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EQUILIBRIUM AND NON-EQUILIBRIUM IN STATISTICAL ENERGY ANALYSIS

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ABSTRACT

This paper is a discussion of the assumptions of Statistical Energy Analysis within the framework of non-equilibrium statistical mechanics. It is shown that Statistical Energy Analysis is analogous to statistical mechanics of systems in thermal equilibrium when equipartition of vibrational energy is reached. The extension of SEA which aims to relax the diffuse field assumption is analogous to non-equilibrium statistical mechanics. The transition from non-equilibrium to equilibrium is controlled by three dimensionless parameters, the number of modes, the modal overlap and the normalized attenuation factor.

1 INTRODUCTION

Statistical Energy Analysis is largely inspired from statistical mechanics. By considering that modes are numerous and that their exact frequency is not known, SEA splits the structure into sub-systems defined as group of modes. The exchange of vibrational energy is then porportional to the difference of modal energies.

SEA is the simplest way to adapt statistical mechanics to vibroacoustics. And since SEA assumes the equipartition of energy, SEA is the theory of global equilibrium of vibrational energy. The recent developments of SEA follow the historical developments of statistical mechanics. The study of fluctuations in SEA [1, 2] can be linked the numerous fluctuation theorems in statistical mechanics. The various extensions of SEA [3–5] which aim to relax the diffuse field assumption are analogous to non-equilibrium statistical mechanics. The introduction of entropy in SEA [6, 7] is also an unavoidable consequence of the application of statistical mechanics ideas.

In this study, the assumptions of SEA are discussed in connection with equilibrium and non-equilibrium statistical mechanics with a particular attention paid to the transition from equilibrium to non-equilibrium.

2 EQUILIBRIUM STATISTICAL ENERGY ANALYSIS

SEA is a statistical method applied to the audio frequency range (macroscopic vibrations) in the same manner that thermodynamics is a statistical method applied to thermal vibrations at the molecular scale (microscopic vibrations). While in thermodynamics the statistical population is composed by a large number of molecules, atoms or any other sites which store the vibrational energy, in SEA, the energy is localized in a large number N of modes. The number of molecules

of thermodynamical systems is of order of Avogadro's number (10^{23}). But in SEA, the number of modes may be only of order of several thousands in the best case [8]. Even if SEA is applied to a statistical population of similar systems (the so-called canonical Gibbs ensemble) where N denotes the cumulative number of modes, the population still remains relatively low. This highlights that SEA is a statistical method applied to small populations, and indeed, this fact can cause some difficulties. Fluctuations around the mean are more important for small populations. In Ref. [1, 2], the variance is found to be of order of $\log N/N^2$. The size of the population is therefore the first criterion for the applicability of SEA. The mode count N that is the number of modes within the frequency band $\Delta\omega$ is,

$$N = n\Delta\omega, \tag{1}$$

where n is the modal density and a large population of modes reads,

$$N >> 1. \tag{2}$$

This is the first criterion of validity of SEA.

SEA is the study of incoherent vibrational energy in the same manner that thermodynamics is the study of 'degraded' mechanical energy. This state of degradation for energy only arises when the disorder prevails in the statistical population. Disorder is inherent to the statistical method. Simple laws can emerge from the behaviour of a large population provided that all 'individuals' are similar and that any of them may influence the population more than other ones. For thermal vibrations of solids, disorder means that the vibration of atoms are uncorrelated. While in the kinetic theory of gases, disorder means that the velocities of molecules before the shock are statistically independent. This is the so-called molecular chaos or *Stosszahlanstaz* introduced by Boltzmann in 1872 [9]. Disorder in vibroacoustics rather means that mode amplitudes, considered as random variables, are uncorrelated. This state is reached when no mode dominates the dynamics of the system that is when the frequency response function is smooth. The modal overlap defined as,

$$M = n\eta\omega, \tag{3}$$

is a measure of the overlapping of successive modes in the frequency response function. Thus, the criterion for disorder in SEA is,

$$M >> 1. \tag{4}$$

The diffuse field assumption of SEA means that the vibrational energy density is homogeneous and isotropic in all sub-systems. This assumption is equivalent to the equipartition of modal energy (the vibrational energy is equally shared among all modes) [8]. But the modal energy plays the same role as the energy per molecule in thermodynamics which is exactly the definition of the temperature. This is why the modal energy can be called the vibrational temperature. Thus, the diffuse field assumption means that the vibrational temperature is the same at any point of the sub-system or, in other words, that the sub-system is in thermal equilibrium. To reach this equilibrium state, it is necessary that rays are mixed. The general mathematical conditions under which a diffuse field can emerge are studied in billiard's theory [10]. But at least, rays must cross several times the sub-system before to be attenuated. If c_g is the group speed of waves and $m = \eta\omega/c_g$ designates the attenuation factor of wave per meter,

$$\bar{m} = \frac{\eta\omega}{c_g} l, \tag{5}$$

can be called the normalized attenuation factor [11], l being the mean free path of the sub-system. Its value must be low enough to ensure mixing of rays that is the thermal equilibrium,

$$\bar{m} << 1. \tag{6}$$

All sub-systems are in thermal equilibrium. But two adjacent sub-systems may have different vibrational temperatures. This is the assumption of local equilibrium. The same situation arises in non-equilibrium thermodynamics. The notion of local temperature makes sense providing that a local equilibrium is reached. The thermal energy flows higher temperature to lower temperature. In SEA, the linearity of the net exchanged power with vibrational temperatures is simply the expression of linearity between fluxes and forces in linear irreversible thermodynamics [12], $\omega\eta_{ij}n_i$ being the appropriate transport coefficient. It is well-known in non-equilibrium thermodynamics that the linearity of fluxes and forces is valid for systems which are not too far from equilibrium. This is the light coupling assumption. In the context of SEA, the light coupling assumption means that the flow of exchanged vibrational energy is small compared with the internal dissipation of energy. This can be enunciated as $\eta_{ij} << \eta_i$. Although some authors have studied the possibility to extend SEA to strong coupling [13], the usual relationships for coupling loss factors are derived under the light coupling assumption. Following Smith [14], the coupling strength is defined as $\gamma_{ij} = \eta_{ij}/\eta_i$. In case of assembled plates with a coupling length b, perimeters P_i and a transmission efficiency τ_{ij}, it yields,

$$\gamma_{ij} = \frac{\tau_{ij} b}{\bar{m}_i P_i}. \tag{7}$$

The light coupling condition reads,

$$\gamma_{ij} << 1. \tag{8}$$

A complete transmission ($\tau_{ij} = 1$) over a small length in a large plate ($b << P_i$) leads to a light coupling.

Indeed, the symmetric condition must also apply,

$$\gamma_{ji} << 1. \tag{9}$$

But the conditions Eqs. (8, 9) are not independent. They are related by,

$$\gamma_{ij} M_i = \gamma_{ji} M_j. \tag{10}$$

The set of dimensionless parameters γ_{ij}, γ_{ji}, M_i and M_j is therefore dependent.

3 NON-EQUILIBRIUM STATISTICAL ENERGY ANALYSIS

When equipartition of energy is not verified, classical thermodynamics is no longer valid and as a consequence, classical SEA does not apply. This is the domain non-equilibrium statistical mechanics. Non-equilibrium statistical mechanics is mainly based on Boltzmann's equation. The distribution function $f(\mathbf{x}, \mathbf{v})$ introduced by Boltzmann is the density in the phase space of particles at any point $\mathbf{x}$ with any velocity $\mathbf{v}$. Boltzmann's equation gives the spatial and time evolution of this distribution function f by writing that particles in volume $d\tau$ with velocity in solid angle $d\Omega$ have either travelled with a constant velocity or result from a shock between two particle with velocities $\mathbf{v}'$ and $\mathbf{v}'_1$ (Figure 1).

This suggests that equilibrium hypothesis in SEA can be relaxed with an analogous of Boltzmann's equation. But for steady-state SEA, rays are more appropriate than particles. Furthermore, rays can cross without colliding. The mixing of rays is therefore rather due to multiple reflections on boundaries rather than collisions inside the domain (Figure 2). The analogous of the distribution function f is the radiative intensity $I(\mathbf{x}, \mathbf{u})$ defined as the power per unit length and unit angle normal to the ray $\mathbf{u}$. The balance of rays being reflected and incident on the boundary is,

$$I(\mathbf{x}, \mathbf{u}) = \int R(\mathbf{x}, \mathbf{u}', \mathbf{u}) \cos\theta' I(\mathbf{x}, \mathbf{u}') du' + I_0 \tag{11}$$

where R is the bi-directional reflectivity of the boundary, θ' the incident angle and I_0 is the source term. Indeed, the energy density and therefore, the total vibrational energy of sub-systems can

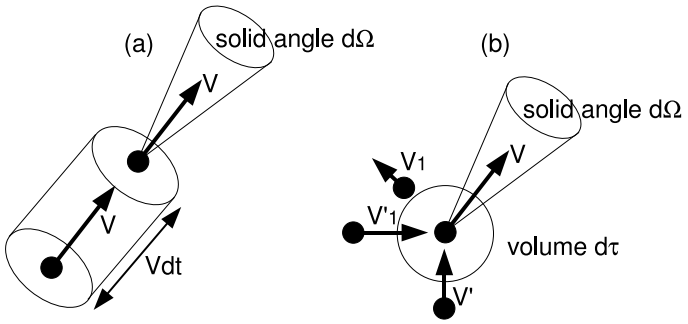

Figure 1: Boltzmann's equation. Particles in $d\tau$ with velocity $\mathbf{v}$ in $d\Omega$ result from, (a) a direct transport over $\mathbf{v}dt$ or, (b) a shock between two particles with velocities $\mathbf{v}'$ and $\mathbf{v}'_1$.

calculated from the knowledge of radiative intensity on the boundary. The complete theory is derived in References [3, 15, 16].

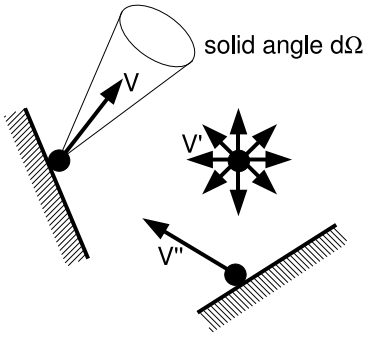

Figure 2: Mixing of rays. Rays reflected on the boundary within solid angle $d\Omega$ about $\mathbf{v}$ can stem from vibrational sources with velocity $\mathbf{v}'$ or from another part of the boundary with velocity $\mathbf{v}''$. In the latter case, a multiple reflection occurs.

Such a theory is analogous to non-equilibrium statistical mechanics. And, as well, equilibrium is a particular case of non-equilibrium. Thus SEA equation can be derived from Eq. (11) [11].

The validity of this theory is the same as any energetic ray theory. The assumptions are large number of modes $N >> 1$, large modal overlap to ensure the additivity of energy $M >> 1$. But the mixing of rays $\bar{m} << 1$ is no longer required neither is the light coupling $\gamma_{12} << 1$ and $\gamma_{21} << 1$.

4 VALIDITY DOMAIN OF SEA

The dimensional analysis provides a useful tool to define the validity domain of Statistical Energy Analysis. Let develop the reasoning on the example of two coupled plates. The system is shown in Figure 3. It is constituted by two rectangular plates whose exterior boundary are simply supported and the common boundary ensures the continuity of deflection, rotation, moment and force.

The governing equation for the out-of-plane vibration v_i of plate i is,

$$\Delta^2 v_i - k_i^4(1 - j\eta_i)v_i = f_i, \tag{12}$$

where the imaginary part $-j\eta_i$ is the contribution of damping to the wavenumber k_i. The boundary

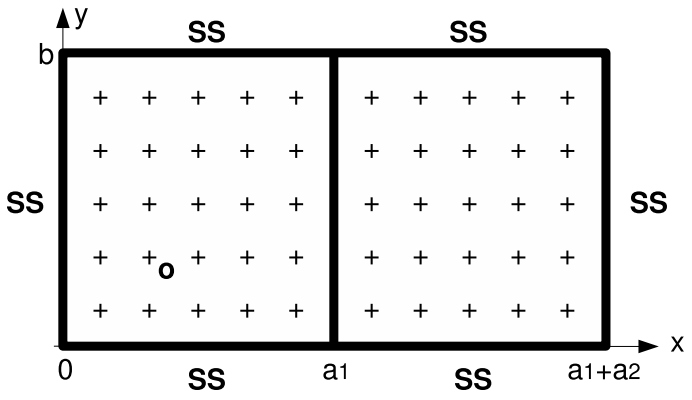

Figure 3: Two rectangular coupled plates. Their lengths are a_1 and a_2 and their width is b. Exterior edges are simply supported and the plates are coupled along the common edge. Position of the driven force (o) and the receiver points (+).

conditions for simply supported edges at $y = 0$, $y = b$ are,

$$v_i(x,0) = v_i(x,b) = 0, \tag{13}$$

$$\frac{\partial^2 v_i}{\partial y^2}(x,0) + \nu_i \frac{\partial^2 v_i}{\partial x^2}(x,0) = \frac{\partial^2 v_i}{\partial y^2}(x,b) + \nu_i \frac{\partial^2 v_i}{\partial x^2}(x,b) = 0. \tag{14}$$

Similar boundary conditions must also hold at $x = 0$ and $x = a_1 + a_2$. The continuity conditions at the interface are,

$$v_1(a_1,y) = v_2(a_1,y), \tag{15}$$

$$\frac{\partial v_1}{\partial y}(a_1,y) = \frac{\partial v_2}{\partial y}(a_1,y) \tag{16}$$

$$D_1\left[\frac{\partial^2 v_1}{\partial x^2}(a_1,y) + \nu_1 \frac{\partial^2 v_1}{\partial y^2}(a_1,y)\right] = D_2\left[\frac{\partial^2 v_2}{\partial x^2}(a_1,y) + \nu_2 \frac{\partial^2 v_2}{\partial y^2}(a_1,y)\right], \tag{17}$$

$$D_1\left[\frac{\partial^3 v_1}{\partial x^3}(a_1,y) + \nu_1 \frac{\partial^3 v_1}{\partial y^2 \partial x}(a_1,y)\right] = D_2\left[\frac{\partial^3 v_2}{\partial x^3}(a_1,y) + \nu_2 \frac{\partial^3 v_2}{\partial y^2 \partial x}(a_1,y)\right]. \tag{18}$$

So, the only physical parameters of this set of equations are the wavenumbers k_1, k_2, the damping loss factors η_1, η_2, the length a_1, a_2, the common width b, the Poisson's coefficients ν_1, ν_2 and the bending stiffnesses D_1 and D_2. There is 11 physical parameters. Their only physical units are the length and the Newton. The theorem of Vaschy-Buckingham [17, 18] gives the number of dimensionless parameters of this problem, $11 - 2 = 9$. These dimensionless parameters can be chosen arbitrarily provided that they are independent. A possible choice is the dimensionless wavenumber $\kappa_i = k_i l_i / 2\pi$ where $l_i = \pi a_i b / 2(a_i + b)$ is the mean free path, the shape ratio $\epsilon_i = (a_i + b)/\sqrt{\pi a_i b}$ defined as the ratio between the perimeter of the plate and that of a circle of the same area, the damping loss factor η_i, the Poisson's coefficient ν_i and the transmission efficiency τ_{12} which is a function of the ratio D_1/D_2. This set of dimensionless parameters is well-suited to rewrite the governing equation (12) and the related boundary conditions (13-18) in dimensionless form. As well, it will be called 'primary' set of dimensionless parameters.

But any other choice of independent dimensionless parameters is possible. The set of dimensionless SEA parameters introduced in Section 2 that is, N_1, N_2, M_1, M_2, $\bar{m}_1$, $\bar{m}_2$, ν_1, ν_2 and γ_{12}, is acceptable provided that a one to one map can be found,

$$\kappa_1, \eta_1, \epsilon_1, \kappa_2, \eta_2, \epsilon_2, \tau_{12} \longrightarrow N_1, M_1, \bar{m}_1, N_2, M_2, \bar{m}_2, \gamma_{12}. \tag{19}$$

These relationships are easily found,

$$N_i = 2\sqrt{2}\kappa_i^2\epsilon_i^2, \quad (20)$$

$$M_i = 4\eta_i\kappa_i^2\epsilon_i^2, \quad (21)$$

$$\bar{m}_i = \pi\eta_i\kappa_i, \quad (22)$$

$$\gamma_{12} = \frac{\tau_{12}}{\pi\eta_1\kappa_1}\mu_1(\epsilon_1). \quad (23)$$

where $\mu_1 = b/2(a_1 + b)$ is a function of ϵ_1. The problem of a pair of vibrating plates is mathematically fully determined by the only nine dimensionless parameters N_i, M_i, $\bar{m}_i$, ν_i and γ_{12}. Since SEA is a theory included in the Love's theory of plate in the sense that equation of SEA can be derived from the governing equation of Love's plate and considering that Poisson's coefficient is of a low importance in SEA, the validity domain of SEA is necessarily confined into the 7-dimensional space N_1, M_1, $\bar{m}_1$, N_2, M_2, $\bar{m}_2$ and γ_{12}.

Eqs. (2), (4) and (6) give the boundary of the validity domain of SEA for plates and Eq. (8) for the coupling. The last condition (9) ($\gamma_{21} << 1$) can be expressed as,

$$\gamma_{12}\frac{M_1}{M_2} << 1. \quad (24)$$

All these conditions are curved varieties in the 7-dimensional space.

5 NUMERICAL SIMULATION

The numerical simulation aims to compare results from direct numerical simulation and SEA.

Simulation	O	$\triangle$	∇	$\times$	+
κ	5.0	10.0	2.9	0.91	5
η	0.01	0.1	0.0001	0.08	0.01
ϵ	1.13	1.13	1.96	1.96	1.13
τ	0.125	0.125	0.0012	0.68	1.0
N	90	360	90	9	90
M	1.3	51	0.013	1	1.3
$\bar{m}$	0.16	3.14	0.001	0.23	0.16
γ	0.2	0.01	0.62	0.14	1.6

Table 1: Dimensionless parameters. Both plates have same dimensionless numbers. They are acceptable for SEA in the reference simulation (o) but not for other simulations: The attenuation $\bar{m}$ is too high ($\triangle$), the modal overlap M is too low (∇), the mode count N is too low ($\times$) and the coupling strength is high (+).

The direct numerical simulation is based on the solving of Eq. (12) for each plate with conditions (13-18). The solution is developped as a Fourier's series along the y-axis. The series is truncated to 2000 terms. Plate 1 is excited by an out-of plane force located at x_0 and y_0. The response is computed on a 30×30 grid of receiver points on each plate. The vibrational energy E_i of plates in the octave band is assessed by computing the frequency response function for 1000 frequencies and by summing the contributions of the 900 receiver points.

The SEA computation is quite simple. It is based on the set of linear equations,

$$\omega \begin{pmatrix} n_1(\eta_1 + \eta_{12}) & -n_2\eta_{21} \\ -n_1\eta_{12} & n_2(\eta_2 + \eta_{21}) \end{pmatrix} \begin{pmatrix} E_1/n_1 \\ E_2/n_2 \end{pmatrix} = \begin{pmatrix} P_1^{\text{inj}} \\ P_2^{\text{inj}} \end{pmatrix} \quad (25)$$

The modal density is given by,

$$n = \frac{S\omega}{2\pi c_g c_\varphi}, \tag{26}$$

where S is the area of the plate, c_g and c_φ the group speed and the phase speed of the flexural wave. The coupling loss factors is,

$$\eta_{ij} = \frac{b c_{g_i}}{\pi S_i \omega} \tau_{ij} \tag{27}$$

where τ_{ij} is the mean transmission efficiency from plate i to plate j. The injected power is,

$$P_1^{\text{inj}} = \frac{|F|^2}{16\sqrt{mD}}. \tag{28}$$

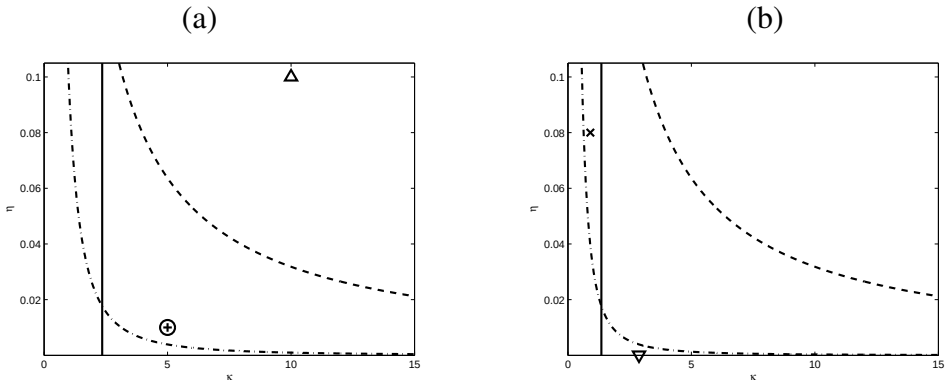

Figure 4: Validity domain of SEA for a pair of coupled rectangular plates. The lines $\bar{m} = 1$ (–), $M = 0.5$ (-.-), $N = 20$ (-), $\gamma_{12} = 0.5$ and $\gamma_{21} = 0.5$ are plotted in the κ, η-cut of 'primary' dimensionless space in cases (a), $\epsilon = 1.13$ and (b), $\epsilon = 1.96$. Numerical simulations are positioned in the reference case (o), for large attenuation ($\triangle$), low modal overlap (∇), few modes ($\times$) and strong coupling (+).

Five numerical simulations have been realized. The respective dimensionless numbers are shown in Table 1. They have been chosen as follows. A first simulation (symbol o) is done with dimensionless numbers N, M and $\bar{m}$ chosen in such a way that they are correct for SEA. They verify the criteria (2), (4), (6), (8), (9). All subsequent simulations violate a criteria. The second simulation (symbol $\triangle$) has a large normalized attenuation factor $\bar{m} > 1$, the third simulation (symbol ∇) has a small modal overlap ($M < 1$), the fourth simulation (symbol $\times$) has a small number of modes $N = 9$ and the fifth simulation (symbol +) has a strong coupling $\gamma > 1$. The positions of these five simulations are plotted in Figures 4a and 4b respectively for $\epsilon = 1.13$ and $\epsilon = 1.96$. In these diagrams, the validity domain of SEA is delimited by the lines $N = 20$, $M = 0.5$ and $\bar{m} = 1$. The position of symbols relative to these lines clearly show which assumption is violated.

Numerical results are shown in Figure 5. The relative error between the direct numerical simulation and SEA is plotted for the power being injected by the driven force and the total vibrational energies of plates 1 and 2. It is clear that the reference calculation (symbol o) shows a fine agreement between SEA and direct numerical simulation. It means that the injected power is well estimated by Eq. (28), but also that the power being exchanged between the two plates is well estimated. For the following three simulations $\triangle$, ∇ and $\times$, the energy of plate 2 is not well estimated. The discrepancy is large (from 200% up to 400%) and cannot be explained by the error on the injected power. Both injected power and exchanged power given by SEA are wrong. Also interesting is the last simulation (+) with strong coupling. Although the assumption $\gamma << 1$ is violated, results are good. It seems that the assumptions (8, 9) are not so important, at least for this case of two coupled plates.

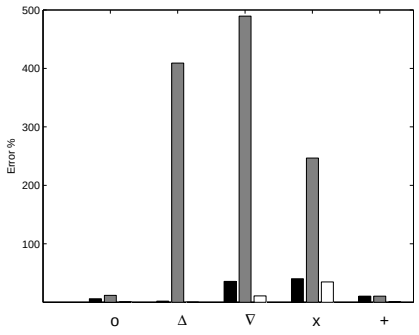

Figure 5: Relative errors(%) of SEA results compared with the direct numerical simulation. Errors on energy of plate 1 (black), energy of plate 2 (grey), injected power (white). The reference situation (o) and strong coupling (+) lead to good results on injected power as well as vibrational energies. But, large attenuation (Δ), low modal overlap (∇) and low number of modes (x) show discrepancies between SEA and direct numerical simulation.

6 CONCLUDING REMARKS

In this paper, it has been shown that SEA is analogous to statistical mechanics of systems in thermal equilibrium when equipartition of vibrational energy is reached. More exactly, each sub-system must be in thermal equilibrium but, two adjacent sub-systems may have different modal energies and therefore, may exchange vibrational energy. The conditions under which equilibrium is reached are large mode count $N >> 1$, large modal overlap $M >> 1$, low attenuation $\bar{m} << 1$ and light couplings $\gamma_{12} << 1$ and $\gamma_{21} << 1$. These conditions define the validity domain of SEA in the κ,ϵ,η,τ-space. The numerical simulations highlight that they are necessary since a discrepancy is observed between direct numerical simulation and SEA if any of these conditions is violated (excepted light coupling may be).

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