (resp. $4/r$) instead of $1/r$ (resp. $2/r$). This difference comes from the relationship (20) (resp. (22)). It shows clearly that energy flow is not proportional to the gradient of energy density. Thus the analogy with the Fourier's law is no longer valid.

Now, consider two numerical simulations. The first concerns a circular plate whith radius $r_{\max}$. Three calculations have been made. The first is a classic calculation. The equation of motion is solved with an excitation at the center of the plate clamped at r_{max}. The energy density and the energy flow are deduced from this solution. This calculation is used as a reference. Secondly, the heat conduction equation (17) of EFA is solved. Boundary conditions are applied on energy flow. Finally, the third calculation is to solve energy equations (20) and (21) of SEF. Figures 2 and 3 show the results. The energy density predicted by EFA is under-estimated near the excitation point and over-estimated in farfield. The decreasing of this solution is clearly too weak. In opposition, the energy density predicted by SEF is a smooth estimation of the classic solution. This result well agrees with the averaging procedure over a wavelengh introduced by Wohlever & Bernhard [5] in the one dimensional case. The second simulation concerns an acoustical spherical enclosure. Results are shown on Figures 4 and 5.

Let give some asymptotic developments of these three solutions. In the case of infinite plate, an analytical far-field solution of the equation of motion is the Hankel function of order zero and second kind. By applying an asymptotic development for large argument, the energy density obtained is proportional to $\mathbf{e}^{-\frac{\eta\omega}{c_g}r} / r$. The decreasing is $1/r$. An analytical solution for the heat conduction equation (17) for infinite system is the modified Bessel function of order zero and second kind $K_0\left(\frac{\eta\omega}{c_g}r\right)$. An asymptotic development of this function for large argument is proportional to $\mathbf{e}^{-\frac{\eta\omega}{c_g}r} / \sqrt{r}$. The decreasing is $1/\sqrt{r}$. Finally, an analytical solution of the energy equation (21) for infinite system is $\mathbf{e}^{-\frac{\eta\omega}{c_g}r} / r$. It leads to the right decreasing $1/r$. So the energy equation (21) is better than the heat conduction equation (17).

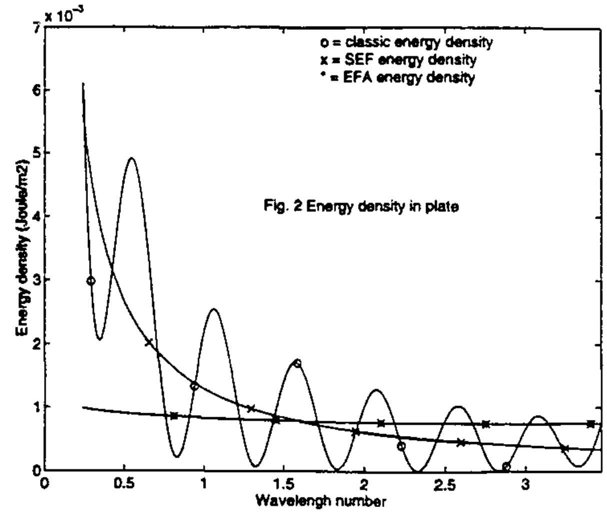
Finally, Burrel, Warner & Chernuka [7] are interested by circular axisymetric plate. The demonstration they proposed is closed than the one explained here at the begining with $1/r$ for particular value of the factor Γ. They wrote correctly the energy balance, sum of (4) and (5) with factor $1/r$, but not the difference (4) minus (5). They forgot factor $1/r$. Thus energy flow becomes proportional to the gradient of energy density and they obtained the heat conduction equation. The numerical simulation proposed shows clearly the default of the heat conduction equation. Without this mistake, they would have obtained the equations (20) and (21).

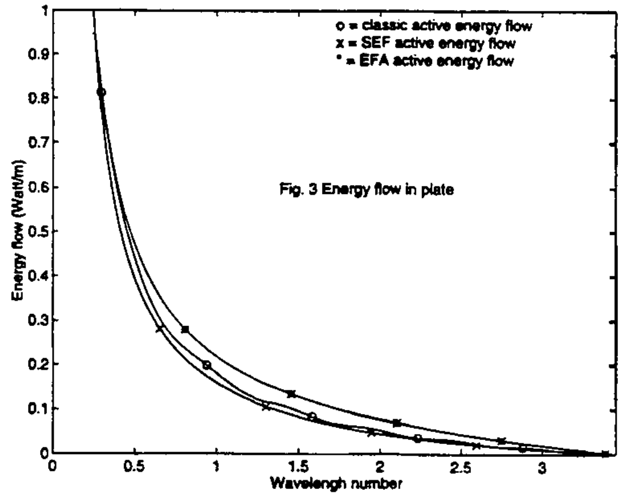
CONCLUSION

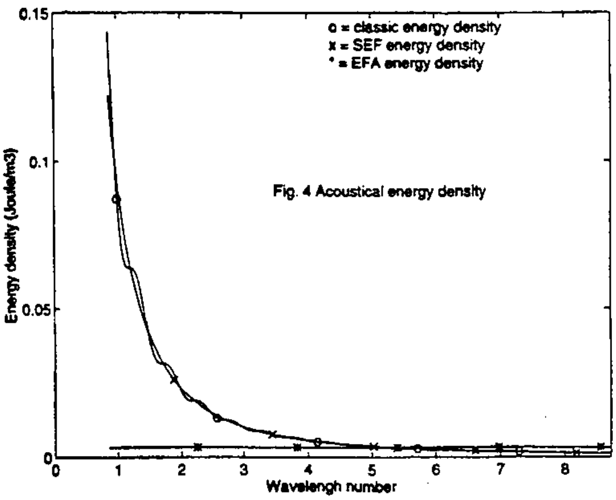
In this study, we have proposed a proof of smooth energy equations to model the behavior of vibratory systems in medium and high frequencies domain. This demonstration was previously suggested by Nefske & Sung [1] in the particular case of one dimensional structures. But Nefske & Sung and other authors generalized these equations for multi-dimensional system by translating the first derivative respect to the abscissa into a gradient. However, Langley [3] raised an objection against this equation. He remarked that for an infinite system, the decreasing predicted does not agree with those predicted by the equation of motion.

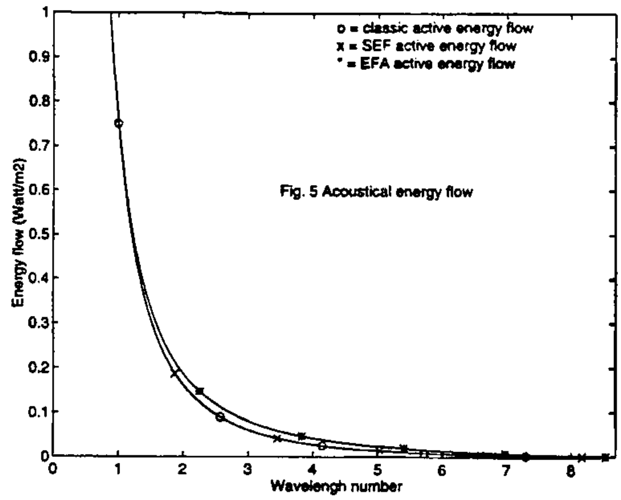
The generalization proposed here is based on differential equations written along the energy flow stream lines. The energy equation has been generalized in a weak sense. The smooth energy equations obtained are able to predict the magnitude of energy density and energy flow but not the direction of energy flow. So, the direction and the geometry of the stream lines have to be *a priori* known. This is a consequent limitation.

But, there exists at least one case for which stream lines are known: infinite system. We have shown then that the smooth energy equation is different from the heat conduction equation. The decreasing predicted by this equation well agrees with the one predicted by the equation of motion: this is our explanation of the paradox raised by Langley.









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