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ENERGY METHODS APPLIED TO TRANSVERSE VIBRATIONS OF BEAMS

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ABSTRACT
This paper discusses two methods for modelling the spatial distribution of energy density in structures. Each method provides its own equation on energy quantities. The resolution of these equations are described. Numerical and experimental results obtained with each method and the classical one based on equation of motion are compared.

RESUME
Ce papier présente deux methodes de modélisation des répartitions spatiales des densités d'énergies au sein des structures. Des équations sur les grandeurs énergétiques sont développées pour chaque méthode ainsi que leur résolution. Des comparaisons numériques et expérimentales avec les équations classiques du mouvement sont présentées.

1- INTRODUCTION

Structural dynamic analysis is generally based on strengh and displacement governing equations. Along with computing advances, numerical techniques such as finite element method or boundary element method were developed to solve these equations. In both cases, the continuous structure is represented with a finite number of degrees of freedom. In order to reach the same accuracy, the higher the frequency, the bigger the number of degrees of freedom. So, in practise, computer power limits frequency scope : these numerical methods are thus suited for low frequencies.

Statistical Energy Analysis (SEA) is well suited for high frequencies. Based on a statistical approach, the SEA describes the structure with one global energy quantity for each subsystem, that is, it does not provide any information about the spatial energy distribution inside each subsystem.

Few investigations have been made about medium frequencies. The structures must be described with global quantities without neglecting the modal behavior. During the last decade, the heat conduction equation has been proposed in order to predict the spatial variation of total energy density within structures. Some investigations demonstrates the feasibility of this approach.

In the following study, two methods of energetical behavior prediction are presented. On one hand, general energy formulation (GEF) deals with lagrangian energy and reactive energy flow in order to take the modal behavior into account. On the other hand, smooth energy formulation (SEF) uses the heat conduction equation. Some numerical and experimental comparisons between each method and the classical one are presented.

2- THEORICAL DEVELOPMENT

We study the steady state forced oscillations of linear systems harmonically excited with frequency $\omega/2\pi$. Each point of the system has an instantaneous kinetic energy density $E_c(t)$, potential energy density $E_p(t)$ and an instantaneous energy flow $P(t)$. The sum and the difference of these energies give respectively the total energy $W(t) = E_c(t) + E_p(t)$ and the lagrangian $L(t) = E_c(t) - E_p(t)$. In the harmonic case, the time is removed by taking time-averaged quantities,

$$T = \frac{2\pi}{\omega}, \qquad E_c = \frac{1}{T}\int_0^T E_c(t)dt, \qquad E_p = \frac{1}{T}\int_0^T E_p(t)dt, \qquad W = E_c + E_p, \; L = E_c - E_p.$$

The energy flow becomes complex with active part P and reactive part Q. The general energy formulation deals with the four energy quantities W, L, P, and Q, whereas the smooth energy formulation only uses W, and P.

Now let us consider the power balance with instantaneous quantities at point M:

$$-\vec{\nabla}.\vec{P}(M,t) - p_{diss}(M,t) = \frac{\partial W}{\partial t}(M,t),$$

where $\vec{P}$ is the energy flow at point M, p_{diss} the power density being dissipated, and $\frac{\partial W}{\partial t}$ the time rate of change of energy density. In the harmonic case, this relationship contains complex values. So by separating real and imaginary parts, it yields:

$$\vec{\nabla}.\vec{P} + p_{diss} = 0, \qquad \vec{\nabla}.\vec{Q} + 2\omega L = 0. \qquad (1), (2)$$

The interpretation of the first equation is straightforward: in steady state conditions, the active energy flow received by the system is equal to the dissipated power density (the time-averaged rate of change of the energy density vanishes). The second equation shows that the reactive energy flow received by the system is proportional to the lagrangian. So, it has a nonzero divergence even though the system is conservative. Now, in order to formulate an energy equation from (1) and (2), we have to determine the power density dissipated and energy flow in terms of energies W and L.

It is usefull to consider hysteretic damping. One subtitutes in Hooke's law a complex Young's modulus. It follows that the power being dissipated is proportional to both, the potential energy and the frequency. So,

$$p_{diss} = \eta\omega(W - L). \qquad (3)$$

This relationship is suited for the GEF but the SEF needs to use W solely. Hence, we add the well-known assumption that kinetic and potential energies are equal. In fact, in most cases, it is a very accurate assumption:

$$p_{diss} = \eta\omega W. \qquad (4)$$

These last relationship is more convenient than (3) for the smooth energy formulation.

3- GENERAL ENERGY FORMULATION

Energy flow must now be assessed in terms of energies. To carry out this analysis, let us consider the specific case of an Euler-Bernoulli's beam. Let u be the transverse deflection of the beam. The classical equation of bending vibration for such a beam is :

$$\frac{d^4u}{dx^4} - k^4u = 0, \qquad (5)$$ where k is the complex wave number.

By neglecting the potential energy of the shear force and the kinetic energy of rotation, it follows:

$$E_c = \frac{1}{4}\rho S\omega^2 uu^*, \qquad E_p = \frac{1}{4}E_0 I\frac{d^2u}{dx^2}\frac{d^2u^*}{dx^2}. \qquad (6), (7)$$

Here, two kinds of energy flow are present : the first one, $\Pi_t = P_t + jQ_t$, is carried by shear force and the second one, $\Pi_f = P_f + jQ_f$ by the moment. Their expressions in terms of displacement are:

$$\Pi_f = -\frac{j\omega}{2}EI\frac{d^2u}{dx^2}\frac{du^*}{dx}, \qquad \Pi_t = \frac{j\omega}{2}EI\frac{d^3u}{dx^3}u^*,$$

where $E=E_0(1+j\eta)$ is the complex Young's modulus.

By calculating the third and seventh derivatives of energy densities and by using equation (5) each time a fourth derivative appears, we can express the energy flows in terms of energies:

$$\begin{pmatrix} P_f \\ Q_f \end{pmatrix} = \frac{-\omega}{16\eta^3}\left\{ \frac{(1+\eta^2)^2}{k_0^8} \begin{pmatrix} 8+\eta^2 & -\eta^2 \\ 6\eta & 0 \end{pmatrix} \frac{d^7}{dx^7}\begin{pmatrix} W \\ L \end{pmatrix} + \frac{(1+\eta^2)}{k_0^4} \begin{pmatrix} -128-64\eta^2 & 36\eta^2 \\ -96\eta-34\eta^3 & 34\eta^3 \end{pmatrix} \frac{d^3}{dx^3}\begin{pmatrix} W \\ L \end{pmatrix} \right\}, \quad (8)$$

$$\begin{pmatrix} P_t \\ Q_t \end{pmatrix} = \frac{-\omega}{16\eta^3}\left\{ \frac{(1+\eta^2)^2}{k_0^8} \begin{pmatrix} 8+3\eta^2 & -3\eta^2 \\ 2\eta & 0 \end{pmatrix} \frac{d^7}{dx^7}\begin{pmatrix} W \\ L \end{pmatrix} + \frac{(1+\eta^2)}{k_0^4} \begin{pmatrix} -128-96\eta^2 & 28\eta^2 \\ -32\eta-14\eta^3 & 14\eta^3 \end{pmatrix} \frac{d^3}{dx^3}\begin{pmatrix} W \\ L \end{pmatrix} \right\}, \quad (9)$$

where k_0 is the real part of the complex wave number.
By adding these two relationships, the total energy flow is obtained.

To derive the energy equations of the GEF, we have to substitute the damping model (3) and the energy flow relationships (8) and (9) into the power balance (1) and (2). It follows:

$$\frac{d^8}{dx^8}\begin{pmatrix} W \\ L \end{pmatrix} - k_0^4 \begin{pmatrix} 16 & -6\eta^2 \\ 6\eta^2 & 0 \end{pmatrix} \frac{d^4}{dx^4}\begin{pmatrix} W \\ L \end{pmatrix} + k_0^8 \begin{pmatrix} 0 & 4\eta^2 \\ -4\eta^2 & 16 \end{pmatrix} \begin{pmatrix} W \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (10)$$

Then we obtain a system of two differential equations of the eighth order governing the behavior of energy densities. To solve this system, it must be transformed into a system of the first order, like $\frac{d\mathbf{Z}}{dx} = \mathbf{AZ}$. The matrix **A** has sixteen eigenvalues which are:

$$2k_0,\ -2k_0,\ 2jk_0,\ -2jk_0,\ k_0\left(1-\frac{\eta}{4}\right)(1-j),\ k_0\left(1-\frac{\eta}{4}\right)(1+j),\ -k_0\left(1-\frac{\eta}{4}\right)(1-j),\ -k_0\left(1-\frac{\eta}{4}\right)(1+j)$$
$$\frac{\eta}{2}k_0,\ -\frac{\eta}{2}k_0,\ j\frac{\eta}{2}k_0,\ -j\frac{\eta}{2}k_0,\ k_0\left(1+\frac{\eta}{4}\right)(1-j),\ k_0\left(1+\frac{\eta}{4}\right)(1+j),\ -k_0\left(1+\frac{\eta}{4}\right)(1-j),\ -k_0\left(1+\frac{\eta}{4}\right)(1+j)$$

With the sixteen eigenvalues, the numerical stability is very bad. But one can improve it by remarking that, in fact, less than sixteen eigenvalues are necessary to calculate the energies. Four eigenvalues are almost equal to another four and can thus be excluded. Moreover, the eigenvalues $-j\frac{\eta}{2}k_0, j\frac{\eta}{2}k_0$ lead to circular functions with a very large period, that is, almost constant. They can also be excluded. Finally, the following expressions for energy densities are chosen:

$$W(x) = \sum_{i=1}^{6} A_i e^{\lambda_i x} \quad \text{with} \quad \lambda_i = 2k_0, -2k_0, 2jk_0, -2jk_0, \frac{\eta}{2}k_0, -\frac{\eta}{2}k_0, \quad (11)$$

$$L(x) = \sum_{i=7}^{10} A_i e^{\lambda_i x} \quad \text{with} \quad \lambda_i = k_0(1+j), -k_0(1+j), k_0(1-j), -k_0(1-j). \quad (12)$$

At last, the boundary conditions must be determined. Ten constants have to be calculated. Thus, at each end, five boundary conditions are necessary. It is adviced, for better numerical stability, not to use (8) and (9) as boundary conditions, but to call the relationships on energy derivatives with smaller order. For example, at clamped end:

$$(W+L) = \frac{d(W+L)}{dx} = \frac{d^2(W+L)}{dx^2} = \frac{d^3(W+L)}{dx^3} = \frac{d^3(W-L)}{dx^3} = 0, \quad (13)$$

and at free end:

$$(W-L) = \frac{d(W-L)}{dx} = \frac{d^2(W-L)}{dx^2} = \frac{d^3(W-L)}{dx^3} = \frac{d^3(W+L)}{dx^3} = 0. \quad (14)$$

Let numerically compare the results obtained with the proposed formulations. Consider a free clamped beam as shown in figure 1.

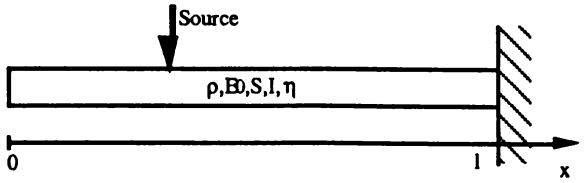

fig. 1 Free-clamped beam

On figures 2 and 3, active and reactive energy flow are plotted. The components carried by shear force and moment also appear.

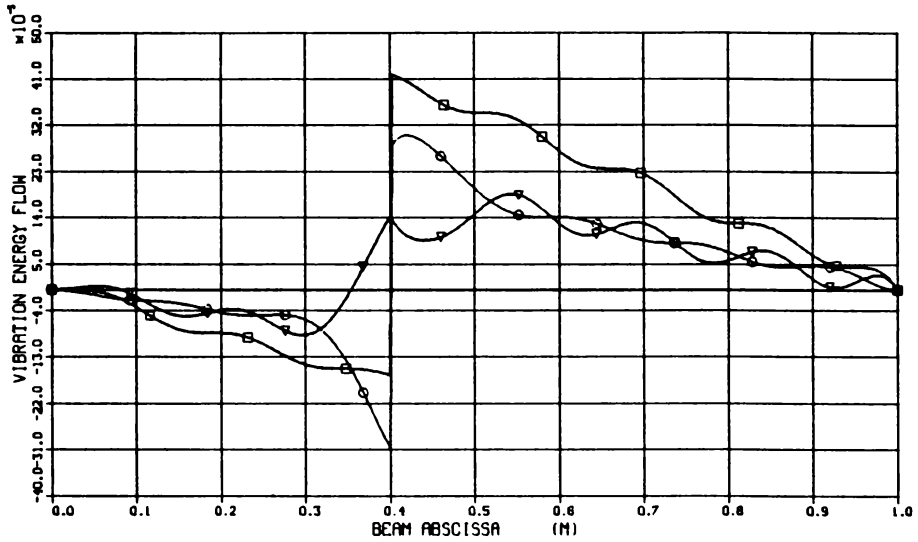

fig. 2 Active energy flows

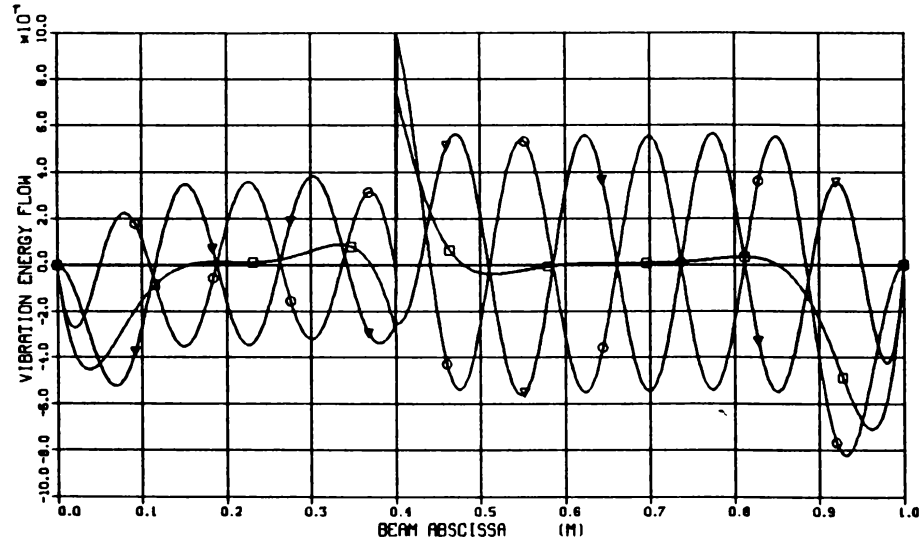

fig. 3 Reactive energy flows

It can be observed that the far-field is characterized by no reactive energy flow. More precisely, the shear energy flow and the bending energy flow are opposite. So,

$$Q_t = -Q_f.$$

For the active energy flow, in farfield conditions, the shear energy flow is almost equal to the bending energy flow. By taking the space averaged quantities over one wavelength, that we note <.> with brackets, we have:

$$\langle P_t \rangle = \langle P_f \rangle.$$

More concisely:

$$\langle \Pi_t \rangle = \langle \Pi_f \rangle^*. \tag{15}$$

In farfield condition, the complex shear energy flow and bending energy flow are conjugate to each other.

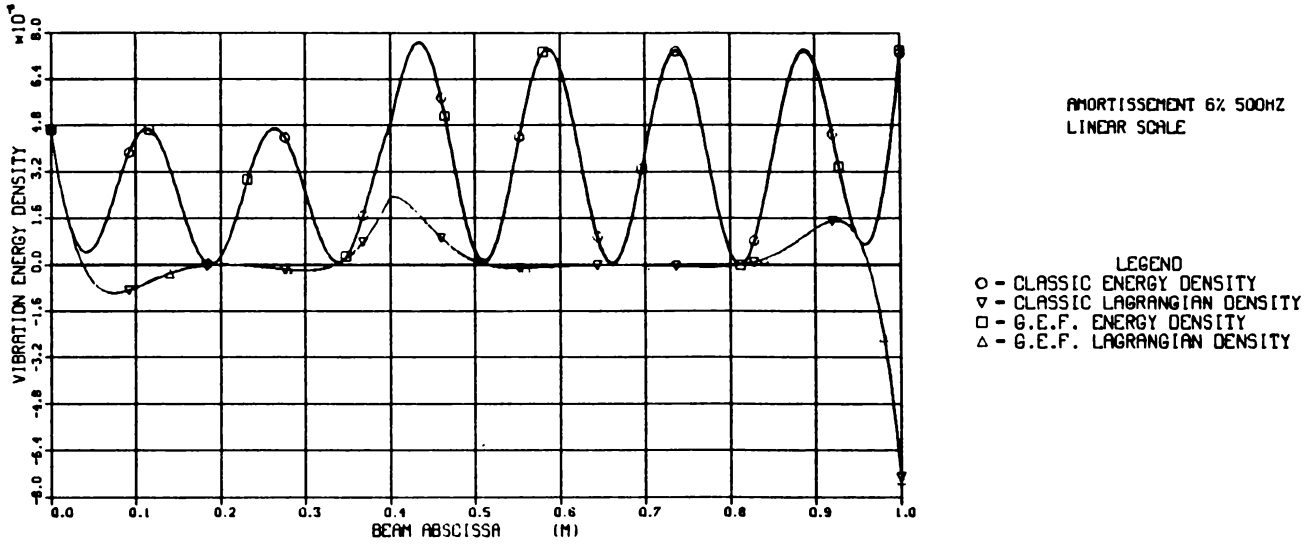

fig. 4 Comparison between classical and GEF energy densities.

On figure 4, two kinds of calculation are visualized. The first one, the classical method, is based on equation (5). Energy densities are determined by using (6) and (7). The second one is based on solutions (11) and (12) of equation (10). Boundary conditions (13) and (14) are used at the end of the beam. We can observe that there is no difference beween the results of both types of calculation. In fact, the general energy formulation assumes nothing more than the displacement formulation.

4- SMOOTH ENERGY FORMULATION

For simplication purposes, let us consider space averaged quantities as discussed previously and beside this, we alsoassume the farfield conditions. Then the lagrangian vanishes and the third derivative of the total energy becomes proportionnal to the first one, $\frac{d^3\langle W \rangle}{dx^3}(x) = \frac{\eta^2 k_0^2}{4}\frac{d\langle W \rangle}{dx}(x)$. Thus, we obtain a simplified relationship between active energy flow and total energy density:

$$\langle P \rangle(x) = -\frac{4\omega}{\eta k_0^2}\frac{d\langle W \rangle}{dx}(x). \tag{16}$$

By substituting this relationship and damping model (4) into power balance (1), it yields the following governing equation on total energy density:

$$\frac{d^2\langle W \rangle}{dx^2}(x) - \frac{\eta^2 k_0^2}{4}\langle W \rangle(x) = 0. \tag{17}$$

The general analytic solution of this equation is exacly the function given by the two last eigenvalues of relationship (11).

$$W(x) = A_5 e^{\frac{\eta}{2}k_0 x} + A_6 e^{-\frac{\eta}{2}k_0 x}. \tag{18}$$

However, equation (17) contains more information than the smooth energy response given by (18). To see it, let us consider the general solution for movement equation (5).

$$u(x) = \alpha_1 \mathbf{e}^{-jkx} + \alpha_2 \mathbf{e}^{jkx} + \alpha_3 \mathbf{e}^{-kx} + \alpha_4 \mathbf{e}^{kx}$$

By summing relationships (6) and (7), the total energy density is obtained. With the following approximation $\rho S\omega^2 \approx E_0 I|k|^4$ and by identifying the result with (11), coefficients A_i are given in terms of coefficients α_i. Thus,

$$A_5 = \frac{1}{2}\rho S\omega^2 \alpha_1 \alpha_1^* \qquad A_6 = \frac{1}{2}\rho S\omega^2 \alpha_2 \alpha_2^* \qquad A_3 = \frac{1}{2}\rho S\omega^2 \alpha_2 \alpha_1^* \qquad A_4 = \frac{1}{2}\rho S\omega^2 \alpha_1 \alpha_2^*$$

Coefficients A_3 and A_4 are those of the varying component of the energy. One relationship between the range H of these variations and coefficients A_5 and A_6 can be established.

$$H = 2\sqrt{\frac{1}{2}\left(A_3 A_3^* + A_4 A_4^*\right)} = \rho S\omega^2 \sqrt{\alpha_1 \alpha_2^* \alpha_1 \alpha_2^*} = 2\sqrt{A_5 A_6} \qquad (19)$$

So equation (17) provides information about the space-averaged energy density and also the range of variations of this energy density.

Now, let try to evaluate the near-field. The eigenvalues $-2k_0$, $-k_0(1+j)$, $-k_0(1-j)$ are associated to the near-field of left side singularity while $2k_0$, $k_0(1+j)$, $k_0(1-j)$ are associated to right side singularity. So, at a given point, just one type of nearfield cannot be neglected. By following the same pattern as previously done to obtain (19), the coefficient of the near-field for total energy density can be expressed for some specific ends.

-clamped end: A_1 or $A_2 = A_5 + A_6$
-free end: A_1 or $A_2 = A_5 + A_6$
-simply supported end: A_1 or $A_2 = 0$

Unfortunately, these relationships cannot be estabished for more general boundary conditions.

As an example, the beam shown in figure 1 may be cut into two parts: from the free-end to the source and from the source to the clamped-end. Let assume that the power injected into, separately, the left part and the right part of the structure, are known. The solution of equation (17) for the left part is given by (18). The boundary conditions are $\langle P \rangle(0) = 0$ and $\langle P \rangle(l) = -P_{inj}$. Thus,

$$\langle W \rangle(x) = \frac{k_0}{2\omega} P_{inj} \frac{ch\frac{\eta}{2}k_0 x}{sh\frac{\eta}{2}k_0 l}, \qquad \langle P \rangle(x) = -P_{inj} \frac{sh\frac{\eta}{2}k_0 x}{sh\frac{\eta}{2}k_0 l}. \qquad (20), (21)$$

At the left singularity, the near-field can be determined, that is $\frac{k_0 P_{inj}}{2\omega sh\frac{\eta}{2}k_0 l}\mathbf{e}^{-2k_0 x}$. By adding it with relationship (20), the non-varying component of the energy density is given by:

$$W(x) = \frac{k_0 P_{inj}}{2\omega sh\frac{\eta}{2}k_0 l}\left[\mathbf{e}^{-2k_0 x} + ch\frac{\eta}{2}k_0 x\right]. \qquad (22)$$

Moreover, at the free-end the varying component is equal to zero. So, the relationship is exact at point x=0:

$$W(0) = \frac{k_0 P_{inj}}{\omega sh\frac{\eta}{2}k_0 l}.$$

At last, the lagrangian and the reactive energy flow can be derived near the free end. Let consider the simplified expression for the reactive energy flow:

$$Q(x) = \mathrm{e}^{-k_0 x}\left(A_1 \sin k_0 x + A_2 \cos k_0 x\right).$$

The lagrangian is issued from (2). Therefore,

$$L(x) = \frac{-k_0}{2\omega}\mathrm{e}^{-k_0 x}\left(-A_1 \sin k_0 x - A_2 \cos k_0 x + A_1 \cos k_0 x - A_2 \sin k_0 x\right).$$

Finally, the two following boundary conditions must be applied $Q(0) = 0$ and $L(0) = W(0)$. So,

$$L(x) = W(0)\mathrm{e}^{-k_0 x}\left[\cos k_0 x - \sin k_0 x\right], \qquad Q(x) = -\frac{2\omega}{k_0} W(0)\mathrm{e}^{-k_0 x} \sin k_0 x.$$

In fact, we have determined the energy density, lagrangian density, active and reactive energy flows by knowing the injected power solely.

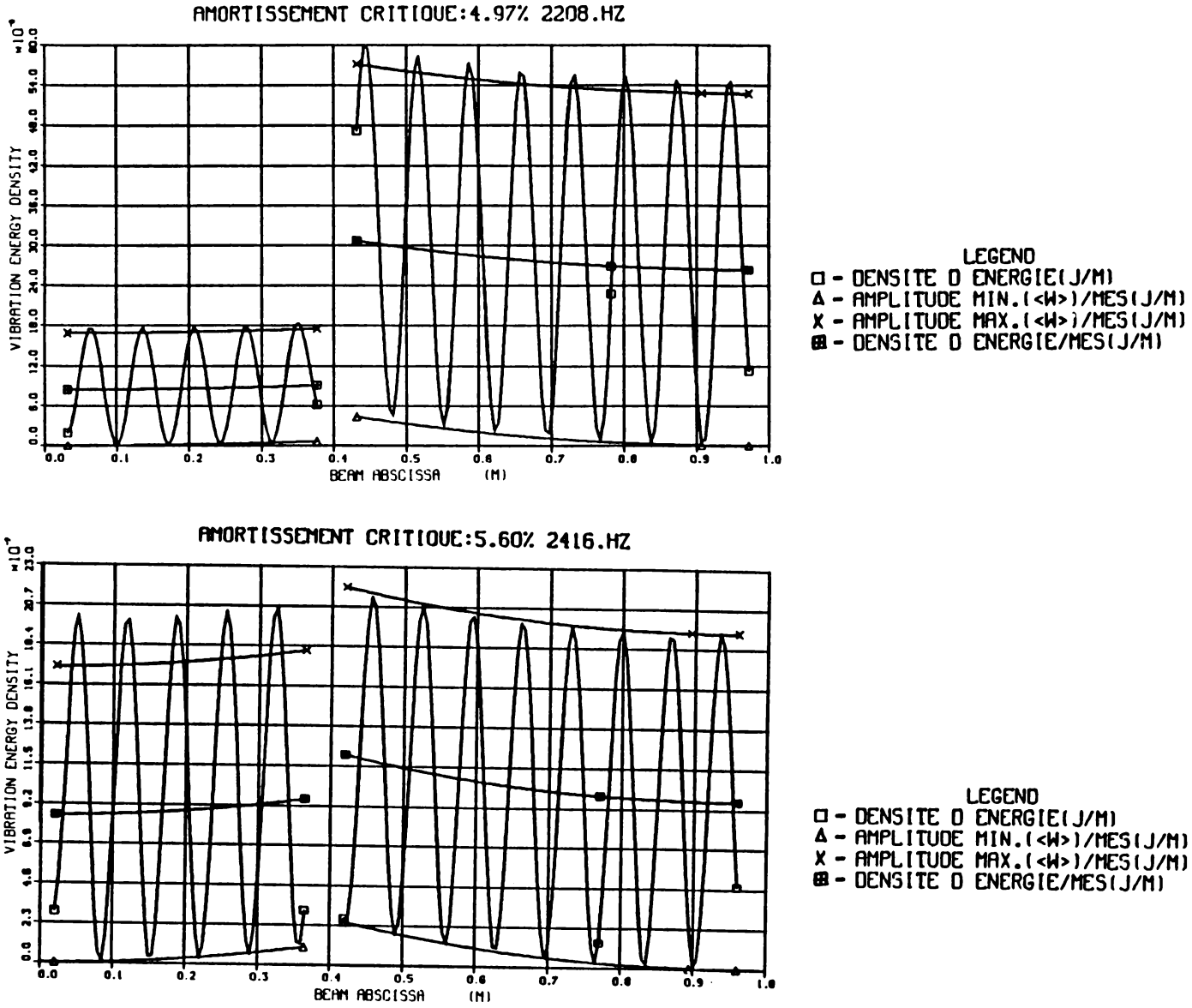

fig. 5 and 6 Comparison between experimentation and SEF

In figure 5 and 6 are compared the results from a measurement and a computation with the smooth energy formulation. First of all, the displacement has been measured along the beam at two hundred points. The energy density has been deduced from this measurements by applying the

relationships (6) and (7). Then, the injected power in both directions has been determined from few points of measurement near the source. And, by using (18) with its boundary conditions, and (19), the energy density and its enveloppes have been computed.

5- CONCLUSION

Two energy formulations have been presented. The first one, the GEF, assumes nothing more than the classical formulation does. The solutions of equation (10) are as accurate as the energies calculated from (6) and (7). For example, it is possible to solve coupled problem with GEF. There is no numerical advantage in using the GEF when compared to the classical formulation. But the SEF can be derived from the former. There is a great advantage to use equation (17) rather than the displacement formulation. When solved by finite element method, a smaller number of degrees of freedom is required. It is thus well suited for modelling high frequencies vibrations behavior. But it is difficult to solve coupled problem with SEF. It should be very convenient to use both methods, the GEF near the singularities to deal with coupled problems, and the SEF for the farfield with its numerical advantage.

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