Statistical energy analysis

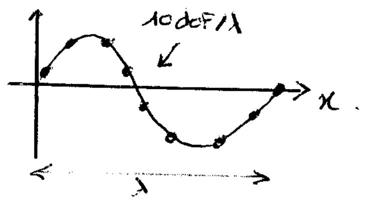
Alain Le Bot - July 2015.

Reference : Foundation of statistical energy analysis in vibroacoustics - Alain Le Bot - Oxford University Press - May 2015 - 978-0-19-872923-5.

Contents

① Introduction

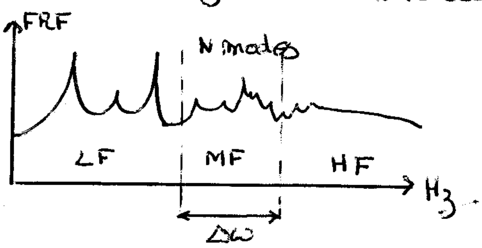
- At high frequencies, the wavelength λ is short. In finite element method, the number of degrees of freedom (dof) per wavelength is fixed (~ 10)



system	limit
car	1000 Hz
aircraft	100 Hz
ship	10 Hz
launcher	1 Hz

The number of dof increases with frequency. Beyond a certain limit, computations become untractable on actual computers.

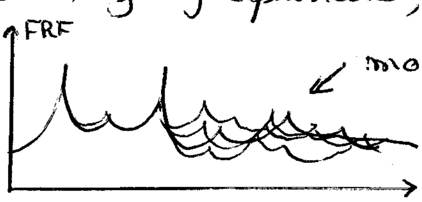
- The number of modes increases with frequency



modal density $n = \dfrac{N}{\Delta\omega}$ modal overlap $M = \eta\omega n$

Modes are numerous, they overlap. Modes loss their individuality.

- At high frequencies, sensitivity of systems is high.

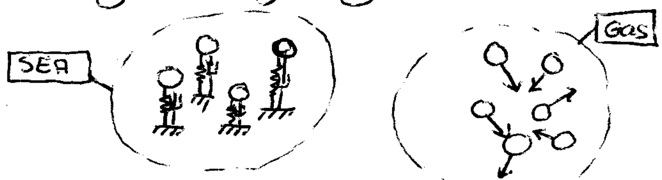


modes become unpredictable

Two samples of a similar structure have different modes in HF.

The structure becomes non deterministic

1) Statistical energy analysis is a statistical theory of sound and vibration when the number of modes is large and vibration is sufficiently disorganized.



large number of modes randomly excited — large number of molecules in random movement

modes $\leftrightarrow$ molecules.
diffuse field $\leftrightarrow$ thermal equilibrium

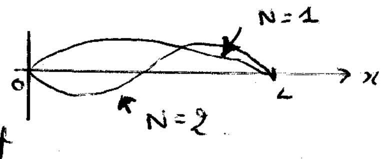
② Modal density

Each finite structure (or bounded cavity) has a sequence of natural frequencies which tends to infinite.

Ex. string : $\omega_i = i\frac{\pi}{L}c$ and $\Psi_i(x) = \sin(i\frac{\pi x}{L})$ for $i = 1, 2, \ldots$

Dimension 1

Resonance = integer number of half wavelengths



wavenumber

$$\frac{k}{\pi}L = N \leftarrow \text{\# mode}$$

length

dispersion relationship $k(\omega)$

modal density $n(\omega) = \frac{dN}{d\omega} = \frac{L}{\pi}\frac{dk}{d\omega}$
group speed $c_g = \frac{d\omega}{dk}$
$\Rightarrow$

dimension 1

$$\boxed{n(\omega) = \frac{L}{\pi c_g}}$$

length, group speed

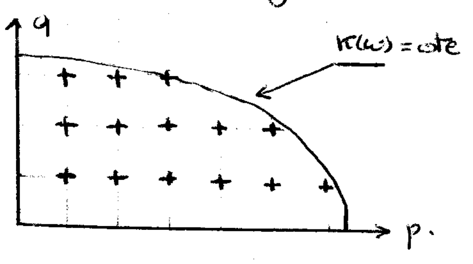
Dimension 2

Resonance = integer number of half wavelengths in <u>each</u> direction

Wavenumber vector

$\vec{k} = (k_x, k_y)$ $\qquad \frac{k_x}{\pi}a = p$ (integer; length) $\qquad \frac{k_y}{\pi}b = q$ (integer; width) $\qquad (p,q) \in \mathbb{N}^{*2}$

$N(\omega)$ number of modes below ω = $\text{card}\{(p,q) \in \mathbb{N}^{*2} ; (\frac{p\pi}{a})^2 + (\frac{q\pi}{b})^2 \leq k^2(\omega)\}$



$N(\omega)$ is approximated by the area under the ellipse

$$1 = \left(\frac{p\pi}{ak}\right)^2 + \left(\frac{q\pi}{bk}\right)^2$$

$$N(\omega) \simeq \frac{1}{4}\pi \times \frac{ak}{\pi} \times \frac{bk}{\pi} = \frac{abk^2}{4\pi}$$

modal density $n(\omega) = \frac{dN}{d\omega} = \frac{ab}{2\pi}k\frac{dk}{d\omega}$
phase speed $c_p = \frac{\omega}{k}$
$\Rightarrow$

dimension 2

$$\boxed{n(\omega) = \frac{S\omega}{2\pi c_p c_g}}$$

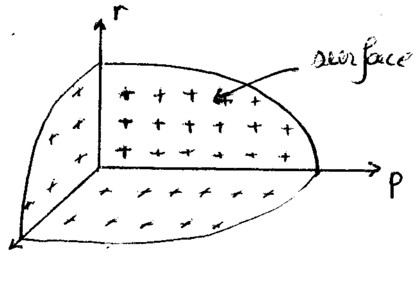
surface, phase speed

Dimension 3

Wavenumber vector $\vec{k} = (k_x, k_y, k_z)$

Resonance condition: $\frac{k_x}{\pi} a = p$; $\frac{k_y}{\pi} b = q$; $\frac{k_z}{\pi} c = r$ $(p,q,r) \in \mathbb{N}^{*3}$

Number of modes below ω $N(\omega) = \text{card}\left\{(p,q,r) \in \mathbb{N}^{*3} \ \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 + \left(\frac{r\pi}{c}\right)^2 \leq k^2(\omega)\right\}$



$N(\omega)$ is approximated by the volume enclosed by the ellipsoid surface:

$$1 = \left(\frac{p\pi}{ak}\right)^2 + \left(\frac{q\pi}{bk}\right)^2 + \left(\frac{r\pi}{ck}\right)^2$$

$$N(\omega) = \frac{1}{8} \times \frac{4}{3} \pi \times \frac{p\pi}{ak} \times \frac{q\pi}{bk} \times \frac{r\pi}{ck} = \frac{abc\,k^3}{6\pi^2}$$

modal density $n(\omega) = \frac{dN}{d\omega} = \frac{abc}{2\pi^2} k^2 \frac{dk}{d\omega}$

phase speed $c_p = \frac{\omega}{k}$ group speed $c_g = \frac{d\omega}{dk}$ $\Rightarrow$

dimension 3

$$\boxed{n(\omega) = \frac{V\omega^2}{2\pi^2 c_p^2 c_g}}$$

volume

phase speed group speed

③ Damping

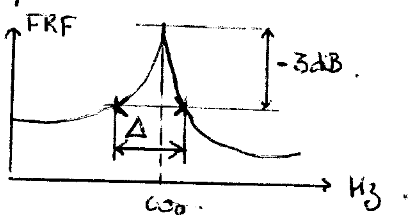
At high frequencies, the vibrational level is controlled by damping

- viscous damping coefficient c
- modal damping ratio ζ
- damping loss factor η
- half-power bandwidth Δ

$\zeta = \frac{c}{2m\omega_0}$ $\omega_0 = \sqrt{\frac{k}{m}}$

$\eta = 2\zeta$. $\Delta = \eta\omega_0$

i) Half-power bandwidth



half-power bandwidth Δ

damping loss factor $\eta = \frac{\Delta}{\omega_0}$

This is a low frequency method.

ii) Reverberation time

- establish a diffuse field
- switch-off all sources.

Reverberation time = time for a decrease of 60 dB of vibrational level (or SPL).

time-decrease of energy $E(t) = E_0 e^{-\eta\omega t}$ $10\log_{10} E(T_r) - 10\log_{10} E_0 = -60$

$T_r = \frac{2{,}2 \times 2\pi}{\eta\omega}$

in structures

$T_r = 0{,}16 \frac{V}{\alpha S}$ (Sabine's formula).

in rooms

iii) Power balance

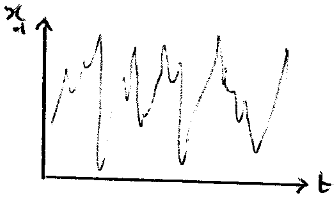
(steady-state condition) $P_{inj} = \eta \omega E$

injected power — vibrational energy.

Measurement of P_{inj} with an impedance head.
Measurement of with accelerometres or vibrometre
This is a high frequency method.

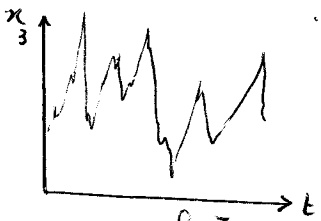
④ Summary of random functions

A random function is a map $t \mapsto x(t)$ where $x(t)$ is a random variable.



sample 1

sample 2



sample 3.

probabilistic expectation
$\langle \cdot \rangle$

Auto-correlation - power spectral density

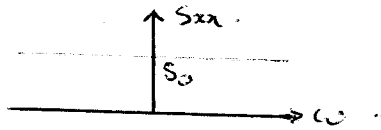
t: initial time
τ: time delay

$$\underbrace{\langle x(t)\, x(t+\tau) \rangle}_{\text{auto-correlation}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{S_{xx}(\omega)}_{\text{power spectral density (PSD)}} e^{i\omega\tau}\, d\omega$$

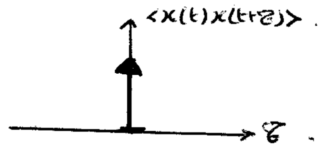
At $\tau = 0$

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega .$$

White noise



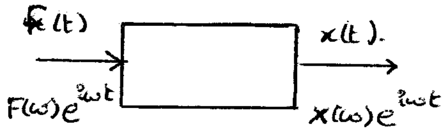
or



$$\begin{cases} S_{xx}(\omega) = S_0 \\ \langle x(t)\, x(t+\tau) \rangle = \frac{S_0}{2\pi} \delta(\tau) \end{cases}$$

Uncorrelation: x and y are said uncorrelated if $\langle x(t)\, y(t+\tau) \rangle = 0$ or $S_{xy}(\omega) = 0$

Response to a linear system

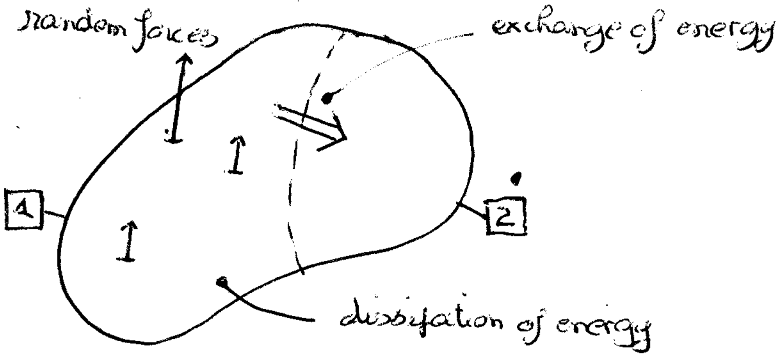


Frequency response function

$$H(\omega) = \frac{X(\omega)}{F(\omega)} .$$

PSD of output: $S_{xx}(\omega) = |H|^2(\omega) \times S_{FF}(\omega)$; $S_{\dot{x}\dot{x}}(\omega) = \omega^2 |H|^2 S_{FF}$; $S_{F\dot{x}} = i\omega H\, S_{FF}(\omega)$

5 Modal approach of statistical energy analysis



flexible structure excited by random forces

- flexible structure in vibration
- random forces
- dissipative processes

The goal of statistical energy analysis is to provide an analysis of vibrating structures in terms of energy and power.

E_i : vibrational energy in subsystem i

K_i : kinetic energy in subsystem i $\qquad E_i = K_i + V_i$

V_i : elastic energy in subsystem i

How is the vibrational energy shared between the elastic and kinetic forms?

$P_{inj,i}$: power injected in subsystem i by external forces

$P_{diss\,i}$: power dissipated in subsystem i

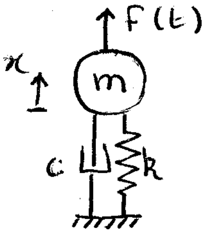
How to compute the injected power from spectrum of forces?

Is the dissipated power related to vibrational energy?

P_{ij} : power exchanged between subsystems i and j.

Is there a relation between P_{ij} and E_i, E_j ?

Single resonator



$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\xi = \frac{c}{2m\omega_0}$$

- linear mechanical oscillator
- stationary random force $F(t)$
- white noise force of spectrum S_0

governing equation:

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

frequency response function

$$H(\omega) = \frac{1}{m\omega_0^2\left[1 + 2i\xi\frac{\omega}{\omega_0} - \frac{\omega^2}{\omega_0^2}\right]}$$

power balance

$$\frac{d}{dt}\Big[\underbrace{\underbrace{\tfrac{1}{2}m\dot{x}^2}_{\text{kinetic energy}} + \underbrace{\tfrac{1}{2}kx^2}_{\text{elastic energy}}}_{\text{vibrational energy}}\Big] + \underbrace{c\dot{x}^2}_{\text{dissipated power } P_{diss}} = \underbrace{F\dot{x}}_{\text{injected power } P_{inj}}$$

i) Equality of kinetic and elastic energies

$$\langle V\rangle = \frac{1}{2}k\langle x^2\rangle = \frac{k}{4\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega = \frac{kS_0}{4\pi}\int_{-\infty}^{\infty}|H|^2\,d\omega = \frac{kS_0}{8\xi m^2\omega_0^3}$$

$$\langle K\rangle = \frac{1}{2}m\langle\dot{x}^2\rangle = \frac{m}{4\pi}\int_{-\infty}^{\infty} S_{\dot{x}\dot{x}}(\omega)\,d\omega = \frac{mS_0}{4\pi}\int_{-\infty}^{\infty}\omega^2|H|^2\,d\omega = \frac{mS_0}{8\xi m^2\omega_0}$$

$$\Rightarrow \boxed{\langle V\rangle = \langle K\rangle}$$

ii) Injected power

$$\langle P_{inj}\rangle = \langle F\dot{x}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{F\dot{x}}(\omega)\,d\omega = \frac{S_0}{2\pi}\int_{-\infty}^{\infty} i\omega H(\omega)\,d\omega = \frac{S_0}{2m}$$

$$\boxed{\langle P_{inj}\rangle = \frac{S_0}{2m}}$$

iii) Mean power balance

$$\left\langle\frac{dE}{dt}\right\rangle = m\langle\dot{x}\ddot{x}\rangle + k\langle x\dot{x}\rangle = \frac{m}{2\pi}\int_{-\infty}^{\infty} S_{\dot{x}\ddot{x}}(\omega)\,d\omega + \frac{k}{2\pi}\int_{-\infty}^{\infty} S_{x\dot{x}}(\omega)\,d\omega = \frac{mS_0}{2\pi}\int_{-\infty}^{\infty} i\omega^3|H|^2\,d\omega + \frac{kS_0}{2\pi}\int_{-\infty}^{\infty} i\omega|H|^2\,d\omega = 0$$

odd functions

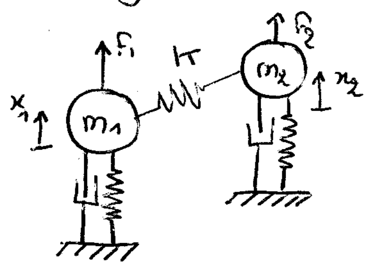
$$\langle P_{diss}\rangle = c\langle\dot{x}^2\rangle = \frac{2c}{m}\times\frac{1}{2}m\langle\dot{x}^2\rangle = \frac{2c}{m}\langle K\rangle = \frac{c}{m}\langle E\rangle$$

$$\boxed{\langle P_{diss}\rangle = \frac{c}{m}\langle E\rangle}$$

$$\langle P_{diss}\rangle = \langle P_{inj}\rangle$$

$$\boxed{\frac{c}{m}\langle E\rangle = \langle P_{inj}\rangle}$$

Pair of resonators



Lyon and Scharton 1968

- linear mechanical oscillators
- stationary random forces
- white noises of spectrum S_1, S_2
- uncorrelated forces $S_{12} = S_{21} = 0$.
- conservative coupling (inertial, gyroscopic and elastic).

governing equation

$$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{M}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}}_{C}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} k_1 & -K \\ -K & k_2 \end{pmatrix}}_{K}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

frequency response matrix

$$H(\omega) = [-M\omega^2 + i\omega C + K]^{-1}$$

power balance

$$\frac{d}{dt}\left[\frac{1}{2} m_1 \dot{x}_1^2 + \underbrace{\frac{1}{2} k_1 x_1^2 - \frac{1}{2} K x_1 x_2}_{\text{elastic energy } V_i} \right] + \underbrace{c_i \dot{x}_i^2}_{\text{dissipated power}} + \underbrace{\frac{1}{2} K (x_1 \dot{x}_2 - \dot{x}_1 x_2)}_{\text{exchanged power } P_{12}} = \underbrace{F_1 \dot{x}_1}_{\text{injected power}}$$

kinetic energy K_i

i) Equality of kinetic and elastic energies

$$\boxed{\langle V_i \rangle = \langle K_i \rangle}$$

- This equality holds for each oscillator
- The elastic energy of the coupling has been splitted and attributed to oscillators.

ii) Injected power.

$$\boxed{\langle P_{inj\, i} \rangle = \frac{S_i}{2m_i}}$$

iii) Exchanged power.

Since each product $\langle x_i^{(p)} x_j^{(q)} \rangle$ is a linear combination of the PSDs,

$$\langle P_{12} \rangle = (A_1 \;\; A_2)\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \text{ and } \begin{pmatrix} \langle E_1 \rangle \\ \langle E_2 \rangle \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \Rightarrow \langle P_{12} \rangle = (A_1 \;\; A_2)\left(E\right)^{-1}\begin{pmatrix} \langle E_1 \rangle \\ \langle E_2 \rangle \end{pmatrix}$$

So, by expanding $\langle P_{12} \rangle = B_1 \langle E_1 \rangle - B_2 \langle E_2 \rangle$

After calculating all integrals by the residue theorem

$$\boxed{\begin{aligned} B = B_1 = B_2 &= \frac{K^2 (\Delta_1 + \Delta_2)}{[(\Omega_1^2 - \Omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1 \Omega_2^2 + \Delta_2 \Omega_1^2)]\, m_1 m_2} \\ \langle P_{12} \rangle &= B(\langle E_1 \rangle - \langle E_2 \rangle) \end{aligned}}$$

"blocked" natural frequencies

$\Omega_i = \sqrt{k_i / m_i}$

power bandwidth

$\Delta_i = c_i / m_i$

Set of resonators

Newland 1966

- white noise forces of spectrum S_i, $i = 1, 2 \dots N$
- uncorrelated forces
- conservative coupling
- light coupling

governing equation

$$M\ddot{X} + C\dot{X} + KX = F$$

light coupling $K \ll k_1, k_2$

$$K = \begin{pmatrix} k_1 & 0 \\ 0 & k_N \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & \times \\ \times & 0 \end{pmatrix}$$

small parameter

perturbation technique

$$x_i(t) = x_{i0}(t) + \varepsilon x_{i1}(t) + \varepsilon^2 x_{i2}(t) + o(\varepsilon^2)$$

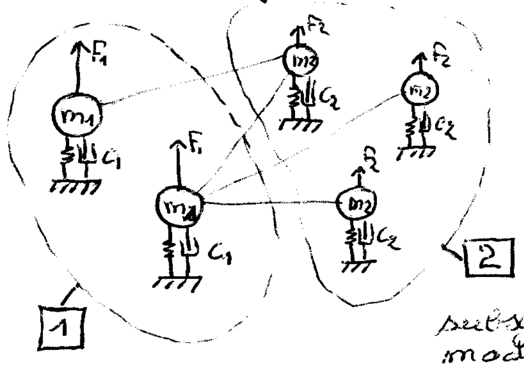
i) Equality of kinetic and elastic energies $\langle V_i \rangle = \langle K_i \rangle$

ii) Injected power $\langle P_{inj\,i} \rangle = \dfrac{S_i}{2m_i}$

iii) Exchanged power.

$$\begin{cases} \langle P_{ij} \rangle = B(\langle E_i \rangle - \langle E_j \rangle) + o(\varepsilon^2) \\ B = B_0^\circ \varepsilon^2 \end{cases}$$

Subsystems of resonators



subsystem i, j
mode α, β

- conservative and light coupling
- subsystem = {
  - uncoupled resonators
  - same m_i, c_i }
- "rain-on-the-roof" forces
  - white noise forces
  - uncorrelated forces
  - PSD constant within subsystems

Canonical problem.

$$\begin{cases} m_i \ddot{x}_{i\alpha} + c_i \dot{x}_{i\alpha} + k_{i\alpha} x_{i\alpha} = f_{i\alpha} + \sum\limits_{j \neq i} \sum\limits_{\beta} k_{i\alpha, j\beta}\, x_{j\beta} \\ E_{i\alpha} = \frac{1}{2} m_i \dot{x}_{i\alpha}^2 + \frac{1}{2} k_{i\alpha} x_{i\alpha}^2 - \frac{1}{2} k_{i\alpha, j\beta}\, x_{i\alpha} x_{j\beta} \\ P_{i\alpha, j\beta} = \frac{1}{2} k_{i\alpha, j\beta} (x_{i\alpha} \dot{x}_{j\beta} - \dot{x}_{i\alpha} x_{j\beta}) \end{cases}$$

$$E_i = \sum_\alpha E_{i\alpha}$$

$$P_{ij} = \sum_{\alpha,\beta} P_{i\alpha, j\beta}.$$

By previous result $\langle P_{ij} \rangle = \sum\limits_{\alpha,\beta} \varepsilon^2 B^\circ_{i\alpha, j\beta} (\langle E_{i\alpha} \rangle - \langle E_{j\beta} \rangle) + o(\varepsilon^2)$.

But the energy of uncoupled resonators $(\varepsilon = 0)$ is $S_i / 2c_i$. Thus

$$\langle E_{i\alpha} \rangle = \frac{S_i}{2c_i} + o(1).$$

PSD constant

damping constant

$\langle E_{i\alpha} \rangle = \dfrac{\langle E_i \rangle}{N_i} + o(1)$ equipartition of energy.

Coupling power proportionality

$$\langle P_{ij} \rangle = \Big(\sum_{\alpha,\beta} B_{i\alpha, j\beta} \Big) \left(\frac{\langle E_i \rangle}{N_i} - \frac{\langle E_j \rangle}{N_j} \right) + o(\varepsilon^2).$$

Random resonator

The coupling factor $\sum_{\alpha,\beta} B_{i\alpha,j\beta}$ is a very complicated function that depends on "blocked" natural frequencies and half-power bandwidth of all resonators. In practice, its effective computation is costly ($N \sim 10^6$) and requires the knowledge of all details of the subsystems.

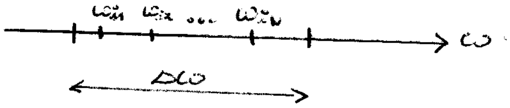
↗ This is exactly what we want to avoid in SEA.

To "forget" the exact positions of eigenfrequencies, we need to approximate the discrete sum by an integral.

$$\frac{1}{N}\sum_{\alpha=1}^{N} f(\omega_\alpha) \simeq \int_{-\infty}^{\infty} f(\omega)\,p(\omega)\,d\omega \qquad \frac{1}{N_i N_j}\sum_{\alpha,\beta} f(\omega_\alpha,\omega_\beta) \simeq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\omega,\omega')\,p_i(\omega)\,p_j(\omega')\,d\omega\,d\omega'$$

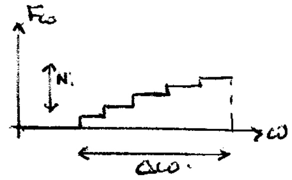
well-chosen pdf

How to choose the pdf?



empirical cumulative distribution function

$$F_{\omega_i}(\omega) = \frac{1}{N_i}\,\mathrm{Card}\{\alpha / \omega_{i\alpha} \leq \omega\}$$



$$p_{\omega_i}(\omega) = \frac{d}{d\omega} F_{\omega_i} = \frac{n_i(\omega)}{N_i}$$

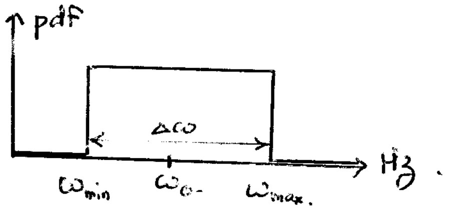
modal density

$$n(\omega) = \frac{N_i}{\Delta\omega}$$

← number of eigenfrequencies

← frequency bandwidth

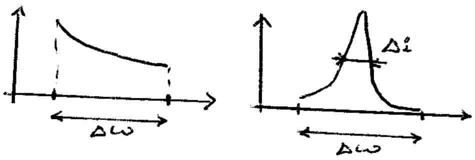
Case of uniform modal density (Newland 1968)



- light damping $\Delta_i \ll \Delta\omega$

half-power bandwidth — frequency bandwidth

$$\sum_{\alpha,\beta} B_{i\alpha,j\beta} = \frac{K^2}{m_i m_j}\int_{\omega_{min}}^{\omega_{max}}\int_{\omega_{min}}^{\omega_{max}} \frac{\Delta_i+\Delta_j}{\underbrace{\left[(\omega_i+\omega_j)^2+\frac{(\Delta_i+\Delta_j)^2}{4}\right]}\left[(\omega_i-\omega_j)^2+\frac{(\Delta_i+\Delta_j)^2}{4}\right]}\,d\omega_i\,d\omega_j$$



$$\boxed{\sum_{\alpha,\beta} B_{i\alpha,j\beta} = \frac{\pi K^2 N_i N_j}{2 m_i m_j \omega_c^2 \Delta\omega}}$$

Coupling power proportionality

$$\boxed{\langle P_{ij}\rangle = \omega_c \eta_{ij} N_i \left(\frac{\langle E_i\rangle}{N_i} - \frac{\langle E_j\rangle}{N_j}\right)}$$

coupling loss factor

Reciprocity

$$\boxed{\eta_{ij} N_i = \eta_{ji} N_j}$$

Ensemble of similar systems.

$$\frac{1}{N_i N_j}\sum_{\alpha,\beta} — \;\rightsquigarrow\; \int_{\Delta\omega}\int_{\Delta\omega} — \, d\omega\, d\omega'$$

- System with a large number of modes $N_i \gg 1$ and $N_j \gg 1$
  SEA applies to a unique system
- Population of similar systems with random variations (Gibb's ensemble)
  SEA applies to the average system

$$N_e \cdot N_m \gg 1 .$$

number of members in the ensemble — number of modes in each member

Coupled beams. Lotz and Crandall 1971

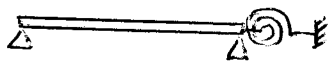
- non resonant modes are neglected

governing equation

$$m_i \frac{\partial^2 u_i}{\partial t^2} + c_i \frac{\partial u_i}{\partial t} + E_i I_i \frac{\partial^4 u_i}{\partial x^4} = f_i(x,t) + K \frac{\partial u_j}{\partial x}\delta'(x) .$$

external force — coupling force

blocked modes $\Psi_{i\alpha}(x)$



- $E_i I_i \dfrac{d^4}{dx^4}\Psi_{i\alpha}(x) = m_i \omega_{i\alpha}^2 \Psi_{i\alpha}(x)$
- modes are orthogonal.
- they form a complete set $u_i(x,t) = \sum_\alpha u_{i\alpha}(t)\Psi_{i\alpha}(x)$

reduction to the canonical problem

$$\begin{cases} m_i \ddot{u}_{i\alpha} + c_i \dot{u}_{i\alpha} + m_i\omega_{i\alpha}^2 u_{i\alpha} = F_{i\alpha} + \sum_\beta K \Psi'_{i\alpha}(0)\Psi'_{j\beta}(0)\, u_{j\beta} \\ E_{i\alpha} = \frac{1}{2} m_i \dot{u}_{i\alpha}^2 + \frac{1}{2} m_i \omega_{i\alpha}^2 u_{i\alpha}^2 - \frac{1}{2}\sum_\beta K \Psi'_{i\alpha}(0)\Psi'_{j\beta}(0)\, u_{i\alpha} u_{j\beta} . \\ P_{i\alpha,j\beta} = \frac{1}{2} K \Psi'_{i\alpha}(0)\Psi'_{j\beta}(0)\left[u_{i\alpha}\dot{u}_{j\beta} - \dot{u}_{i\alpha} u_{j\beta}\right] \end{cases}$$

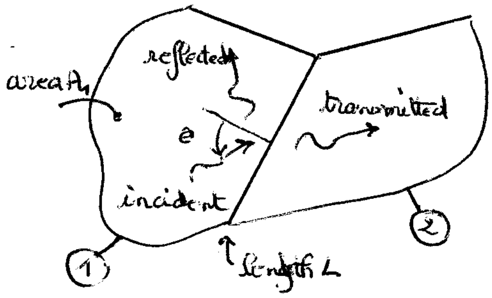
coupling power proportionality

$$\langle P_{ij}\rangle = \omega\, \eta_{ij}\, n_i \left(\frac{\langle E_i\rangle}{n_i} - \frac{\langle E_j\rangle}{n_j}\right) \qquad \eta_{ij} = \frac{K^2}{L_i (m_i E_i I_i)^{1/2} m_j^{1/4} (E_j I_j)^{3/4}\omega^{3/2}}$$

CLF — modal density — beam length — mass per unit length — bending stiffness

6 Coupling loss factors

Wave approach



• Diffuse field (homogeneous + isotropic)

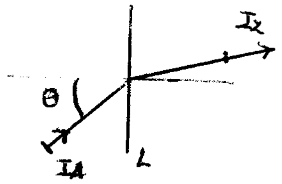
$$P_{1\to 2} = \omega\, \eta_{12}\, E_1$$

power transmitted from 1 to 2 — vibrational energy

specific intensity : $I_1 = c_{g1} \dfrac{E_1}{2\pi A_1}$

group speed — surface

transmission efficiency : $T(\theta) = \dfrac{P_{transmitted}}{P_{incident}}$ ← transmitted power / ← incident power.



$$P_{1\to 2} = \int_L \int I_1 \cos\theta\, T(\theta)\, d\theta\, dL$$

$$P_{1\to 2} = \frac{E_1}{2\pi A_1} c_{g1} L \int_{-\pi/2}^{\pi/2} T(\theta)\cos\theta\, d\theta$$

Hence

dimension 2

$$\boxed{\eta_{12} = \frac{L\, c_{g1}}{\pi\omega A_1}\int_0^{\pi/2} T(\theta)\cos\theta\, d\theta}$$

coupling length — plate area

dimension 3

$$\boxed{\eta_{12} = \frac{S\, c_{g1}}{4\pi\omega V_1}\int_0^{2\pi}\int_0^{\pi/2} T(\theta,\varphi)\cos\theta\sin\theta\, d\theta\, d\varphi}$$

coupling surface — room volume

The problem reduces to the computation of the mean transmission efficiency.

Identification procedure

System = { n subsystems }

The subsystems are alternatively excited by a source of unit power.

E_j^i : energy in subsystem j for a unit power in subsystem i

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ P_i = 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} & & \\ & \eta_{ij} & \\ & & \end{pmatrix} \begin{pmatrix} E_1^i \\ \vdots \\ E_n^i \end{pmatrix} \quad \text{n equations}$$

measured — unknown — measured

Repeating the experiment for other source subsystems gives n^2 equations.

The coupling loss factors η_{ij} are obtained by solving these equations.

Remark: reciprocity of CLF has not been used.

⑦ Energy balance

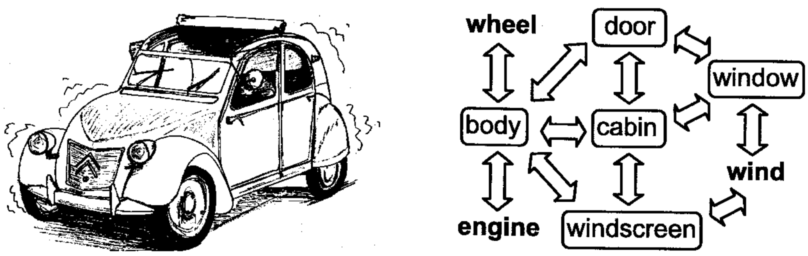


Fig. 10.1 Example of substructuring a truly vibrating system.

For each subsystem, the energy balance reads

$$P_i = P_{diss,i} + \sum_{j \neq i} P_{ij}$$

injected power   dissipation   exchanged power

loss by damping $P_{diss,i} = \eta_i \omega E_i$

loss by coupling $P_{ij} = \omega \eta_{ij} E_i - \omega \eta_{ji} E_j$

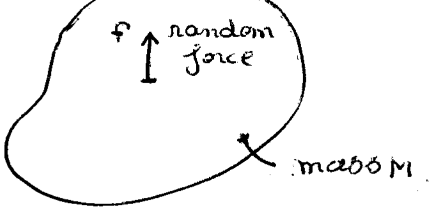
Hence

$$P_i = \eta_i \omega E_i + \sum_{j \neq i} \left(\eta_{ij} \omega E_i - \eta_{ji} \omega E_j \right)$$

In a matrix form

$$\boxed{\omega \begin{pmatrix} \sum_j \eta_{1j} & & -\eta_{ji} \\ & \ddots & \\ -\eta_{ji} & & \sum_j \eta_{nj} \end{pmatrix} \begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix} = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}}$$

⑧ Injected power



Injected power $P = \langle f v \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{Fv}(\omega)\, d\omega$

force   velocity

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ff}(\omega)\, Y(\omega)\, d\omega$$

mobility

$$P = \langle f^2 \rangle \frac{1}{\Delta\omega} \int_{\Delta\omega} \mathrm{Re}\left[Y(\omega) \right] d\omega .$$

A statistical estimation of the mean conductance (real part of the mobility) gives,

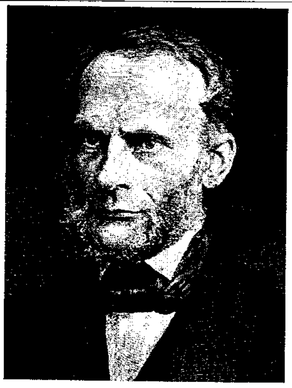
$$\boxed{P = \langle F^2 \rangle \times \frac{\pi\, n(\omega)}{2M}}$$

modal density

mass of subsystem

square mean force

9 Entropy in statistical energy analysis



R. Clausius

Clausius' postulate:

Heat cannot be tranferred from cold body to hot body without converting heat into work.

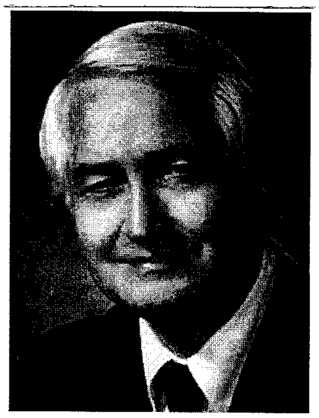
$$\boxed{T_2 > T_1} \xrightarrow{\delta Q} \boxed{T_2}$$

Heat spontaneously flows from hot body to cold body.

Coupling power proportionality:

$$P_{ij} = \omega \eta_{ij} N_i \left(\frac{E_i}{N_i} - \frac{E_j}{N_j} \right)$$

vibrational power   high modal energy   low modal energy



R.H. Lyon

Vibrational energy flows from high modal energy to low modal energy

Analogy: heat $\delta Q \leftrightarrow$ vibrational energy dE

temperature $T \leftrightarrow$ modal energy E/N

Clausius entropy: $dS = \frac{\delta Q}{T} \longleftrightarrow$ vibrational entropy $\boxed{dS = N \frac{dE}{E}}$

variation of energy

variation of entropy   # of modes

i) Production of entropy

Sources warm up the system by injecting energy

$$\frac{dS_{inj,i}}{dt} = \frac{N_i}{E_i}\frac{dE_i}{dt} = \frac{N_i P_{inj,i}}{E_i} > 0$$

ii) Dissipation of entropy

Dissipation cool down the system by extracting energy

$$\frac{dS_{diss,i}}{dt} = \frac{-N_i}{E_i} P_{diss,i} \quad \text{but} \quad P_{diss,i} = \eta_i \omega E_i \quad \text{therefore} \quad \frac{dS_{diss,i}}{dt} = -\eta_i \omega N_i < 0.$$

iii) Mixing entropy

$$\frac{dS_{ij}}{dt} = P_{ij}\left(\frac{N_j}{E_j} - \frac{N_i}{E_i}\right) \quad \text{but} \quad P_{ij} = \omega(\eta_{ij}E_i - \eta_{ji}E_j) \quad \text{therefore} \quad \frac{dS_{ij}}{dt} = \omega(\eta_{ij}E_i - \eta_{ji}E_j)\left(\frac{N_j}{E_j} - \frac{N_i}{E_i}\right) > 0$$

iv) Entropy balance

produced entropy   dissipated entropy   mixing entropy

$$\frac{dS}{dt} = \sum_i \left[\frac{N_i}{E_i} P_{inj,i} - \frac{N_i}{E_i}\eta_i \omega E_i \right] + \sum_{i>j} \left(\frac{N_j}{E_j} - \frac{N_i}{E_i} \right) P_{ij} = 0.$$